On the Schrödinger Operator with a Periodic PT-symmetric Matrix Potential

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AAMP XIX, Prague, 07/09/2022

The main aim of my talk is to discuss the spectrum of the one-dimensional Schrödinger operator $L(Q)$ generated in the space $L_2^m(-\infty,\infty)$ of the vector functions by the differential expression

$$-y^{''}+Qy, \tag{1}$$

where $Q=(q_{i,j})$ is a $m\times m$ matrix with the PT-symmetric π-periodic locally square integrable entries $q_{i,j}$. In other words,

$$\overline{q_{i,j}\left(-x\right)}=q_{i,j}\left(x\right),\ \ q_{i,j}\left(x+\pi\right)=q_{i,j}\left(x\right),\ \ q_{i,j}\in L_2\left[0,\pi\right]. \tag{2}$$

It is well-known that [Rofe-Beketov (1963), McGarvey (1965)] the spectrum $\sigma(L(Q))$ of the operator $L(Q)$ is the union of the spectra $\sigma(L_t(Q))$ of the operators $L_t(Q)$ for $t\in(-1,1]$ generated in $L_2^m[0,\pi]$ by the differential expression (1) and the quasiperiodic conditions

$$y^{'}\left(\pi\right)=e^{i\pi t}y^{'}\left(0\right),\ \ y\left(\pi\right)=e^{i\pi t}y\left(0\right). \tag{3}$$

The spectrum of $L_t(Q)$ consists of the eigenvalues $\Lambda_1(t)$, $\Lambda_2(t)$, ... that are the roots of the characteristic determinant

$$\Delta(\lambda, t) = \det(Y_j^{(\nu-1)}(\pi, \lambda) - e^{i\pi t} Y_j^{(\nu-1)}(0, \lambda))_{j,\nu=1}^{2} = \tag{4}$$

$$e^{i2m\pi t} + f_1(\lambda)e^{i(2m-1)\pi t} + f_2(\lambda)e^{i(2m-2)\pi t} + ... + f_{2m-1}(\lambda)e^{i\pi t} + 1,$$

where $Y_1(x, \lambda)$ and $Y_2(x, \lambda)$ are the solutions of the matrix equation

$$-Y^{''}(x) + Q\,(x)\,Y(x) = \lambda Y(x) \tag{5}$$

satisfying $Y_1(0, \lambda) = O_m$, $Y_1^{'}(0, \lambda) = I_m$ and $Y_2(0, \lambda) = I_m$, $Y_2^{'}(0, \lambda) = O_m$. Here O_m and I_m are $m \times m$ zero and identity matrices respectively. The set $\{\Lambda_n(t) : t \in (-1, 1]\}$, in the self-adjoint case, is the nth band of the spectrum, while in the non-self-adjoint case is the curve in the complex plane. Thus the spectrum of $L(Q)$ consists of the curves.

In the first papers [Bender et al (1999)] about the PT-symmetric periodic potential, the disappearance of real energy bands for some complex-valued PT-symmetric periodic potentials have been reported. Shin (2004) showed that the disappearance of such real energy bands implies the existence of nonreal band spectra. He involved some condition on the Hill discriminant to show the existence of nonreal curves in the spectrum.
I proved that the main part of the spectrum of $L(q)$ is real and contains the large part of $[0, \infty)$. However, in general, the spectrum contains also infinitely many nonreal acts. In my papers, the necessary and sufficient condition on the potential for finiteness of the number of the nonreal arcs is determined. Moreover, I find necessary and sufficient conditions for the equality of the spectrum of $L(q)$ to the half line $[c, \infty)$. Besides, I find the explicit conditions on PT-symmetric periodic complex-valued potential q for which the number of the gaps in $\mathrm{Re}\,(\sigma(L(q)))$ is finite (see Monograph Veliev 2021).

The steps of my talks are the followings:

1. On the general properties of the spectrum of $L(Q)$ with PT-symmetric matrix potential Q.

2. On the asymptotic formulas for the eigenvalues of $L_t(Q)$.

3. On the condition from which it follows that the real component $\sigma(L(Q)) \cap \mathbb{R}$ of the spectrum $\sigma(L(Q))$ is contained in a finite interval $[a,b]$. Sufficient conditions under which $\sigma(L(Q)) \cap \mathbb{R}$ contains most of $[0, \infty)$.

4. On the conditions on the potential, for which the spectrum of $L(Q)$ contains all half line $[H, \infty)$ for some H and hence the number of gaps in $\sigma(L(Q)) \cap \mathbb{R}$ is finite.

