Bifurcations of the detuned 2 : 4 resonance

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Outline

- 2D Hamiltonian systems around symmetric resonances
- Normal forms
- Geometric reduction

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We consider a family of $\mathbb{Z}_2 \times \mathbb{Z}_2$–symmetric Hamiltonian systems in two degrees of freedom, i.e. invariant with respect to the reflectional symmetries

$$\begin{aligned} \varrho_1 &: (x_1, x_2, p_1, p_2) &\mapsto& (-x_1, x_2, -p_1, p_2) \\ \varrho_2 &: (x_1, x_2, p_1, p_2) &\mapsto& (x_1, -x_2, p_1, -p_2) \end{aligned}$$

where (x, p) denote the canonical coordinates. We assume the system to be close to an elliptic equilibrium at the origin and consider

$$H(x, p; \varepsilon) = \sum_{j=0}^{\infty} \varepsilon^{2j} H_{2j}(x, p).$$

Here H_{2j} are homogeneous polynomials of degree $2(j+1)$ in the coordinates (x, p), ε is a small parameter and

$$H_0(x, p) = \frac{\omega_1}{2}(x_1^2 + p_1^2) + \frac{\omega_2}{2}(x_2^2 + p_2^2)$$

so the system can be treated as a symmetric perturbed oscillator.

The Hamiltonian

$$H = \frac{\omega_1}{2}(x_1^2 + p_1^2) + \frac{\omega_2}{2}(x_2^2 + p_2^2) + \sum_{j=1}^{\infty} \varepsilon^{2j} H_{2j}$$

is in general not integrable. Let us introduce a detuning parameter δ by assuming

$$\omega_1 = \left(\frac{m}{n} + \varepsilon^2\delta\right)\omega_2, \quad m, n \in \mathbb{N}$$

and put the term with the detuning into the perturbation, so to see the system as a perturbation of a $m{:}n$ resonant oscillator invariant under the reflection symmetries.

Then we proceed to a normalization procedure w.r.t. the unperturbed $m{:}n$ resonant oscillator: we look for a (formal) coordinate transformation that brings H into the normal form K so that after scaling $t \to \frac{\omega_2}{n} t$,

$$\{K, H_0\} = 0, \quad H_0 = \frac{m}{2}(x_1^2 + p_1^2) + \frac{n}{2}(x_2^2 + p_2^2).$$

In this way the system acquires a (formal) constant of motion $H_0 = \eta$.

The normalized system is therefore integrable.

Why the detuning? Because even if the unperturbed system is non-resonant, the non-linear coupling between the degrees of freedom induced by the perturbation determines a “passage through resonance”. This in turn is responsible for the birth of new orbit families bifurcating from the normal modes or from lower-order resonances. Moreover, in this way we can avoid the presence of terms with small denominators while normalizing the system.

We aim at a general understanding of the phase space structure and the bifurcation sequences of periodic orbits in general position from the normal modes, parametrised by the (generalized) energy E, the detuning δ and the independent coefficients characterising the nonlinear perturbation.

Typical periodic orbits associated to the 2:2 resonance

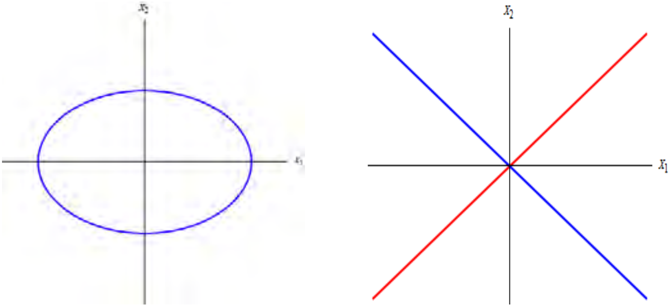


Figure: $2:2$ resonance: loop orbits if $2(\phi_1 - \phi_2) = \pm\pi$, inclined orbits if $\phi_1 - \phi_2 = 0, \pi$.

