Eigenvalues of a perturbed periodic differential system of arbitrary order

Volodymyr Khrabustovskyi

Ukrainian State University of Railway Transport (Kharkiv, Ukraine)

In $[L^2(\mathbb{R})]^m$ we consider the following eigenvalue problem:

$$(L_r + T_r)y(t) = \lambda y(t), \quad -\infty < t < \infty. \tag{1}$$

Here L_r is a self-adjoint operator corresponding to the system of ODEs

$$L_r y \equiv \sum_{\nu=1}^{r} p_\nu(t) y^{(\nu)} = \lambda y, \quad p_\nu(t) \in \mathbb{C}^{m\times m}, \quad p(t+1) = p(t), \tag{2}$$

and $T_r y$ is a **symmetric perturbation** of the form

$$T_r y = \sum_{k=0}^{r} i^k \ell_k[y], \quad \text{where} \tag{3}$$

$$\ell_{2j} = \mathcal{D}^j \eta_{2j}(t) \mathcal{D}^j, \quad \ell_{2j-1} = \mathcal{D}^{j-1}\left(\mathcal{D}^j \eta_{2j-1}(t) + \eta^*_{2j-1}(t)\mathcal{D}^j\right)\mathcal{D}^{j-1},$$

$$\mathcal{D} := \frac{\mathrm{d}}{\mathrm{d}t}, \quad \eta_{2j}(t), \eta_{2j-1}(t) \in \mathbb{C}^{m\times m}, \quad \eta_{2j} = \eta^*_{2j}.$$

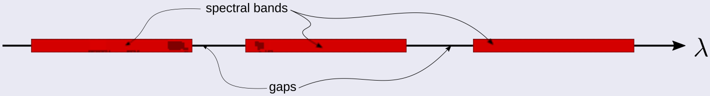
One can also write T_r in a **non-divergence form**:

$$T_r y = \sum_{\nu=0}^{r} \widetilde{\eta}_\nu(t) y^{(\nu)}, \quad \widetilde{\eta}_\nu(t) \in \mathbb{C}^{n\times n}, \quad \det(p_r + \widetilde{\eta}_r) \neq 0. \tag{4}$$

The spectrum of L_r: $\sigma(L_r) = \sigma_{\text{ess}}(L_r)$

Example 1. Scalar Hill's equation

$$-y'' + p_0 y = \lambda y$$



Example 2. Vector equation

$$L_2 y = -\begin{pmatrix} y_1'' \\ y_2'' \end{pmatrix} + \begin{pmatrix} q_1(t) & 0 \\ 0 & q_2(t) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \lambda y$$

$$\sigma(L_2) = \sigma(\ell_1) \cup \sigma(\ell_2), \text{ where } \ell_k y_k = -y_k'' + q_k(t) y_k = \lambda y_k, \ k = 1, 2$$

Remark

In the scalar case one has $\sigma(L_1) = (-\infty, \infty)$.
The operator L_r is **not lower semibounded** either if r is odd or if r is even and $p_r(t)$ is an indefinite matrix-function.

Theorem 1 [K. (1973, 1977)]

1. (T_r is in divergence form (3))

Let L_r be lower-semibounded (hence, $r = 2n$ and $(-1)^n p_{2n}(t) > 0$), while the coefficients of T_{2n} satisfy

$$\exists \alpha, \beta \in \mathbb{R}: \quad 0 < \alpha \mathrm{I}_m < (-1)^n p_{2n}(t) + \eta_{2n}(t) < \beta \mathrm{I}_m$$

$$\exists \gamma \in \mathbb{R}: \quad |\eta_{2n-1}(t)| < \gamma,$$

$$\lim_{|x|\to\infty} \int_x^{x+1} |\eta_j(t)| \, \mathrm{d}t = 0, \; j = 0, \ldots, 2n.$$

Then $\sigma_{\mathrm{ess}}(L_{2n} + T_{2n}) = \sigma_{\mathrm{ess}}(L_{2n})$.

2. (T_r is in non-divergence form (4))

Let

$$\lim_{|t|\to\infty} |\widetilde{\eta}_r(t)| = 0, \tag{5}$$

$$\lim_{|x|\to\infty} \int_x^{x+1} |\widetilde{\eta}_\nu(t)|^2 \, \mathrm{d}t = 0, \; \nu = 0, \ldots, r-1. \tag{6}$$

Then $\sigma_{\mathrm{ess}}(L_r + T_r) = \sigma_{\mathrm{ess}}(L_r)$.

