

Spectrum of the Laplacian on a domain perturbed by small resonators

Andrii Khrabustovskyi



based on joint work with **Giuseppe Cardone** (University of Naples Federico II)

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Introduction

It is known that the spectrum of the **Dirichlet Laplacian** is stable under small perturbations of a domain.



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for any compact set $K \subset \Omega$ (respectively, $K \subset \mathbb{R}^n \setminus \overline{\Omega}$) we have $K \subset \Omega_\varepsilon$ (respectively, $K \subset \mathbb{R}^n \setminus \Omega_\varepsilon$) if ε is sufficiently small.

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The situation becomes more subtle for the Laplacian with the **Neumann** or **mixed** boundary conditions.

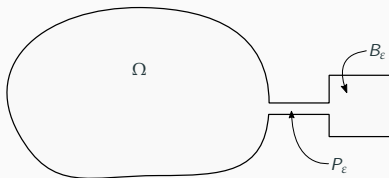
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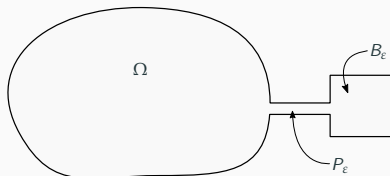
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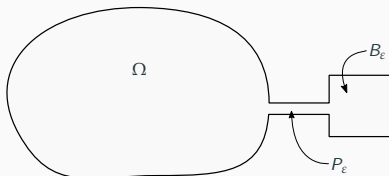
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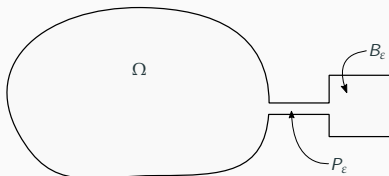
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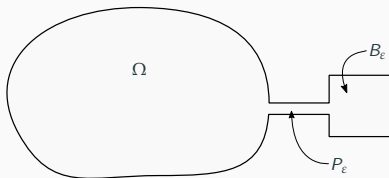
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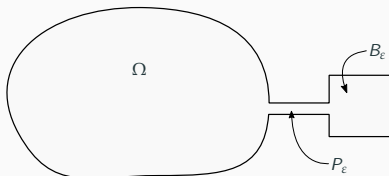
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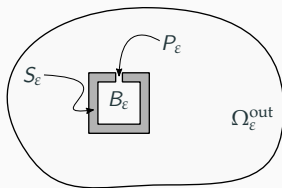
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It was shown that, if $\alpha > (n+1)/(n-1)$, one has

$$\lim_{\varepsilon \rightarrow 0} \lambda_1(-\Delta_{\Omega_\varepsilon}^N) = 0, \quad \lim_{\varepsilon \rightarrow 0} \lambda_k(-\Delta_\Omega^N) = \lambda_{k-1}(-\Delta_\Omega^N), \quad k \in \mathbb{N} \setminus \{1\}.$$

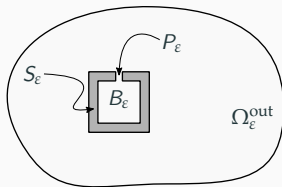


Schweizer, B. *Proc. A. R. Soc. Lond.* **471**, 20140339 (2015)





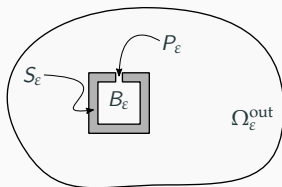
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Let \mathcal{A}_ε be the minus Laplacian on $\Omega_\varepsilon = \Omega \setminus S_\varepsilon = \Omega_\varepsilon^{\text{out}} \cup P_\varepsilon \cup B_\varepsilon$ subject to the Neumann conditions on ∂S_ε and the Dirichlet conditions on $\partial\Omega$.



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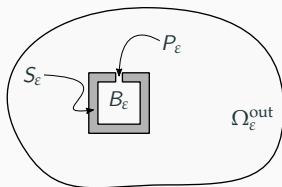
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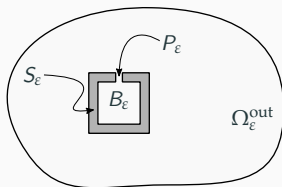
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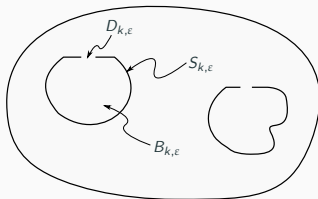
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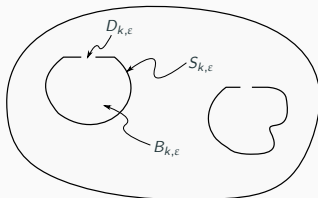
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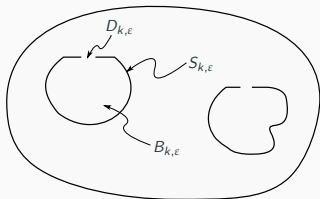
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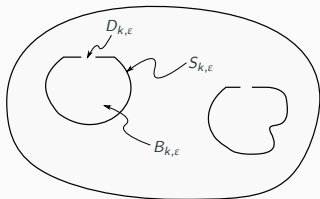
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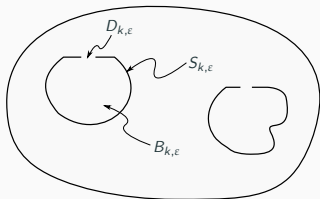
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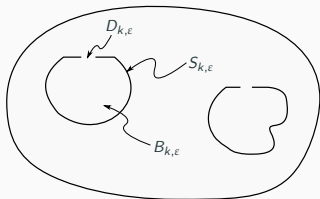
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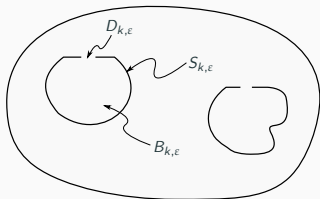
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Main assumption