Here action-angle like variables have been introduced:

$$p_j = \sqrt{2\tau_j}\sin\phi_j, \quad x_j = \sqrt{2\tau_j}\cos\phi_j, \quad j = 1, 2.$$

Typical periodic orbits associated to the 2:4 resonance

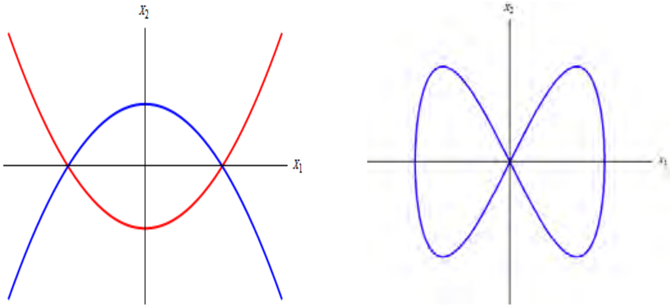


Figure: 2 : 4 resonance: anti-banana orbits if $2\phi_1 - \phi_2 = 0, \pi$, banana orbits: $4\phi_1 - 2\phi_2 = \pm\pi$.

Here action-angle like variables have been introduced:

$$p_j = \sqrt{2\tau_j}\sin\phi_j, \quad x_j = \sqrt{2\tau_j}\cos\phi_j, \quad j = 1, 2.$$

As an example, let us consider the family of systems

$$H(x,p) = \frac{1}{2}(p_1^2 + p_2^2) + V(x),$$

where

$$V(x_1,x_2) \;=\; \frac{1}{a}\left(1 + x_1^2 + \frac{x_2^2}{q^2}\right)^{a/2} \;, \quad 0 < a < 2,\ \tfrac{1}{4} < q \le 1.$$

This gravitational potential is generated by a simple but realistic matter distribution. Its astrophysical relevance is based on its ability to describe in a simple way the gross features of elliptical galaxies. In the limit $a \to 0$ we have the logarithmic potential

$$V(x_1,x_2) \;=\; \log\left(1 + x_1^2 + \frac{x_2^2}{q^2}\right)$$

The Hamiltonian is "prepared" for normalization by setting

$$q \;=\; \frac{\omega_1}{\omega_2} \;=\; \frac{m}{n} \;+\; \varepsilon^2\delta \;,$$

and scaling time and space variables as

$$x \;\mapsto\; \varepsilon x, \;\; t \;\mapsto\; \frac{\varepsilon^2\omega_2}{n}\, t.$$

Let us now consider in general

$$H = \frac{m}{2}(x_1^2 + p_1^2) + \frac{n}{2}(x_2^2 + p_2^2) + \varepsilon^2 n \frac{\delta}{2}(x_1^2 + p_1^2) + \sum_{j=1}^{\infty} \varepsilon^{2j} H_{2j}.$$

The flow $\varphi_t^{H_0}$ of the unperturbed system yields the $\mathbb{S}^1$–action φ^{H_0} on $\mathbb{R}^4 \cong \mathbb{C}^2$ given by

$$\begin{array}{rccc} \varphi^{H_0} : & \mathbb{S}^1 \times \mathbb{C}^2 & \longrightarrow & \mathbb{C}^2 \\ & (\ell, (z_1, z_2)) & \to & (e^{-\mathrm{i}m\ell} z_1, e^{-\mathrm{i}n\ell} z_2) \end{array}$$

where

$$z_j = x_j + \mathrm{i}y_j, \quad j = 1, 2.$$

or, equivalently, in action-angle like variables

$$z_j = \sqrt{2\tau_j} e^{i\phi_j}, \;\; j = 1, 2.$$

The perturbed Hamiltonian is in general not invariant under this action, however we can normalize H so that the resulting normal form K does have the oscillator symmetry, namely

$$\{K, H_0\} = 0.$$

A set of generators of the Poisson algebra of φ^{H_0}–invariant functions is given by

$$\tau_1 = \frac{z_1 \bar{z}_1}{2} , \quad \tau_2 = \frac{z_2 \bar{z}_2}{2}$$

together with

$$\sigma_1 = \frac{\mathrm{Re}\, z_1^n \bar{z}_2^m}{2} , \quad \sigma_2 = \frac{\mathrm{Im}\, z_1^n \bar{z}_2^m}{2}$$

and it is constrained by $\tau_1 \geq 0$, $\tau_2 \geq 0$ and the syzygy

$$R(\tau, \sigma) := 2^{n+m-2} \tau_1^n \tau_2^m - (\sigma_1^2 + \sigma_2^2) = 0.$$

