

Superintegrability in non-inertial reference frames with conic defects

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Based on:

1. L. Inzunza and M. S. Plyushchay, [JHEP 2021, 165 \(2021\)](#) [[arXiv:2012.04613 \[hep-th\]](#)].
2. L. Inzunza and M. S. Plyushchay, [Phys. Rev. D 103, 106004 \(2021\)](#) [[arXiv:2103.07752 \[quant-ph\]](#)].
3. L. Inzunza and M. S. Plyushchay, [JHEP 2022, 179 \(2022\)](#) [[arXiv:2109.05161\[hep-th\]](#)].

Introduction

Symmetries control peculiarities of physical systems at the classical and quantum level^{4,5,6,7}.

Conserved quantities → Conserved operators

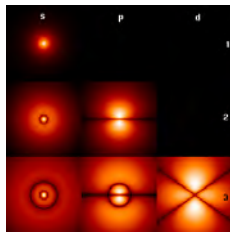
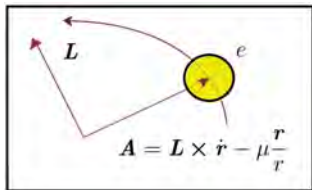


Figure: On the left: Hidden symmetries in the classical Kepler problem. \mathbf{A} is the so-called Laplace-Runge-Lenz vector. On the right: probability densities through the xz -plane for the electron at different quantum numbers (l , across top; n , down side; $m = 0$) ([Wikipedia:Hydrogen atom](#)).

4. W. Pauli, Z Phys 36: 336-363 (1926).

5. J. J. Sakurai (1985). Modern quantum mechanics.

6. J. de Boer, F. Harmsze and T. Tjin, *Phys. Rept.* **272** (1996), 139-214 [[arXiv:hep-th/9503161](#) [[hep-th](#)]].

7. M. Cariglia, *Rev. Mod. Phys.* **86**, 1283 (2014) [[arXiv:1411.1262](#) [[math-th](#)]].

Classical dynamics in curved spaces

Given a space-time metric of the form

$$ds^2 = -c^2 dt^2 + g_{ij}(x) dx^i dx^j ,$$

one can construct the relativistic action

$$S_R = -mc \int \sqrt{c^2 dt^2 - g_{ij}(x) dx^i dx^j} .$$

Symmetries of $S_R \rightarrow$ Killing vectors and tensors associated to ds^2 ^{8,9}.

The non-relativistic limit:

$$S = \int \frac{m}{2} g_{ij}(x) \frac{dx^i}{dt} \frac{dx^j}{dt} .$$

Symmetries of $S \rightarrow$ non-relativistic limits ^{10,11} of the Killing vectors and Killing tensors associated to ds^2 .

8. M. Cariglia, *Rev. Mod. Phys.* **86**, 1283 (2014) [[arXiv:1411.1262\[math-th\]](#)].

9. M. Crampin, *Reports on Math. Phys* **20** (1984) 31.

10. D. T. Son, *Phys. Rev. D* **78** (2008) 046003 [[arXiv: 0804.3972 \[hep-th\]](#)].

11. A. Bagchi and R. Gopakumar, *JHEP* **0907** (2009) 037 [[arXiv: 0902.1385 \[hep-th\]](#)].

- ① Rotating background
- ② A cosmic string background
- ③ A rotating conic background
- ④ Conclusion and outlook

Rotating background ^{12,13,14}

The metric of the $(3+1)$ Minkowski vacuum subjected to a uniform rotation Ω along the z axis in polar coordinates is given by

$$ds^2 = -c^2 dt^2 + d\rho^2 + \rho^2(d\varphi + \Omega dt)^2 + dz^2.$$

The non-relativistic Lagrangian

$$L = L_\Omega + \frac{m}{2}\dot{z}^2, \quad L_\Omega = \frac{m}{2}(\dot{\rho}^2 + \rho^2(\dot{\varphi}^2 + \Omega^2)),$$

formally relates with the flat Lagrangian by $\varphi(t) \rightarrow \varphi(t) + \Omega t$.

In Cartesian coordinates ($g_{ij} = \delta_{ij}$)

$$L_\Omega = g_{ij}\left(\frac{m}{2}\dot{x}^i\dot{x}^i + \frac{m}{2}\Omega^2 x^i x^i + \frac{q_{GM}}{c}\dot{x}^i A^i\right), \quad A_i = \frac{\Omega c}{2}\epsilon^{ij}x^j.$$

Gravitoelectromagnetism

$B_G = \epsilon_{ij}\partial_i A_j = -\Omega c$ is interpret as constant Gravitomagnetic field.
 $q_{GM} = -2m$ is the Gravitomagnetic charge.