We assume that the following limits are finite:

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Assumption $(*)$ implies

$$d_{k,\varepsilon} = C_\varepsilon \varepsilon^{\frac{n}{n-2}} \text{ if } n \geq 3, \quad |\ln d_{k,\varepsilon}|^{-1} = C_\varepsilon \varepsilon^2 \text{ if } n = 2, \quad C_\varepsilon = \mathcal{O}(1).$$

In the Hilbert space $L^2(\Omega) \oplus \mathbb{C}^m$ we introduce the operator \mathcal{A} via

$$\mathcal{A} = (-\Delta_{\Omega}^D) \oplus \Gamma,$$

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Spectrum of \mathcal{A}

$$\sigma(\mathcal{A}) = \sigma(-\Delta_{\Omega}^D) \cup \{\gamma_k, k = 1, \dots, m\}.$$

Hausdorff distance between closed sets $X, Y \subset \mathbb{R}$

$$d_H(X, Y) = \max \left\{ \sup_{x \in X} \inf_{y \in Y} |x - y|; \sup_{y \in Y} \inf_{x \in X} |y - x| \right\}.$$

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- X_ε converges from inside to X ($X_\varepsilon \nearrow X$) if for any $x \in X$ there exists a family $(x_\varepsilon)_{\varepsilon>0}$ with $x_\varepsilon \in X_\varepsilon$ such that $x_\varepsilon \rightarrow x$ as $\varepsilon \rightarrow 0$;
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Proposition

Let $(X_\varepsilon)_{\varepsilon>0}$ be a family of closed sets, $X_\varepsilon \subset [0, \infty)$. Let $X \subset [0, \infty)$ be a closed set. Then $\tilde{d}_H(X_\varepsilon, X) \rightarrow 0$ if and only if $X_\varepsilon \rightarrow X$.

Theorem 1

One has $\sigma(\mathcal{A}_\varepsilon) \rightarrow \sigma(\mathcal{A})$ as $\varepsilon \rightarrow 0$, and the estimate

$$\tilde{d}_H(\sigma(\mathcal{A}_\varepsilon), \sigma(\mathcal{A})) \leq \begin{cases} C \sum_{k=1}^m |\gamma_{k,\varepsilon} - \gamma_k| + C\varepsilon, & n \geq 3, \\ C \sum_{k=1}^m |\gamma_{k,\varepsilon} - \gamma_k| + C\varepsilon |\ln \varepsilon|^{3/2}, & n = 2. \end{cases}$$

Theorem 2

One has $\sigma_{\text{ess}}(\mathcal{A}_\varepsilon) = \sigma_{\text{ess}}(-\Delta_\Omega^D) = \sigma_{\text{ess}}(\mathcal{A})$ and

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The multiplicity is preserved: if $\lambda \in \sigma_{\text{disc}}(\mathcal{A})$ is of multiplicity μ and $[\lambda - L, \lambda + L] \cap \sigma(\mathcal{A}) = \{\lambda\}$ with $L > 0$, then for sufficiently small ε the spectrum of \mathcal{A}_ε in $[\lambda - L, \lambda + L]$ is purely discrete and the total multiplicity of the eigenvalues of \mathcal{A}_ε in $[\lambda - L, \lambda + L]$ equals μ .

Theorem 2

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If, in addition, $\mu = 1$ (i.e. the eigenvalue λ is simple), and $\psi = (\psi_0, \psi_1, \dots, \psi_m)$ with $\psi_0 \in L^2(\Omega)$ and $(\psi_k)_{k=1}^m \in \mathbb{C}^m$ is the corresponding normalized in $L^2(\Omega) \oplus \mathbb{C}^m$ eigenfunction, then there is a sequence of normalized in $L^2(\Omega_\varepsilon)$ eigenfunctions ψ_ε of \mathcal{A}_ε such that

$$\|\psi_\varepsilon - \psi_0\|_{L^2(\Omega_\varepsilon \setminus \bigcup_{k=1}^m B_{k,\varepsilon})}^2 + \sum_{k=1}^m \|\psi_\varepsilon - |B_{k,\varepsilon}|^{-1/2} \psi_k\|_{L^2(B_{k,\varepsilon})}^2 \rightarrow 0.$$

Remark

If Ω is a bounded domain, the spectra of \mathcal{A}_ε and \mathcal{A} are purely discrete; hence, due to Theorem 1, additionally to $\sigma_{\text{disc}}(\mathcal{A}_\varepsilon) \nearrow \sigma_{\text{disc}}(\mathcal{A})$ we also have $\sigma_{\text{disc}}(\mathcal{A}_\varepsilon) \searrow \sigma_{\text{disc}}(\mathcal{A})$.