The (truncated) normal form K is a polynomial in (τ, σ), namely

$$K = m\tau_1 + n\tau_2 + \varepsilon^2 n\, \delta \tau_1 + \sum_{j=1}^{N-1} \varepsilon^{2j} K_{2j}(\tau) + \varepsilon^{2N} K_{2j}(\tau, \sigma).$$

Without symmetries, the minimal truncation order is $m + n - 2$. With both reflection symmetries, the minimal truncation order increases to $2N = 2(m + n) - 2$ and this is why one speaks of 2m:2n resonance.

The normalization allows us to reduce the dynamics to one degree of freedom as the Poisson bracket on $\mathbb{R}^4$ induced by (τ, σ) has two Casimir elements, namely R and $H_0 = m\tau_1 + n\tau_2$.

For a fixed value $\eta \geq 0$ of H_0 we can eliminate $\tau_2 = \frac{1}{2}(\eta - \tau_1)$. The dynamics are then constrained to the reduced phase space

$$\mathcal{V}^\eta = \left\{ (\tau_1, \sigma_1, \sigma_2) \in \mathbb{R}^3 \ : \ R^\eta(\tau_1, \sigma_1, \sigma_2) = 0, \ 0 \leq \tau_1 \leq \eta \right\}$$

with Poisson structure

$$\{f, g\} = \langle \nabla f \times \nabla g, \nabla R^\eta \rangle \ ,$$

where

$$R^\eta(\tau_1, \sigma_1, \sigma_2) = 2^{n-2}(\eta - \tau_1)^m \tau_1^n - (\sigma_1^2 + \sigma_2^2).$$

Let us focus now on the 2 : 4 resonance. The general structure of the normal form, truncated at the minimal order reads

$$K(\tau,\sigma;\delta) \;=\; K_0(\tau)+\varepsilon^2 K_2(\tau;\delta)+\varepsilon^4\left[\mu\frac{\sigma_1^2-\sigma_2^2}{2}\;+\;\nu\sigma_1\sigma_2\;+\;K_4(\tau;\delta)\right]$$

with K_2, K_4 polynomials of degree 2 and 4, respectively,

$$K_0=H_0 \;=\; \tau_1\;+\;2\tau_2\;=\;\eta.$$

We assume at least one of the coefficients μ and ν to be non-vanishing (otherwise we have to consider higher order normal form).

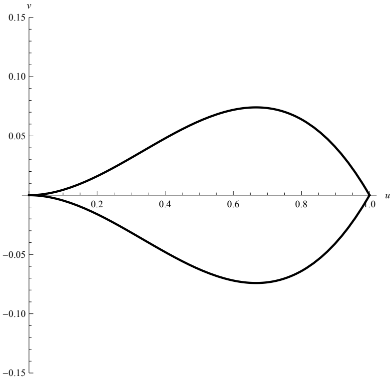
The $\mathbb{Z}_2 \times \mathbb{Z}_2$–symmetry of the original system is inherited by the normal form.

Indeed, none of the invariants (τ, σ) changes under reflectional symmetry with respect to the x_1- axis. The reflectional symmetry with respect to the x_2- axis becomes the symmetry

$$(\tau, \sigma) \quad \mapsto \quad (\tau, -\sigma)$$

We perform a further reduction to explicitly divide out this symmetry, by introducing variables

$$\begin{aligned} u &= \tau_1 \\ v &= \tfrac{1}{2}(\sigma_1^2 - \sigma_2^2) \\ w &= \sigma_1 \sigma_2 \quad . \end{aligned}$$



The (twice) reduced phase space is given by

$$\mathcal{P}^{\eta} \;=\; \left\{ (u, v, w) \in \mathbb{R}^3 \;:\; S^{\eta}(u, v, w) = 0,\; 0 \le u \le \eta \right\}$$

where

$$S^{\eta}(u, v, w) \;=\; \frac{(\eta - u)^2}{2} u^4 \;-\; 2(v^2 + w^2) \;.$$

The normal form then becomes (after neglecting constant terms and scaling one more time by ε^2)

$$K^\eta(u,v,w;\delta) \;=\; (2\delta+\alpha\eta)u + \lambda u^2 + \varepsilon^2\left[\mu v + \nu w + K_4^\eta(u;\delta)\right]$$

Note that, since the reduced phase space is a surface of revolution, by rotation we can always eliminate one of the two variables v, w from the Hamiltonian (we do not consider the case $\mu = \nu = 0$ here).