12. T. Kibble, *J. Phys. A* **9** (1976) 1387.

13. A. Vilenkin, *Phys. Rev. D* **23** (1981) 852.

14. A. Vilenkin, *Phys. Rept.* **121** (1985) 263.

Free motion

The geodesic Hamiltonian in polar coordinates

$$H_{\Omega} = H_0 - \Omega p_{\varphi}, \quad H_0 = \frac{1}{2m} g^{ij} p_i p_j, \quad p_{\varphi} = x^1 p_2 - x^2 p_1,$$

shows that the energy is not restricted from below (#).

The equations of motions are solved by a rotation of the straight line,

$$x_+(t) = x^1 + ix^2 = \rho e^{i\varphi} = e^{-i\Omega t} (Bt + C), \quad B, C \in \mathbb{C}.$$

If the minimum of the trajectory occurs at t_* , then, the value of $x_+(t_*) = \rho_* e^{i\varphi_*}$ is controlled by the the integrals of motion.

$$\rho_* = \frac{|p_{\varphi}|}{\sqrt{2mH_0}}, \quad e^{\pm i\varphi_*} \approx I_{\pm} = \exp\left(\pm \frac{\Omega \rho p_{\rho}}{2H_0}\right) (p_1 \pm ip_2).$$

The system is ill-defined at the quantum level (#).

Harmonically confined motion

We supplement L_Ω with an harmonic potential

$$L_\gamma = L_\Omega - \frac{1}{2}m\gamma^{-2}\Omega^2 g_{ij}x^i x^j. \quad \gamma \in \mathbb{R}.$$

When introducing a new frequency ω by the equation $\gamma\omega = -\Omega$, the Hamiltonian takes the form

$$H_\gamma = H_{\text{osc}} + \gamma\omega p_\varphi, \quad H_{\text{osc}} = H_0 + \frac{1}{2}m\omega^2 g_{ij}x^i x^j.$$

In terms of “spherical” ladder operators classical analogs,

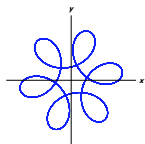
$$b_j^\pm = \frac{1}{\sqrt{2}}(a_1^\pm \mp i(-1)^j a_2^\pm), \quad \{b_j^\pm, b_k^\mp\} = -i\delta_{jk}, \\ a_i^\pm = \sqrt{\frac{m\omega}{2}}(x_i \pm \frac{i}{m\omega}p_i),$$

the Hamiltonian and the equations of motion looks like

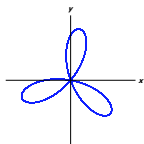
$$H_\gamma = \omega(\ell_1 b_1^+ b_1^- + \ell_2 b_2^+ b_2^-), \quad \ell_1 = 1 + \gamma, \quad \ell_2 = 1 - \gamma, \\ \dot{b}_i^\pm = \{b_i^\pm, H\} = \pm i\omega \ell_i b_i^\pm \quad \Rightarrow \quad b_i^\pm(t) = e^{\pm i\omega \ell_i t} b_i^\pm(0).$$

In terms of $x_+(t)$:

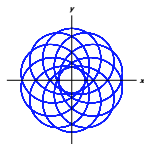
$$x_+(t) = R_1 e^{i\vartheta_1} e^{i\omega \ell_1 t} + R_2 e^{-i\vartheta_2} e^{-i\omega \ell_2 t}.$$



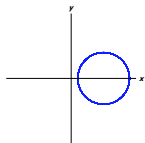
(a) $\gamma = 2/3$, $R_1 < R_2$



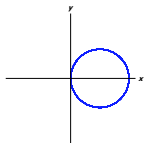
(b) $\gamma = 1/3$, $R_1 = R_2$



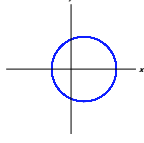
(c) $\gamma = 4/5$, $R_1 > R_2$



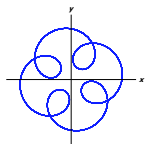
(d) $\gamma = 1$, $R_1 < R_2$



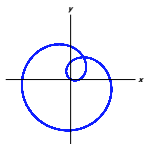
(e) $\gamma = 1$, $R_1 = R_2$



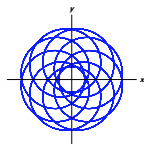
(f) $\gamma = 1$, $R_1 > R_2$



(g) $\gamma = 3/2$, $R_1 < R_2$



(h) $\gamma = 3$, $R_1 = R_2$



(i) $\gamma = 5/4$, $R_1 > R_2$

Figure: Roulettes: Trajectories for some rational values of γ . In cases b), e) and h), $p_\varphi = 0$ and trajectories pass through the origin.