- $\sigma(L_r) = \{\lambda : \text{ Eqn. } L_r y = \lambda y$ has **multipliers** with absolute value $1\}$
- **Multiplicators** are eigenvalues of the Wronski matrix $\|Y_i^{(k-1)}(1,\lambda)\|_{i,k=1}^{r}$ for Eqn. $L_r y = \lambda y$ at $t = 1$; here $Y_t(t,\lambda)$ are $m \times m$ matrix solutions of Eqn. $L_r y = \lambda y$ such that $\|Y_i^{(k-1)}(0,\lambda)\| = \mathrm{I}_{mr}$

Theorem 2 [K. (1977)]

Let μ_0 be one of the endpoints of a spectral gap Δ of the operator L_r. Then the eigenvalues of the problem (1) lying in Δ **do not accumulate** at the point μ_0 if (5) holds and

$$\overline{\lim}_{|x|\to\infty} \left| x^{2p-1} \sum_{\nu=0}^{r} \int_x^{\infty \operatorname{sgn}(x)} \left(|\widetilde{\eta}_\nu(t)| + |\widetilde{\eta}_\nu(t)|^2\right) \mathrm{d}t \right| < \delta,$$

where $2p$ is the size of the largest Jordan block, which correspond to unimodular multipliers of Eqn. (2) for $\lambda = \mu_0$, and the constant $\delta = \delta(L_r, \mu_0)$.

Remark

It is known that the sizes of Jordan blocks corresponding to unimodular multipliers of Eqn. (2) at the endpoints of spectral gaps are **even**.

In the theorem below we assume that $\eta_k(t) \equiv 0$ for $k > 2n$, where $n = [\frac{r}{2}]$.

Theorem 3 [K. (2022)]

Let $\Delta = (\lambda_-, \lambda_+)$ be either finite or infinite gap in $\sigma_{\mathrm{ess}}(L_r)$.

Then $\sigma_{\mathrm{ess}}(L_r + T_r) = \sigma_{\mathrm{ess}}(L_r)$, and, if $|\lambda_\pm| < \infty$, the eigenvalues of $L_r + T_r$ lying in Δ **accumulate** to $\lambda_\pm$, provided the following two conditions hold:

1° $\displaystyle \lim_{|x|\to\infty} \int_x^{x+1} (|\eta_{2k}^{(i)}(t)| + |\eta_{2k-1}^{(i)}(t)|^2)\,\mathrm{d}t = 0,\ i = 0, 1, \dots, k + r.$

2° There is such t_0 that for any $t \le t_0 < 0$ or $t \ge t_0 > 0$ and for any $y \in C_0^\infty(\mathbb{R})$ with $\mathrm{supp}(y) \subset (-\infty, t_0)$ or $\mathrm{supp}(y) \subset (t_0, \infty)$ one has

$$\exists k_\pm \ge 0 : \quad \pm \int_{-\infty}^{\infty} (T_r y, y)\,\mathrm{d}t \le \pm \int_{-\infty}^{\infty} (\tau_\pm y, y)\,\mathrm{d}t, \tag{7}$$

where $\tau_\pm = \mp C_\pm (-1)^{k_\pm} \mathcal{D}^{k_\pm} \frac{1}{t^{2p_\pm}} 1_m \mathcal{D}^{k_\pm}$ with $2p_\pm$ being the maximal of the orders of Jordan blocks associated with unimodular multipliers of equation

$$L_r y = \lambda_\pm y, \tag{8}$$

at that, if $k_\pm > 0$, only such multipliers are taken into consideration that among the corresponding to them Floquet solutions there exists non-constant ones; $C_\pm = C_\pm(L_r, \lambda_\pm, k_\pm) > 0$ are some constants.

We build $m(n+1) \times m(n+1)$ matrix from the coefficients of T_r:

$$H(t) = \begin{pmatrix} \eta_0 & i\eta_1^* & 0 & \dots & \dots & \dots \\ -i\eta_1 & \eta_2 & i\eta_3^* & \dots & \dots & \dots \\ 0 & -i\eta_3 & \eta_4 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \\ \dots & \dots & \dots & \dots & \eta_{2(n-1)} & \eta_{2n-1}^* \\ \dots & \dots & \dots & \dots & -i\eta_{2n-1} & \eta_{2n} \end{pmatrix}$$

Remark

Condition (7) holds, in particular, if for $t \leq t_0 < 0$ or for $t \geq t_0 > 0$ one has

$$\pm \left(\mathrm{diag}(0_m, \dots 0_m, \frac{\mp C_\pm}{t^{2p_\pm}} 1_m, 0, \dots, 0_m) - H(t) \right) \geq 0. \tag{9}$$

This inequality holds for λ_+ if, for example,

$$\nu(H(t)) \leq -\frac{C_+}{t^{2p_+}},$$

where $\nu(H(t))$ is the largest eigenvalue of the matrix $H(t)$; similar statement holds for λ_-.