For definiteness we assume from now on $\mu > 0$ *and* $\nu = 0$.

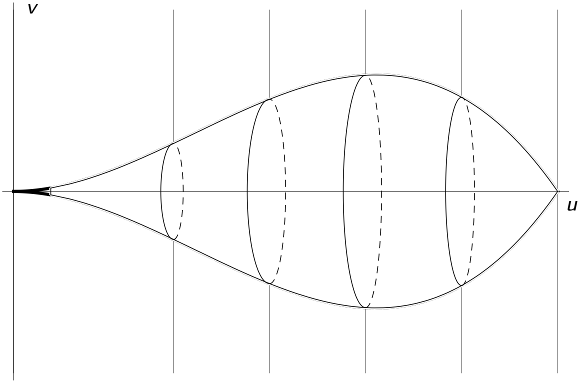
Several parameters: δ, η, the coefficients $\alpha, \lambda, \mu, \ldots$ And by varying the parameters of course bifurcations might occur... How to describe the dynamics in terms of all these parameters?

To understand the dynamics of the normal form we follow a geometric approach: we look at the intersections between the level sets of the Hamiltonian and the reduced phase space.

At first order the level sets determined by constant values of the normal form

$$\mathcal{K}^{\eta}_{\delta}(h) \;\; := \;\; \left\{ (u,v,w) \in \mathbb{R}^3 \;:\; (2\delta + \alpha\eta)u + \lambda u^2 = h \right\}$$

are generically double planes. Their intersections with the reduced phace space correspond to stable singular equilibria or to periodic orbits.



Here we depict only level sets of the normal form that do not degenerate into a double plane (section $w = 0$). Their intersections with the reduced phase space correspond to stable singular equilibria or to periodic orbits.

Singular equilibria are at the singular points of the reduced phase space, namely at

$$\mathcal{Q}_1 = (0,0,0) \quad \text{(cuspidal)}, \quad \textit{and} \; ; \mathcal{Q}_2 = (0,0,\eta) \quad \text{(conical)}.$$

From the equations of motion

$$\dot{u} = 0, \quad \dot{v} = 4w\frac{\partial K_\delta^\eta}{\partial u}, \quad \dot{w} = -4v\frac{\partial K_\delta^\eta}{\partial u}$$

we infer that the intersections between the level sets

$$\mathcal{K}_\delta^\eta(h) \ := \ \left\{ (u, v, w) \in \mathbb{R}^3 \ : \ K_\delta^\eta(u, v, w) = h \right\}$$

and the reduced phase space are periodic orbits, except when $\mathcal{K}_\delta^\eta(h)$ is a double plane where the circle consists of equilibria. Since

$$\frac{\partial K_\delta^\eta}{\partial u} \ = \ 2\delta \ + \ \alpha\eta \ + \ 2\lambda u$$

the corresponding double root is given by

$$u \ = \ u_0 \ := \ -\frac{2\delta + \alpha\eta}{2\lambda}$$

and it gives a circle of eqilibria on the reduced phase space only if

$$0 \ < \ u_0 \ < \ \eta \ .$$

When we have a double plane, the circle of equilibria can fall on the singular equilibria for

$$u_0 = 0, \quad \text{or} \quad u_0 = \eta$$

at

$$h = h_0 := \ -\frac{(2\delta + \alpha\eta)^2}{4\lambda}.$$

This gives the threshold values for η at

$$\eta = \eta_{01} \ := \ -\frac{2\delta}{\alpha} \quad \text{at} \quad Q_1 = (0, 0)$$

and

$$\eta = \eta_{02} \ := \ -\frac{2\delta}{\alpha + 2\lambda} \quad \text{at} \quad Q_2 = (\eta, 0).$$

This requires $\delta\alpha \leq 0$ and $\delta(\alpha + 2\lambda) \leq 0$, respectively, since η cannot be negative.