Integrability in rotating harmonic trap

The classical ladder operators in polar coordinates are

$$b_j^- = \frac{1}{2} e^{(-1)^j i\varphi} (\sqrt{m\omega} \rho - (-1)^j \frac{p_\varphi}{\sqrt{m\omega} \rho} + \frac{i p_\rho}{\sqrt{m\omega}}).$$

Integrals of motion appears for $\gamma \in \mathbb{Q}$.

$$\begin{array}{ll} \gamma = \frac{s_2 - s_1}{s_1 + s_2} & |\gamma| \leq 1 \quad \mathcal{L}_{s_1, s_2}^\pm = (b_1^\pm)^{s_1} (b_2^\mp)^{s_2} \\ \gamma = \frac{s_2 + s_1}{s_1 - s_2} & |\gamma| > 1 \quad \mathcal{J}_{s_1, s_2}^\pm = (b_1^\pm)^{s_1} (b_2^\pm)^{s_2} \end{array}.$$

Poisson brackets:

$$\begin{array}{ll} \{H_{\text{osc}}, \mathcal{L}_{s_1, s_2}^\pm\} = \mp i\omega(s_1 - s_2) \mathcal{L}_{s_1, s_2}^\pm & \{p_\varphi, \mathcal{L}_{s_1, s_2}^\pm\} = i(s_1 + s_2) \mathcal{L}_{s_1, s_2}^\pm \\ \{H_{\text{osc}}, \mathcal{J}_{s_1, s_2}^\pm\} = \mp i\omega(s_1 + s_2) \mathcal{J}_{s_1, s_2}^\pm & \{p_\varphi, \mathcal{J}_{s_1, s_2}^\pm\} = i(s_1 - s_2) \mathcal{J}_{s_1, s_2}^\pm \end{array}.$$

and so (I_{s_1, s_2}^\pm equal to $\mathcal{L}_{s_1, s_2}^\pm$ or $\mathcal{J}_{s_1, s_2}^\pm$),

$$\begin{aligned} \{H_\gamma, I_{s_1, s_2}^\pm\} &= \{H_{\text{osc}}, I_{s_1, s_2}^\pm\} + \gamma\omega\{p_\varphi, I_{s_1, s_2}^\pm\} = 0. \\ \{I_{s_1, s_2}^+, I_{s_1, s_2}^-\} &= P(H_{\text{osc}}, p_\varphi). \end{aligned}$$

Quantum picture

Canonical quantization

$$b_j^\pm(x, p) \rightarrow \hat{b}_j^\pm = \hbar^{-\frac{1}{2}} b_j^\pm(\hat{x}, \hat{p}), \quad [\hat{b}_i^\mp, \hat{b}_j^\pm] = \delta_{ij}, \\ \hat{H}_\gamma = \hbar\omega(\ell_1 \hat{b}_1^+ \hat{b}_1^- + \ell_2 \hat{b}_2^+ \hat{b}_2^- + 1), \quad \hat{p}_\varphi = \hbar(\hat{b}_1^+ \hat{b}_1^- - \hat{b}_2^+ \hat{b}_2^-).$$

Stationary eigenstates Ψ_{n_1, n_2} with $n_1, n_2 = 0, \dots$, that satisfy

$$\hat{b}_1^- \Psi_{n_1, n_2} = \sqrt{n_1} \Psi_{n_1-1, n_2}, \quad \hat{b}_1^+ \Psi_{n_1, n_2} = \sqrt{n_1 + 1} \Psi_{n_1+1, n_2}, \\ \hat{b}_2^- \Psi_{n_1, n_2} = \sqrt{n_2} \Psi_{n_1, n_2-1}, \quad \hat{b}_2^+ \Psi_{n_1, n_2} = \sqrt{n_2 + 1} \Psi_{n_1, n_2+1}.$$

Eigenvalue equations are:

$$\hat{H}_\gamma \Psi_{n_1, n_2} = E_{n_1, n_2}^{(\gamma)} \Psi_{n_1, n_2}, \quad E_{n_1, n_2}^{(\gamma)} = \hbar\omega(\ell_1 n_1 + \ell_2 n_2 + 1) \\ \hat{p}_\varphi \Psi_{n_1, n_2} = \hbar(n_1 - n_2) \Psi_{n_1, n_2}.$$

In coordinate representation, Ψ_{n_1, n_2} is a linear combination of Hermite polynomials times $e^{-\frac{m\omega}{2\hbar}(x_1^2 + x_2^2)}$.

Degeneracy

$$\text{For } \gamma = \frac{s_2 - s_1}{s_1 + s_2} \quad \hat{\mathcal{L}}_{n_1, n_2}^+ \Psi_{n_1, n_2} = \sqrt{\frac{n_2!}{n_1!} \frac{\Gamma(n_1 + s_1 + 1)}{\Gamma(n_2 - s_2 + 1)}} \Psi_{n_1 + s_1, n_2 - s_2}$$

$$\hat{\mathcal{L}}_{n_1, n_2}^- \Psi_{n_1, n_2} = \sqrt{\frac{n_1!}{n_2!} \frac{\Gamma(n_2 + s_2 + 1)}{\Gamma(n_1 - s_1 + 1)}} \Psi_{n_1 - s_1, n_2 + s_2}$$

- $E_{s_1, s_2}^{(\gamma)}$ is bounded from below.
- $E_{s_1, s_2}^{(\gamma)}$ is invariant under $n_1 \rightarrow n_1 \pm s_1$ and $n_2 \rightarrow n_2 \mp s_2$.
- $|\gamma| < 1$, there are finite number of eigenstates for each energy level.
- For $|\gamma| = 1$, each energy level is infinite degenerate (Landau problem).