Remark

The assumption on unimodular multipliers standing at the definition of $p_{\pm}$ (that, if $k_{\pm} > 0$, there exist non-constant Floquet solutions corresponding to these multipliers) cannot be omitted, as the example below demonstrates.

Example. We consider the scalar operator $L_4 = y^{(4)}$. One has $\sigma(L_4) = [0, \infty)$, thus $\lambda_+ = 0$. All multipliers of the equation $y^{(4)} = 0$ equal to 1; the corresponding Jordan blocks are of the size 4; thus $p_+ = 4$. The perturbed operator

$$(L_4 + T_2)y = y^{(4)} + \underbrace{\left[-\left(\frac{C}{t^4}y'\right)'\right]}_{T_2} \qquad (|t| > 1)$$

does not have infinite many eigenvalues below 0 **for any** C, although the inequality (9) holds with $p_+ = 2$, $k_+ = 1 > 0$, $-C_+ \geq C$. The equation $y^{(4)} = 0$ has only one Floquet solution $y \equiv \mathrm{const}$.

Remark

One can show that there are no constant Floquet solutions at the endpoinds of gaps located sufficiently far away from zero.

Discrete spectrum in a semi-infinite gap

In the case of semi-infinite gap Theorem 3 can be specified. Namely, we can "distinguish" a part in the periodic operator L_r, which corresponds to unimodular multipliers, and generate infinitely many eigenvalues in this gap perturbing **only this part**, and not the whole operator.

Let $\inf L_r = \lambda_0 > -\infty$ (consequently, $r = 2n$ and the leading coefficient p_{2n} satisfies $(-1)^n p_{2n}(t) > 0$).

To formulate the next Theorem 4 we first present some auxiliary results.

Proposition

Eqn. (1) for $r = 2n$ reduces to the system

$$x' = \begin{pmatrix} h_1(t) & h_2(t) \\ h_3(t) & -h_1^*(t) \end{pmatrix} x, \quad h_1,\, h_2 = h_2^*,\, h_3 = h_3^* \in \mathbb{C}^{mn\times mn} \tag{10}$$

by means of the substitution

$$x = \mathrm{col}\{y, y', \dots, y^{(n-1)}, y^{[2n-1]}, \dots, y^{[n]}\},$$

where $y^{[j]}$ are quasi-derivatives

Theorem [K.]

Let λ_0 be the edge of a semi-infinite gap. Then the system (11), corresponding to Eqn. (2) at $\lambda = \lambda_0$, possesses $2mn \times mn$-matrix Floquet-type solution:

$$\begin{pmatrix} U(t) \\ V(t) \end{pmatrix}, \quad U, V \in \mathbb{C}^{mn \times mn},$$

where $U(t) = Z(t) \exp(\Lambda t)$, $Z(t+1) = Z(t)$, $\det Z(t) \neq 0$.

For the matrix Λ the sizes of Jordan blocks corresponding to imaginary characteristic numbers are equal to the half of the sizes of the Jordan blocks corresponding to unimodular multipliers of Eqn. (2) for $\lambda = \lambda_0$.

Let the matrix S reduces Λ to the following block-diagonal form:

$$S^{-1} \Lambda S = \mathrm{diag}(\Lambda_1, \Lambda_2), \quad \Lambda_1 \in \mathbb{C}^{q \times q},\ \Lambda_2 \in \mathbb{C}^{\ell \times \ell},$$

where $\sigma(\Lambda_1) \subset i\mathbb{R}$, $\sigma(\Lambda_1) \cap i\mathbb{R} = \emptyset$.

Let $\Lambda_1 = D + N$ be the Dunford decomposition of the matrix Λ_1, where D is a diagonalizable matrix and N is a nilpotent matrix.