What does it happen at $\eta = \eta_{01}$ and $\eta = \eta_{02}$? We need to look at the higher order terms.

Let us look at the system near $h = h_0$, i.e. we consider the level sets

$$K^{\eta,h_0}_{\delta,\varepsilon}(k) \ := \ \{ (u,v,w) \in \mathbb{R}^3 \ : \ K^\eta(u,v,w;\delta) = h_0 + \varepsilon^2 k \}$$

which give a family of third order curves when intersecting with the (u,v)–plane, with equation

$$v(u) \ = \ \frac{1}{\mu}\left[k - \frac{\lambda}{\varepsilon^2}(u-u_0)^2 - K^\eta_4(u;\delta)\right] \ ,$$

where

$$u_0 \ = \ -\frac{2\delta + \alpha\eta}{2\lambda}.$$

The ε^2 in the denominator lets the parabolic part of the curve dominate over the cubic part K^η_4. Thus, we have to understand the intersections between the parabola

$$v = \left[k - \frac{\lambda}{\varepsilon^2}(u-u_0)^2\right]$$

and the reduced phase space section. Tangency points correspond to regular equilibria.

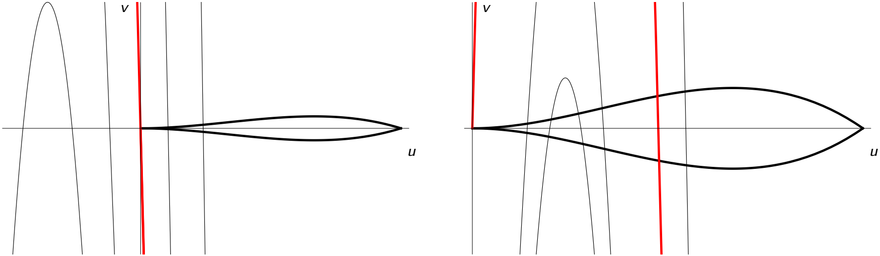


Figure: Possible configurations between the phase space section $\mathcal{P}^{\eta} \cap \{w = 0\}$ and a second order approximation of the level sets of the Hamiltonian for $\delta = 0.25$, $\alpha = -1$, $\lambda = 0.35$, $\mu = 0.25$, $\varepsilon = 0.2$ and $\eta = 0.4$ (left), $\eta = 0.6$ (right).

For values of k corresponding to the red curve we have a stable equilibrium at the origin (left) or a stable equilibrium at the origin and a periodic orbit around it (right). For values of k slightly different (gray curves) we can have periodic orbits around the origin or no dynamics; in the right figure we furthermore have periodic orbits around a regular equilibrium.

At $(0,0)$ the reduced phase space section has a cuspidal singularity.

The equilibrium $\mathcal{Q}_1 = (0,0,0)$ can be unstable only if the parabola passes through the origin $(u,v) = (0,0)$ with vanishing first derivative. This happens for

$$v'(0) \;=\; -\frac{1}{\mu}\left[\frac{2\delta+\alpha\eta}{\varepsilon^2} + \beta_2\delta\eta + \gamma_3\eta^2\right] \;=\; 0 \;.$$

Since we are following a perturbative approach, we look for a solution of this equation in the form of a power series in ε.

We find just one solution, namely

$$\eta = \bar{\eta} := \eta_{01} + \varepsilon^2\eta_{11}$$

where

$$\eta_{01} = -\frac{2\delta}{\alpha},$$

acceptable for $\alpha \neq 0$ and $\delta\alpha \leq 0$ (i.e. η not negative).

For $\alpha \neq 0$ and $\delta\alpha \leq 0$, at

$$\eta = \bar{\eta}$$

two families of periodic orbits namely banana and anti-banana orbits, bifurcate for the two-degree-of-freedom system defined by the normal form and up to second order terms in the perturbation this happens simultaneously, at the *same* critical value of η. Since Q_1 is a cusp point this has a geometric reason and in particular subsists through all orders of the perturbation.