$$\text{For } \gamma = \frac{s_2 + s_1}{s_2 - s_2} \quad \hat{\mathcal{J}}_{n_1, n_2}^- \Psi_{n_1, n_2} = \sqrt{\frac{n_1! n_2!}{\Gamma(n_1 - s_1 + 1) \Gamma(n_2 - s_2 + 1)}} \Psi_{n_1 - s_1, n_2 - s_2}$$

$$\hat{\mathcal{J}}_{n_1, n_2}^+ \Psi_{n_1, n_2} = \sqrt{\frac{\Gamma(n_1 + s_1 + 1) \Gamma(n_2 + s_2 + 1)}{n_1! n_2!}} \Psi_{n_1 + s_1, n_2 + s_2}$$

- $E_{s_1, s_2}^{(\gamma)}$ is not bounded from below.
- $E_{s_1, s_2}^{(\gamma)}$ is invariant under $n_1 \rightarrow n_1 \pm s_1$ and $n_2 \rightarrow n_2 \mp s_2$.
- Each energy level has infinite degeneracy.

A cosmic string background ^{12,13,14}

The metric produced by a cosmic string is

$$dS^2 = -c^2 dt^2 + ds^2, \quad ds^2 = \left(1 - \frac{8\mu G}{c^2} \ln\left(\frac{r}{r_0}\right)\right) (dr^2 + r^2 d\varphi^2),$$

G : Newton constant.

c : Speed of light.

μ : Linear mass density of the cosmic string.

r_0 : Cosmic string radius.

By introducing the new coordinate

$$r'^2 = \left(1 - \frac{8\mu G}{c^2} \ln\left(\frac{r}{r_0}\right)\right) r^2, \quad \alpha^2 dr'^2 = \left(1 - \frac{8\mu G}{c^2} \ln\left(\frac{r}{r_0}\right)\right) dr^2, \\ \alpha = \frac{1}{1 - \frac{4\mu G}{c^2}} > 0,$$

one gets (renaming $r' \rightarrow r$)

$$ds^2 = \alpha^2 dr^2 + r^2 d\varphi^2.$$

12. T. Kibble, *J. Phys. A* **9** (1976) 1387.

13. A. Vilenkin, *Phys. Rev. D* **23** (1981) 852.

14. A. Vilenkin, *Phys. Rept.* **121** (1985) 263.

The conical metric

$$ds^2 = \alpha^2 dr^2 + r^2 d\varphi^2.$$

$$g_{ij} = \delta_{ij} + (\alpha^2 - 1) \frac{1}{(x^1)^2 + (x^2)^2} \begin{pmatrix} (x^1)^2 & x^1 x^2 \\ x^1 x^2 & (x^2)^2 \end{pmatrix},$$

$$\alpha > 1: \left\{ \begin{array}{l} \bullet \text{ Reduction from } ds_E^2 = dx^2 + dy^2 + dz^2 \\ \text{to the conical surface } z = \lambda_E r. \alpha = 1 + \lambda_E^2. \\ \bullet \alpha > 1 \text{ implies } \mu > 0. \end{array} \right.$$

$$0 < \alpha < 1: \left\{ \begin{array}{l} \bullet \text{ Reduction from } ds_M^2 = -c^2 d\tau^2 + dx^2 + dy^2 \\ \text{to the surface } \tau = \lambda_M r \text{ (non-causal), } \alpha = 1 - \lambda_M^2. \\ \bullet 0 < \alpha < 1 \text{ implies } \mu < 0. \\ \text{Acquires a different interpretation in condensed matter }^{15,16}. \end{array} \right.$$

ds^2 is formally obtained from the plane metric by means of the transformation $\rho \rightarrow \alpha\rho$ and $\varphi \rightarrow \alpha^{-1}\varphi$.

15. G. E. Volovik, (Oxford Science Publications, 2003).

16. N. S. Manton, J. Phys. A **50** (2017) 125403 [arXiv: 1612.06710 [hep-th]].

Non-relativistic motion in a cone

Consider the systems

$$I^{(\alpha)} = \int L^{(\alpha)} dt, \quad L^{(\alpha)} = \frac{m}{2} g_{ij} \frac{dx_i}{dt} \frac{dx_j}{dt} = \frac{m}{2} (\alpha^2 \dot{r}^2 + r^2 \dot{\varphi}^2),$$
$$I_{\text{osc}}^{(\alpha)} = \int L_{\text{osc}}^{(\alpha)} dt, \quad L_{\text{osc}}^{(\alpha)} = \frac{m}{2} (\alpha^2 \dot{r}^2 + r^2 \dot{\varphi}^2) - \frac{m\omega^2 \alpha^2}{2} r^2,$$

where g_{ij} is the conical metric.