5. Comparison the PT-symmetric and self-adjoint cases.

First we note that if λ is an eigenvalue of multiplicity p of the operator $L_t(Q)$, then $\overline{\lambda}$ is also an eigenvalue of the same multiplicity of $L_t(Q)$. This is the characteristic property of the Schrödinger operator with PT-symmetric potential. Then we consider the operator $L_t(O_m)$ for an unperturbed operator and the operator of multiplication by Q for a perturbation and prove that there exists a constant c such that the eigenvalues $\lambda(t)$ of the operator $L_t(Q)$ lie on the c neighborhoods of the eigenvalues of $L_t(O_m)$. Note that the eigenvalues of the operator $L_t(O_m)$ are $(2k+t)^2$ for $k \in \mathbb{Z}$. If $t \neq 0, 1$, then the multiplicity of the eigenvalue $(2k+t)^2$ is m. In the cases $t = 0$ and $t = 1$ the multiplicity of the nonzero eigenvalues $(2k)^2$ and $(2k+1)^2$ is $2m$. Thus the multiplicity of the Bloch eigenvalues of $L(O_m)$ is changed at n^2 for $n > 0$, that is, they are exceptional point of the spectrum in case $Q = O_m$.

To obtain a sharp asymptotic formulas we consider the operator $L_t(Q)$ as perturbation of $L_t(A)$, where

$$A = \int_{(0,\pi)} Q(x)\,dx, \tag{6}$$

by $Q - A$, that is, we take the operator $L_t(A)$ for an unperturbed operator and the operator of multiplication by $Q - A$ for a perturbation. Therefore first let us discuss the eigenvalues and eigenfunctions of $L_t(A)$. Using (2) and the substitution $t = -x$ one can get the equality

$$\overline{\int_0^{\pi} q_{i,j}(x)\,dx} = \int_0^{\pi} q_{i,j}(-x)\,dx = -\int_0^{-\pi} q_{i,j}(t)\,dt = \int_0^{\pi} q_{i,j}(t)\,dt$$

which means that

$$\int_0^{\pi} q_{i,j}(x)\,dx \in \mathbb{R}$$

for all i and j. Hence the entries of the matrix A are the real numbers.

Therefore, the eigenvalues of the matrix A consist of the real eigenvalues and the pairs of the conjugate complex numbers. The distinct eigenvalues of A are denoted by $\mu_1, \mu_2, ..., \mu_p$. Let $u_{j,1},\ u_{j,2}, ...u_{j,s_j}$ be the linearly independent eigenvectors corresponding to the eigenvalue μ_j. It is not hard to see that the eigenvalues and eigenfunctions of $L_t(A)$ are

$$\mu_{k,j}(t) = (2k+t)^2 + \mu_j,\ \Phi_{k,j,s} = u_{j,s}e^{i(2k+t)x}. \tag{7}$$

It readily implies that the spectrum of $L(A)$ consists of the half lines

$$\left\{\mu_j + a : \ a \in [0, \infty)\right\} \tag{8}$$

for $j = 1, 2, ..., p$. We find a sharp and uniform, with respect to the quasimomenta $t \in (-1, 1]$, asymptotic formula for the Bloch eigenvalues of $L(Q)$ in term of the eigenvalues of the matrix A for any matrix potential Q with locally square integrable entries.

Theorem

The eigenvalues of $L_t(Q)$ *are contained in* ε_k *neighborhood* $D\left(\mu_{k,j}, \varepsilon_k\right)$ *of the eigenvalues* $\mu_{k,j}(t)$ *of* $L_t(A)$ *for* $j = 1, 2, ..., m$*, where* $\varepsilon_k \longrightarrow 0$ *as* $k \longrightarrow \infty$*.*

Using this result we classify the spectrum of $L(Q)$. This theorem implies that the spectrum of $L(Q)$ is asymptotically close to the spectrum of the operator $L(A)$. On the other hand, the spectrum of $L(A)$ are the half lines (8). Therefore, if the matrix A has no real eigenvalues then the spectrum of the operator $L(Q)$ with general potential Q approaches the nonreal half lines (8). Thus we have

Theorem

If the matrix A *has no real eigenvalues, then the real component of the spectrum of* $L(Q)$ *is contained in a finite interval* $[a, b]$.

Now let us discuss the cases when the matrix A has the real eigenvalues and investigate the real component of the spectrum of $L(Q)$.

Theorem

Suppose that the matrix A has a real eigenvalue μ_j of odd multiplicity. Then the spectrum of $L(Q)$ contains the main part of $[0,\infty)$ in the sense that

$$\lim_{n\to\infty}\frac{\mu([0,n]\backslash\sigma(L(Q)))}{\mu(\sigma(L(Q))\cap[0,n])}=0. \tag{9}$$

Now suppose that m is an odd number. If the multiplicities of $\mu_1,\mu_2,...,\mu_p$ are $m_1,m_2,...,m_p$, then $m_1+m_2+...+m_p=m$. The nonreal eigenvalues are the pairs of the conjugate complex numbers with the same multiplicity. Therefore, the total multiplicity of the nonreal eigenvalues is an even number. Then the total multiplicity of the real eigenvalues is an odd number. Hence the matrix A has a real eigenvalue μ_j of odd multiplicity and we have the following consequence of the theorem.