At $\mathcal{Q}_2 = (\eta, 0, 0)$ the reduced phase space has a conical singularity. In this case, regular equilibria appear/disappear in two successive bifurcations.

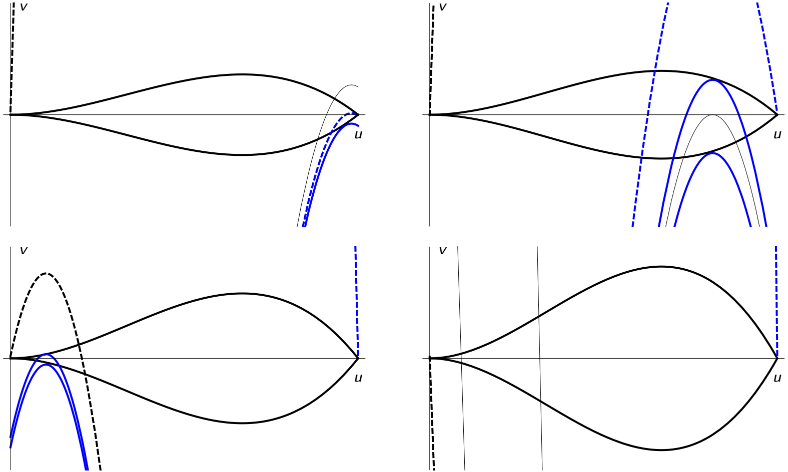


Figure: Possible tangencies between the parabola and the phase space section $\mathcal{P}^\eta \cap \{w = 0\}$ in (2) for increasing values of η and fixed values $\delta = -0.25$, $\mu = 0.25$, $\lambda = 0.1$, $\alpha = 1$ and $\varepsilon = 0.3$ of the detuning and the other parameters. Two regular equilibria appear successively from the conical singularity and subsequently disappear simultaneously on the singular equilbrium at the origin. The equilibrium on the upper contour of the phase space is unstable while the equilibrium on the lower contour is stable.

At $\mathcal{Q}_2 = (\eta, 0, 0)$ the reduced phase space has a conical singularity.

The intersection of the reduced phase space $\mathcal{P}^\eta$ with the (u, v)–plane is given by

$$\mathcal{C}^\eta_\pm \; = \; \mathcal{P}^\eta \cap \{w = 0\} \; = \; \left\{ (u, v) \in \mathbb{R}^2 \; : \; v = \pm\frac{1}{2}(\eta - u)u^2, \; 0 \le u \le \eta \right\}$$

whence the slope of the two contour lines constituting the reduced phase space section at $(u, v) = (\eta, 0)$ is $\mp\frac{1}{2}\eta^2$. The corresponding equilibrium can be unstable only if the slope of the parabola at $(u, v) = (\eta, 0)$ takes values in the interval $(-\frac{1}{2}\eta^2, \frac{1}{2}\eta^2)$. Thus, to find the critical values for η which correspond to stability/instability transitions of the equilibrium, we need to solve

$$v'(\eta) \; = \; \pm\frac{\eta^2}{2} \; .$$

We arrive at the two solutions $\eta = \eta_{2,\pm} := \eta_{02} + \eta_\pm$, $\eta_{02} = -\frac{2\delta}{\alpha+2\lambda}$.

In this case two families of periodic orbits can appear, not together.

Implications for the original system: what the equilibria for the reduced system correspond to?

- The singular equilibria $\mathcal{Q}_1 = (0, 0, 0)$ and $\mathcal{Q}_2 = (0, 0, \eta)$ correspond to $\tau_1 = 0$ and $\tau_1 = \eta$, respectively.