The Hamiltonian are ^{17,18}:

$$H^{(\alpha)} = \frac{1}{2m} \left(\frac{p_r^2}{\alpha^2} + \frac{p_\varphi^2}{r^2} \right), \quad H_{\text{osc}}^{(\alpha)} = H^{(\alpha)} + \frac{m\omega^2 \alpha^2}{2} r^2.$$

Notable!

The local canonical transformation

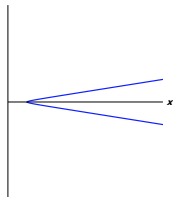
$$r \rightarrow \alpha r, \quad p_r \rightarrow \frac{p_r}{\alpha}, \quad \varphi \rightarrow \frac{\varphi}{\alpha}, \quad p_\varphi \rightarrow \alpha p_\varphi,$$

applied to the Hamiltonians in the Euclidian plane gives us $H^{(\alpha)}$ and $H_{\text{osc}}^{(\alpha)}$.

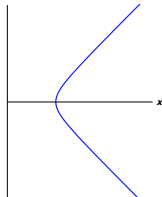
17. G.'t Hooft, [Commun. Math. Phys.](#) **117** (1988) 685.

18. S. Deser and R. Jackiw, [Comm. Math. Phys](#) **118** (1988) 495.

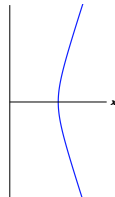
Solutions of the equations of motion



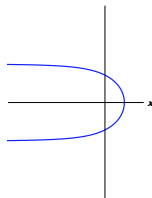
(a) $\alpha = 1/10$



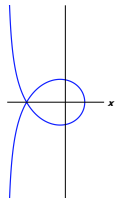
(b) $\alpha = 1/2$



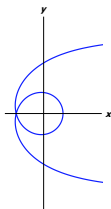
(c) $\alpha = 4/5$



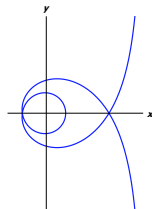
(d) $\alpha = 2$



(e) $\alpha = 3$



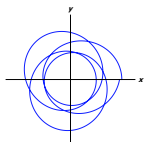
(f) $\alpha = 4$



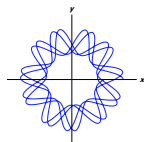
(g) $\alpha = 5$

Figure: Solutions of the equations of motion for different values of α .

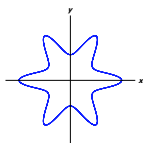
Solutions of the equations of motion



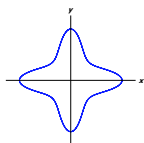
(a) $\alpha = e$



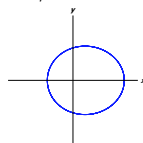
(b) $\alpha = 1/e$



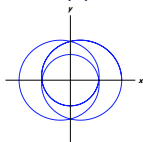
(c) $\alpha = 1/3$



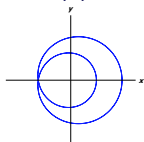
(d) $\alpha = 1/2$



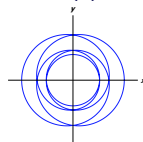
(e) $\alpha = 2$



(f) $\alpha = 3$



(g) $\alpha = 4$



(h) $\alpha = 5$

Figure: Only when α is rational one has closed trajectories.

A rotating conic background

After applying the transformation $\rho \rightarrow \alpha\rho$ and $\varphi \rightarrow \alpha^{-1}\varphi$ (and $\Omega \rightarrow \alpha\Omega$) in to the rotating vacuum metric we get

$$ds^2 = -c^2 dt^2 + \alpha^2 d\rho^2 + \rho^2 (d\varphi + \alpha\Omega dt)^2.$$

The non-relativistic dynamics is obtained by the same transformation:

| Geometric background | Phase space functions |
|----------------------|--------------------------------------------------------------------------|
| Plane | $f(\rho, \varphi, p_\rho, p_\varphi)$ |
| Cone | $f(\alpha\rho, \alpha^{-1}\varphi, \alpha^{-1}p_\rho, \alpha p_\varphi)$ |

Planar solutions transform as $x_\pm(t) \rightarrow X_{\alpha,\pm}(t) = \alpha\rho(t)e^{\frac{i\varphi(t)}{\alpha}}$.

The solution we are looking

$$x_{\alpha,\pm} = \rho(t)e^{i\varphi(t)} = \alpha^{-1}|X_{\alpha,\pm}(t)|^{1-\alpha}(X_{\alpha,\pm})^\alpha$$

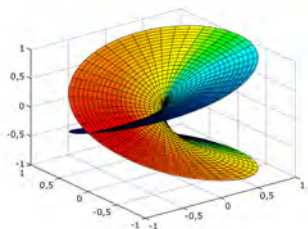
The domain of $(X_{\alpha,\pm})^\alpha$ is a **Riemann surface** depending on the value of α .