Corollary

If m is an odd number, then the real component of the spectrum of $L(Q)$ contains the main part of $[0, \infty)$ and (9) holds.

Finally, we find a condition on the eigenvalues of the matrix A for which the the real component of the spectrum of $L(Q)$ contains a half line $[H, \infty)$ for some H.

Theorem

If the matrix A has at least three real eigenvalues μ_{j_1}, μ_{j_2} and μ_{j_3} of odd multiplicities such that

$$\min_{i_1, i_2, i_3} \left(diam(\{\mu_{j_1} + \mu_{i_1}, \mu_{j_2} + \mu_{i_2}, \mu_{j_3} + \mu_{i_3}\}) \right) = d \neq 0, \qquad (10)$$

where $i_k = 1, 2, ..., s$ for $k = 1, 2, 3$ and $diam(E) = \sup_{x,y \in E} \mid x - y \mid$, then there exists a number H such that $[H, \infty) \in \sigma(L(Q))$.

In the self-adjoint vectorial case we only suppose that there exists a triple (j_1, j_2, j_3) such that (10) holds. Note that in the scalar case $m = 1$ the finite zone potentials are infinitely differentiable functions and have a special form expressed by Riemann θ function, while in the vectorial case we guarantee finite number of gaps under simple algebraic condition on the eigenvalue of the matrix A. Let us explain why we need three different eigenvalues satisfying (10), in order to prove that the number of the gaps in the spectrum of $L(Q)$ is finite. If the matrix A has only one eigenvalue μ with multiplicity m, then it is possible that the spectrum of $L(Q)$ has infinitely many gaps. For example, if $Q = qI_m$, where q is not a finite zone scalar potential, then the spectrum of $L(Q)$ has infinitely many gaps. The multiplicity of $\mu_{k,j}(t)$ is changed, that is, $\mu_{k,j}(t)$ is an exceptional point of the spectrum of $L(A)$ if $(2\pi k + t)^2 + \mu_j = (2\pi n + t)^2 + \mu_i$ for some $(n, i) \neq (k, j)$. If the matrix C has only two eigenvalues μ_1 and μ_2, then the perturbation $Q - A$ may generate a gap at the neighborhood of the exceptional Bloch eigenvalues $(2\pi k + t)^2 + \mu_1 = (2\pi n + t)^2 + \mu_2$

Now let us discuss why gaps in $\sigma(L(Q))$ do not appear in the interval (H, ∞) if H is a large number and condition (10) holds. For each $s \in \{1, 2, 3\}$ the set

$$\sigma_s(L(A)) = \left\{ (2\pi k + t)^2 + \mu_{j_s} : k \in \mathbb{Z},\ t \in (-\pi, \pi] \right\}$$

(let us call it j_s spectrum) cover the interval (H, ∞). The perturbation $Q - C$ may generate a gap in $\sigma_s(L(C))$ only at the neighborhood of the exceptional Bloch eigenvalues $(2\pi k + t)^2 + \mu_{j_s}$ (let us call it j_s exceptional Bloch eigenvalues). On the other hand, condition (10) implies that the j_1, j_2 and j_3 exceptional Bloch eigenvalues have no common points. That is why, for each $\lambda \in (H, \infty)$ there exists $s \in \{1, 2, 3\}$ such that λ does not belong to the neighborhood of j_s exceptional Bloch eigenvalues. Hence the perturbation $Q - C$ does not generate a gap in $\sigma_{j_s}(L(C))$ at the neighborhood of λ.

Now about the multidimensional Schrödinger operator

$$L(q) = -\Delta + q(x),\ x \in \mathbb{R}^d,\ d \geq 2$$

with a periodic, relative to a lattice Ω, potential q. Recall that the lattice Ω is the set of all linear combinations of d linearly independent vectors $\omega_1, \omega_2, ..., \omega_d$ with the integer coefficients:

$$\Omega = \left\{ \omega = \sum_{k=1}^{d} n_k \omega_k : n_1 \in \mathbb{Z},\ n_2 \in \mathbb{Z},\ ...,\ n_d \in \mathbb{Z} \right\}.$$

The parallelotope (d-dimensional parallelepiped)

$$F = \left\{ x = \sum_{k=1}^{d} y_k \omega_k : y_1 \in [0,1),\ y_2 \in [0,1),\ ...,\ y_d \in [0,1) \right\}$$

is called the fundamental parallelotope or the primitive unit cell of the lattice.