  For the original system this are the normal modes

  $$x_1^2 + p_1^2 = 0, \quad x_2^2 + p_2^2 = \eta$$

  and

  $$x_1^2 + p_1^2 = 2\eta, \quad x_2^2 + p_2^2 = 0,$$

  also called short and long axial orbits.
- Tangencies on the lower contour of the reduced phase space are banana orbits:

  $$0 = \sigma_1 = \tau_1 \sqrt{2\tau_2} \cos(2\phi_1 - \phi_2).$$

- Tangencies on the upper contour of the reduced phase space are anti-banana orbits:

  $$0 = \sigma_2 = \tau_1 \sqrt{2\tau_2} \sin(2\phi_1 - \phi_2).$$

- banana and/or anti-banana orbits appear/disappear when the corresponding threshold values for η are acceptable, i.e. not negative. This is always associated with a stability/instability transition of a normal mode. This gives conditions in terms of the coefficients α, λ and the detuning δ, namely

$$\delta\alpha < 0, \quad \text{and} \quad \delta(\alpha + 2\lambda) < 0$$

  for the short and long axial orbit, respectively.

- The difference between the threshold values is

$$\eta_- \; - \; \eta_+ \;\; = \;\; \frac{4\varepsilon^2\delta^2\mu}{(\alpha + 2\lambda)^3} \quad .$$

  Therefore, the sign of $\mu(\alpha + 2\lambda)$ determines the bifurcation order from the long axis orbit.

- η is not a constant for the original system; nevertheless we can use it to find threshold values for the bifurcations in terms of the (generalized) energy E (that is conserved for the original system).

On the long axial orbit ($\tau_1 = \eta$, $\tau_2 = 0$), the normal form reads as

$$K \;=\; \eta \;+\; \varepsilon^2(2\delta + \alpha_1\eta)\eta \;+\; \ldots \;\; .$$

By the scaling of time we have

$$\frac{\omega_2}{n} K \;+\; O(\varepsilon^6) \;=\; H \;=\; E$$

and we can express the (generalized) energy in terms of η as

$$E \;=\; \frac{\omega_2}{n} \left[\eta \;+\; \varepsilon^2 \left(2\delta + \alpha_1\eta\right) \eta \;+ \ldots\right.$$

Substituting the threshold values for η we find the critical energy threshold values that correspond to the bifurcations off/from the long axial orbit.

Coming back to the potential

$$V(x_1, x_2) = \frac{1}{a}\left(1 + x_1^2 + \frac{x_2^2}{q^2}\right)^{p/2}, \quad 0 < a < 2,\ \frac{1}{4} < q \leq 1.$$

relevant to galactic dynamics, concentrating on $q > \frac{1}{2}$, i.e $\delta > 0$, after normalization we have ($q = \frac{1}{2} + \varepsilon^2\delta$)

$$\lambda = -\alpha = \frac{a-2}{8} < 0, \ \ \mu = \frac{1}{32}(a^2 - 4) < 0, \ \ \delta(\alpha + 2\lambda) < 0, \ \ \delta\alpha > 0.$$

Thus, bifurcations occur always from the long normal mode, with bananas appearing at lower energies than anti-bananas ($\mu(\alpha + 2\lambda) > 0$). The critical values of the energy that determine the bifurcations read as

$$E_+ = \frac{16}{2-a}\left(q - \frac{1}{2}\right) + \frac{8(41a - 10)}{3(a-2)^2}\left(q - \frac{1}{2}\right)^2$$

$$E_- = \frac{16}{2-a}\left(q - \frac{1}{2}\right) + \frac{8(53a + 14)}{3(a-2)^2}\left(q - \frac{1}{2}\right)^2$$

for the bifurcation of banana and anti-banana orbits, respectively.

Conclusion and perspectives:

- 3D problems: a geometric reduction is possible also for 3D systems that are close to resonances. However, the outcome of the normalization is in general a normal form possessing only one additional integral, besides the Hamiltonian and therefore it is not integrable. Sometimes a renormalization is possible...
- Indefinite resonances: one could consider more general systems with indefinite quadratic part, so that

$$H_0 = \frac{1}{2}(m_1\tau_1 - m_2\tau_2), \qquad m_1, m_2 \in \mathbb{N}$$

These systems differ from the definite case in several features, even if their analysis can be performed almost in the same way.

K. Efstathiou, H. Hassmann, A. Marchesiello, *Bifurcations and monodromy of the axially symmetric* $1:1:-2$ *resonance*, Journal of Geometry and Physics, **146**, 103493 (2019).

Thank you for your attention

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