Drawing in a Riemann surface (RS)

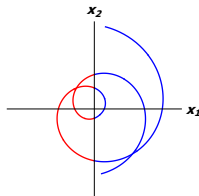
Consider the complex functions:

$$z = |z|e^{i(\vartheta+2\pi n)}, \quad f_\alpha(z) = z^\alpha = |z|^\alpha e^{i\alpha\vartheta} e^{i2\pi n\alpha}$$

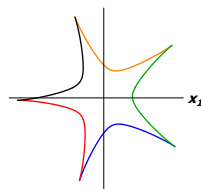
1. $\alpha \in \mathbb{Z}^+$ $f_\alpha(z) \in \mathbb{C}$.
2. $\alpha = \frac{q}{k}, f_\alpha(z) \in \text{RS of } k \text{ sheets.}$
3. $\alpha \in \mathbb{I}, f_\alpha(z) \in \text{RS with infinite sheets (like } \ln(z)).$



(a) \sqrt{z} .



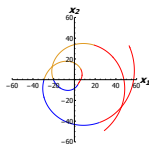
(b) Trajectory F.P
 $\alpha = 1/2$.



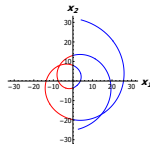
(c) Trajectory H.O
 $\alpha = 3/5, \gamma = 1/3$.

Figure: [Wikipedia: Riemann surfaces](#).

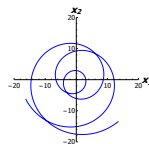
Classical trajectories: free case



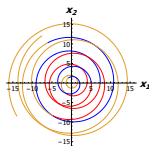
(a) $\alpha = 1/3, \dot{\varphi}_* < 0$.



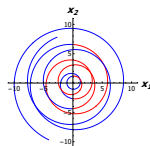
(b) $\alpha = 1/2, \dot{\varphi}_* < 0$.



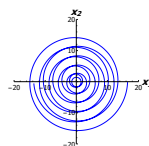
(c) $\alpha = 1, \dot{\varphi}_* < 0$.



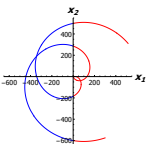
(d) $\alpha = 4/3, \dot{\varphi}_* < 0$.



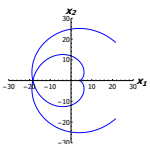
(e) $\alpha = 3/2, \dot{\varphi}_* < 0$.



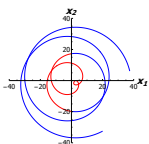
(f) $\alpha = 2, \dot{\varphi}_* < 0$.



(g) $\alpha = 1/2, \dot{\varphi}_* > 0$.

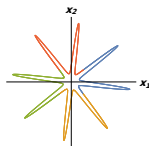


(h) $\alpha = 1, \dot{\varphi}_* = 0$.

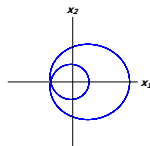


(i) $\alpha = 3/2, \dot{\varphi}_* > 0$.

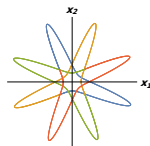
Classical trajectories: Harmonic oscillator case



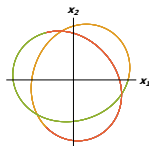
(a) $\gamma = 0, \alpha = 1/4$.



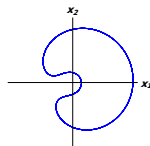
(b) $\gamma = 0, \alpha = 4$.



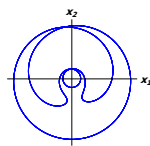
(c) $\gamma = 0, \alpha = 3/4$.



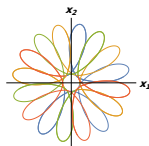
(d) $\gamma = 1, \alpha = 2/3$.



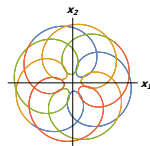
(e) $\gamma = 1, \alpha = 3$.



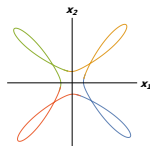
(f) $\gamma = 1, \alpha = 7$.



(g) $\gamma = 1/2, \alpha = 3/4$.



(h) $\gamma = 2, \alpha = 3/4$.



(i) $\gamma = 1/3, \alpha = 3/4$.

Integrals of motion

The complex quantities

$$b_{\alpha,j}^- = \frac{1}{2}e^{(-1)^j i \frac{\varphi}{\alpha}} \left(\alpha \sqrt{m\omega} \rho - (-1)^j \frac{p_\varphi}{\sqrt{m\omega} \rho} + \frac{ip_\rho}{\alpha \sqrt{m\omega}} \right),$$
$$\mathcal{L}_{\alpha,s_1,s_2}^\pm = (b_{\alpha,1}^\pm)^{s_1} (b_{\alpha,2}^\mp)^{s_2}, \quad \mathcal{J}_{\alpha,s_1,s_2}^\pm = (b_{\alpha,1}^\pm)^{s_1} (b_{\alpha,2}^\pm)^{s_2},$$

are not well defined on phase-space for $\alpha \neq 1$.