It is well-known that the spectrum of $L(q)$ is the union of the spectra of the operators $L_t(q)$ generated in primitive cell of the lattice Ω by $-\Delta u(x)+q(x)u(x)$ and the Bloch conditions

$$u(x+\omega) = e^{i\langle t,\omega\rangle}u(x),\ \forall\omega\in\Omega,$$

for all quasimomenta t lying in the primitive cell (Brillouin zone) F^* of the reciprocal lattice Γ. The spectrum of $L_t(q)$ consists of the eigenvalues $\Lambda_1(t) \leq \Lambda_2(t) \leq$ ....that are called Bloch eigenvalues of $L(q)$. The n-th band function $\Lambda_n : t \to \Lambda_n(t)$ is continuous with respect to t and its range

$$\{\Lambda_n(t) : t\in F^*\}$$

is n-th band of the spectrum of $L(q)$. The eigenfunctions of $L_t(q)$ are known as the Bloch functions.

In one-dimensional case it is very easy to explain the arising of the gaps in the spectrum. There are only two Bloch eigenvalues $(-n)^2$ and $(n)^2$ of the free operator lying at the point $\lambda = (n)^2$. Under the perturbation q one eigenvalue goes to the left and one to the right and the gap in the neighborhood of $(n)^2$ emerges as a result of these movings.

In the big contrary of the one-dimensional case, in the multidimensional case the set of all Bloch eigenvalues $\mid \gamma + t \mid^2$ of the unperturbed operator $L(0)$ lying at the same point ρ^2 as much as the points of the sphere $S(\rho) = \{x \in \mathbb{R}^d :\mid x \mid = \rho\}$, since $\mid \gamma + t \mid^2 = \rho^2$ is the Bloch eigenvalue of $L(0)$ and $\mathbb{R}^d = \{\gamma + t : \gamma \in \Gamma, t \in F^*\}$. If the sphere is large, then after the perturbation q the probability that all these eigenvalues go away from the point ρ^2 and the other Bloch eigenvalues do not come to this point is very small. However the rigorous mathematical investigation of the perturbations of all these eigenvalues and to prove that the isoenergetic surface $I(\rho^2, q) =: \{t \in F^* : \exists N,\ \Lambda_N(t) = \rho^2\}$ can not become an empty set are extremely complicated.

Note that the regular perturbation theory does not work, since the Bloch eigenvalues of the free operator are situated very close to each other in high energy region. In general, the perturbation theory is easy if the potential q is smaller than the distance between the eigenvalues of the unperturbed operator $L_t(0)$. The regular perturbation theory breaks down when the potential cannot be considered as a small perturbation. This happens for the large Bloch eigenvalue. For large ρ in the interval (ρ^2-1, ρ^2+1) of length 2 there are, in average, ρ^{d-2} Bloch eigenvalues $\mid \gamma + t \mid^2$ of the free operator. It means that the distance between neighboring eigenvalues is of order ρ^{2-d}. The eigenvalue $\Lambda(\gamma+t) \in (\rho^2-1, \rho^2+1)$ of the operator $L_t(q)$ is a result of moving of the Bloch eigenvalues $\mid \gamma + t \mid^2$ of the free electron under the perturbation q. After the perturbation q all these eigenvalues move and some of them move of order 1 and hence each of the resulting eigenvalues $\Lambda(\gamma+t)$ of $L_t(q)$ may coincide with ρ^2. Thus we need to control the moving of all eigenvalues $\mid \gamma + t \mid^2 \in (\rho^2-1, \rho^2+1)$.

In my papers (1983-1985) for the first time the eigenvalues $|\gamma + t|^2$, for large $\gamma \in \Gamma$, were divided into two groups: non-resonance ones and resonance ones and for the perturbations of each group various asymptotic formulae were obtained. Then using the perturbation theory we constructed a part of isoenergetic surfaces $I(\rho^2, q)$ of the Schrödinger operator $L(q)$ of arbitrary dimension which has the measure asymptotically close to the measure of the sphere $S(\rho)$:

$$\mu(I(\rho^2, q)) = \mu(S(\rho))(1 + O(\rho^{-\alpha})), \ \alpha > 0. \tag{11}$$

The nonemptiness of the isoenergetic surfaces $I(\rho^2, q)$ for large ρ, and hence (11) implies that the there exist only a finite number of gaps in the spectrum of L which is the Bethe-Sommerfeld conjecture. This conjecture was formulated in 1933. For the first time M. M. Skriganov (1984) proved the Bethe-Sommerfeld conjecture under some conditions on the lattice by investigation of the arithmetic and geometric properties of the lattice. My method is a first and unique (for the present) by which the validity of the Bethe-Sommerfeld conjecture for arbitrary lattice and for arbitrary dimension is proved (1985).

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THANK YOU