Denote

$$\widetilde{\Lambda} = \mathrm{diag}(N, \Lambda_2), \quad \widetilde{Z} = Z(t) S \cdot \exp(\mathrm{diag}(D, \mathrm{I}_\ell) t).$$

Let e_j^i $(i = 1, \dots, q_j)$ be the system of vectors consisting of

- eigenvectors e_j^1 of the matrix $\widetilde{\Lambda}$, which correspond to the eigenvalue 0,
- and generalized eigenvectors of $\widetilde{\Lambda}$, i.e., $\widetilde{\Lambda} e_j^i = e_j^{i-1}$, $i = 1, \dots q_j$, $e_j^0 = 0$.

Then we define the subspaces $\mathfrak{M}^s(e_{j_1}^1, \dots, e_{j_k}^1)$, $1 \le s \le \min\{q_{j_1}, \dots, q_{j_k}\}$:

- $\mathfrak{M}^1(e_{j_1}^1, \dots, e_{j_k}^1) := \operatorname{span}\{e_{j_1}^1, \dots, e_{j_k}^1\}$,
- $\mathfrak{M}^s(e_{j_1}^1, \dots, e_{j_k}^1) := \mathfrak{M}^{s-1}(e_{j_1}^1, \dots, e_{j_k}^1) + \operatorname{span}\{e_{j_1}^s, \dots, e_{j_k}^s\}$

With the perturbation $T_{2(n-1)}$ we associate the following block diagonal matrix consisting of its coefficients:

$$H_1(t) = \begin{pmatrix} \eta_0 & i\eta_1^* & 0 & \dots & \dots & \dots \\ -i\eta_1 & \eta_2 & i\eta_3^* & \dots & \dots & \dots \\ 0 & -i\eta_3 & \eta_4 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \\ \dots & \dots & \dots & \dots & \eta_{2(n-2)} & \eta_{2n-1}^* \\ \dots & \dots & \dots & \dots & -i\eta_{2n-1} & \eta_{2(n-1)} \end{pmatrix}$$

and $q \times q$ matrix $\widetilde{H}_{11}(t)$ such that

$$\widetilde{Z}^*(t) H_1(t) \widetilde{Z}(t) = \begin{pmatrix} \widetilde{H}_{11}(t) & * \\ * & * \end{pmatrix} \tag{11}$$

Finally, we assume that for $T_{2(n-1)}$ the condition 1° from Theorem 1 fulfills.

Theorem 4 [K. (2022)]

The eigenvalues of the operator $L_{2n} + T_{2(n-1)}$, which belong to $(-\infty, \lambda)$, **do accumulate at** λ if there is such t_0 that for any $t \le t_0 < 0$ or $t \ge t_0 > 0$ there exists two subspaces $\mathfrak{M}^1(e^1_{j_1}, \dots, e^1_{j_k})$, $\mathfrak{M}^s(e^1_{j_1}, \dots, e^1_{j_k})$ such that

1° $(\widetilde{H}_{11}\xi, \xi) \le 0$, $\forall \xi \in \mathfrak{M}^s(e^1_{j_1}, \dots, e^1_{j_k})$.

2° $\min_{\xi:\, \sum_{\beta=1}^k |\xi_\beta|^2 = 1} \overline{\lim}_{x \to (\operatorname{sgn} t_0)\infty} \left| x^{2s-1} \int_x^{(\operatorname{sgn} t_0)\infty} (\widetilde{H}_{11}(t)\xi, \xi)\, \mathrm{d}t \right| > K$,

where $\xi = \sum\limits_{\beta=1}^{k} \xi_\beta e^1_{j_\beta}$, $K = K(L_{2n}, \lambda_0, \widetilde{Z}, \mathfrak{M}^s) > 0$.

Corollary

The eigenvalues of the operator $L_{2n} + T_{2(n-1)}$, which belong to $(-\infty, \lambda_0)$, **do accumulate** at λ_0 if there is such t_0 that for any $t \le t_0 < 0$ or $t \ge t_0 > 0$:

$$\widetilde{H}_{11}(t) \le 0 \quad \text{and} \quad \overline{\lim}_{x \to (\operatorname{sgn} t_0)\infty} \left| x^{2p-1} \int_x^{(\operatorname{sgn} t_0)\infty} \mu(-\widetilde{H}_{11}(t))\, \mathrm{d}t \right| = \infty, \tag{12}$$

where $\mu(\cdot)$ is the smallest eigenvalue, $2p$ is the size of the largest Jordan block, which correspond to unimodular multipliers of Eqn. (2) for $\lambda = \lambda_0$.

One gets more rough result if in (12) $\widetilde{H}_{11}(t)$ is replaced by $H_1(t)$.

Thank you for your attention!