For $\gamma = \frac{s_2-s_1}{s_2+s_2}$ and $\alpha = q/k$ we have instead

$$\mathcal{L}_{\alpha,s_1,s_2}^{(\epsilon)\pm} = (\mathcal{L}_{\alpha,s_1,s_2}^\pm)^\epsilon, \quad \epsilon = \begin{cases} r & \text{if } q = r(s_1 + s_2), \quad r = 1, 2, \dots, \\ q & \text{if } q \neq r(s_1 + s_2), \end{cases}$$

For $\gamma = \frac{s_2+s_1}{s_2-s_2}$ and $\alpha = q/k$ we have

$$\mathcal{J}_{\alpha,s_1,s_2}^{(\delta)\pm} = (\mathcal{J}_{\alpha,s_1,s_2}^\pm)^\delta, \quad \delta = \begin{cases} r' & \text{if } q = r'|s_2 - s_1|, \quad r' = 1, 2, \dots, \\ q & \text{if } q \neq r'|s_2 - s_1|, \end{cases}$$

General comments about the quantum cases

$$\hat{H}_\gamma^{(\alpha)} = -\frac{\hbar^2}{2m} \left(\frac{1}{\alpha^2 \rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \varphi} \right) + \frac{\alpha^2 m \omega^2}{2} \rho^2 - i\gamma \omega \hbar \frac{\partial}{\partial \varphi},$$

whose eigenstates and spectrum are given by

$$\Psi_{n_\rho, \ell}^{(\alpha, \gamma)}(\rho, \varphi) = \left(\frac{m\omega\alpha^2}{\hbar} \right)^{\frac{1}{2}} \sqrt{\frac{n_\rho!}{2\pi\alpha\Gamma(n_\rho + \alpha|\ell| + 1)}} \zeta^{\alpha|\ell|} L_{n_\rho}^{(\alpha|\ell|)}(\zeta^2) e^{-\frac{\zeta^2}{2} + i\ell\varphi},$$

$$\zeta = \sqrt{\frac{m\alpha^2\omega}{\hbar}} \rho, \quad E_{n_\rho, \ell}^{(\alpha, \gamma)} = \hbar\omega(2n_\rho + \alpha(1 + \gamma \text{sign}(\ell))|\ell| + 1),$$

$$n_r = 0, 1, \dots, \quad \ell = 0, \pm 1, \dots$$

General proprieties

1. New degeneracy when $\alpha, \gamma \in \mathbb{Q}$.
2. Quantum operators $\hat{\mathcal{L}}_{\alpha, s_1, s_2}^{(\epsilon)\pm}$ $\hat{\mathcal{J}}_{\alpha, s_1, s_2}^{(\epsilon)\pm}$ are well defined on Hilbert space only when $\alpha = n$.

Conclusion and outlook

- Geometry and reinterpretations
- i) Non-inertial effects reinterpret as a magnetic fields.
 - ii) Conic background and cosmic strings.
 - iii) Rotating conic background and spinning cosmic strings ds_{sp}^2 ¹⁹?

With $d\varphi \rightarrow \frac{1}{2}(d\varphi - 2\alpha\Omega dt)$, $dt \rightarrow dt + \frac{1}{2\alpha\Omega}d\varphi$ (periodic time), and $J = \frac{c^4}{8\alpha\Omega G}$, the rotating conic metric transform in,

$$ds_{\text{sp}}^2 = -c^2(dt + 4c^{-4}GJd\varphi)^2 + \alpha^2 d\rho^2 + \rho^2 d\varphi^2.$$

- Geometry and classical harmonic oscillator dynamics
- i) Free parameters in geometry (α and Ω) may imply special characteristics of the paths.
 - ii) For rational α and $|\gamma| \in \mathbb{Q}$ (Commensurability of frequencies) exist higher order integrals of motion and closed paths (they are correlated).

- Geometry and quantum harmonic oscillator dynamics
- i) For $\alpha \in \mathbb{N}$ and $|\gamma| \in \mathbb{Q}$, higher order integrals operator describe the spectrum.
 - ii) *Multiparticle generalization:*
Bose Einstein condensation in the cone.
- Open questions:
- i) Interpretation of the higher-order integrals of motion in geometric terms (Killing tensors).
 - ii) Investigate the quantum anomaly under the light of exotic boundary conditions ²⁰.
 - iii) To study different classical and quantum systems in this an other geometries.

20. B.S. Kay, and U.M. Studer, *Commun. Math. Phys.* **139** (1991) 103.



Thank you very much for your
attention !

A multi particle generalization

1. Gross-Pitaevskii formalism

Hamiltonian of N particles in the cone geometry

$$\hat{\mathcal{H}} = \sum_{n=1}^N H_{\gamma}^{(\alpha)}(\mathbf{r}_i) + \mathcal{U}(\mathbf{r}_1, \dots, \mathbf{r}_1), \quad \mathcal{U} = \sum_{i < j} U(\mathbf{r}_i, \mathbf{r}_j).$$

In the Hartree-Fock approximation: $U = \frac{4\pi\hbar a_s}{m} \delta(\mathbf{r}_i - \mathbf{r}_j)$ with $a_s \sim 0$.
Using low temperature approximation (μ chemical potential)

$$\mu\psi(\mathbf{r}) = \left(\hat{H}_{\gamma}^{(\alpha)} + \frac{4\pi\hbar a_s}{m} |\psi|^2 \right) \psi(\mathbf{r}), \quad \psi = \langle \hat{\Psi} \rangle.$$

We take ψ as a linear combination of each solutions associated to the ground state of $\hat{H}_{\gamma}^{(\alpha)}$ such that

$$I = \frac{4\pi\hbar a_s}{m} \int \alpha \rho d\rho d\varphi |\psi|^4 \sim 0.$$

$$\text{Landau: } \gamma = 1, \quad \psi^{(\alpha)} = \sqrt{N} \sum_{l=0}^{l_{\text{cut}}} c_l \Psi_{0,l}^{(\alpha,1)} \quad \sqrt{N} \sum_{l=0}^{l_{\text{cut}}} c_l c_l^* = 1 \quad .$$

Vortices

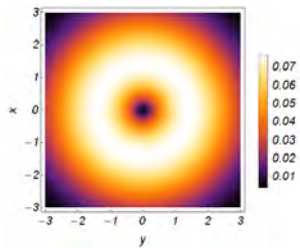
2. The zeros of $\psi^{(\alpha)}$ are interpreted as vortices.

For testing we chose $\ell_{\text{cut}} = 2$ and $\frac{m\omega}{\hbar} = 1$.

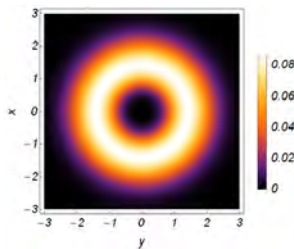
$$\psi^{(\alpha)} = \sqrt{\frac{\alpha N}{\pi(|a_0|^2 + |a_1|^2 + |a_2|^2)}} \left(a_0 + \frac{a_1(\alpha\rho)^\alpha}{\sqrt{\Gamma(\alpha+1)}} e^{i\varphi} + \frac{a_2(\alpha\rho)^{2\alpha}}{\sqrt{\Gamma(\alpha+1)}} e^{i2\varphi} \right) e^{-\frac{\alpha^2 \rho^2}{2}}.$$

| α | a_0 | a_1 | a_2 | I_{a_0, a_1, a_2} |
|----------|-------|-------|-----------------------------------|-------------------------------------------------------------|
| 1/4 | 0 | 0 | 1 | $1/(4\pi) \approx 0.025$ |
| 1/3 | 0 | 0 | 1 | $\Gamma(7/6)/(6\pi^{\frac{3}{2}}\Gamma(5/3)) \approx 0.031$ |
| 1/2 | 0 | 0 | 1 | $1/(8\pi) \approx 0.040$ |
| 1 | 0 | 0 | 1 | $3/(16\pi) \approx 0.060$ |
| 2 | 1 | 0 | $\sqrt{112/19} \approx 2.428$ | $33/(131\pi) \approx 0.080$ |
| 3 | 1 | 0 | $\sqrt{992/199} \approx 2.233$ | $115/(397\pi) \approx 0.092$ |
| 4 | 1 | 0 | $\sqrt{32512/6179} \approx 2.294$ | $12866/(38691\pi) \approx 0.106$ |

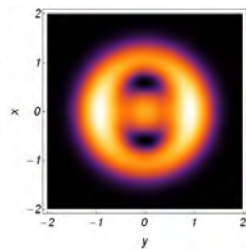
Table: The values of parameters that minimize I_{a_0, a_1, a_2} for different values of α , where a_i are assumed to be real numbers (the imaginary part only contributes with a global sign in this particular case). Note that the value of I_{a_0, a_1, a_2} grows with the increasing value of α .



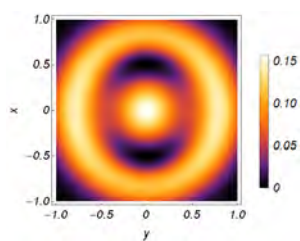
(a) $\alpha = 1/2$



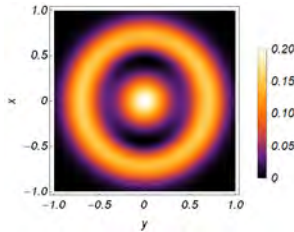
(b) $\alpha = 1$



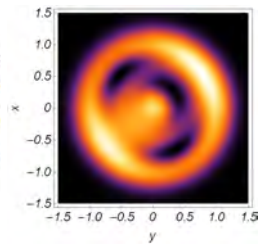
(c) $\alpha = 2$



(d) $\alpha = 3$



(e) $\alpha = 4$



(f) $\alpha = 3$