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However, if Ω is unbounded, the latter property does not necessary hold true: it may happen that there exists a sequence $(\lambda_{\varepsilon_k})_{k \in \mathbb{N}}$ with $\lambda_{\varepsilon_k} \in \sigma_{\text{disc}}(\mathcal{A}_{\varepsilon_k})$ converging to $\lambda_0 \in \sigma_{\text{ess}}(\mathcal{A})$.

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Corollary

Let Ω be a bounded domain (consequently, the spectra of \mathcal{A}_ε and \mathcal{A} are purely discrete). Then $\forall k \in \mathbb{N}$ one has

$$\lambda_k(\mathcal{A}_\varepsilon) \rightarrow \lambda_k(\mathcal{A}) \text{ as } \varepsilon \rightarrow 0.$$

Application: waveguide with predefined eigenvalues

We are inspired by celebrated papers



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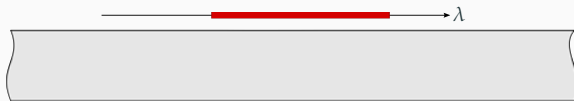
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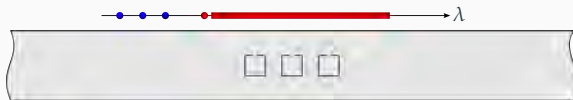
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 Behrndt, J.; K.A. *Math. Nachr.* **295**, 1063–1095 (2022)

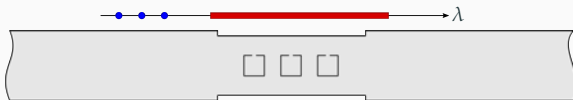
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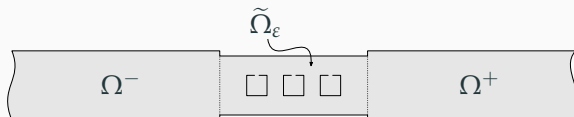
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- $\tilde{\Omega} = \Omega' \times (-L, L)$, where $L > 0$ and Ω' is a bounded Lipschitz domain in \mathbb{R}^{n-1} .
- $\tilde{\Omega}_\varepsilon = \tilde{\Omega} \setminus (\cup_{k=1}^m S_{k,\varepsilon})$, where

$$S_{k,\varepsilon} = \partial B_{k,\varepsilon} \setminus D_{k,\varepsilon}, \quad B_{k,\varepsilon} \cong \varepsilon B, \quad D_{k,\varepsilon} \cong d_{k,\varepsilon} D,$$

$$d_{k,\varepsilon} = d_k \varepsilon^{\frac{n}{n-2}} \text{ if } n \geq 3, \quad \text{and} \quad d_{k,\varepsilon} = \exp(-1/(d_k \varepsilon^2)) \text{ if } n = 2.$$

- $\Omega^+ = \Omega'' \times (L, \infty)$, $\Omega^- = \Omega'' \times (-\infty, -L)$, where Ω'' is a bounded Lipschitz domain in \mathbb{R}^{n-1} such that $\overline{\Omega'} \subset \Omega''$.
- $\Omega_\varepsilon = \Omega^- \cup \tilde{\Omega}_\varepsilon \cup \Omega^+ \cup S^- \cup S^+$, where $S^\pm = \Omega' \times \{\pm L\}$.
- $\mathcal{A}_\varepsilon^{d_1, \dots, d_m}$ is the minus Laplacian on Ω_ε subject to the Dirichlet conditions on $\partial\Omega \setminus \cup_{k=1}^m S_{k,\varepsilon}$ and the Neumann conditions on $S_{k,\varepsilon}$.

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One has $\sigma_{\text{ess}}(\mathcal{A}_{\varepsilon}^{d_1, \dots, d_m}) = [\Lambda, \infty)$, where $\Lambda = \lambda_1(-\Delta_{\Omega''}^D) > 0$.

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We introduce the functions $\mathcal{F}_k : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ via

$$\mathcal{F}_k(t) = \begin{cases} \left(t \frac{4|B_k|}{\text{cap}(D_k)} \right)^{\frac{1}{n-2}}, & n \geq 3, \\ t \frac{2|B_k|}{\pi}, & n = 2. \end{cases}$$

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Theorem

Let $\tilde{\gamma}_k$, $k \in \{1, \dots, m\}$ be arbitrary numbers satisfying

$$0 < \tilde{\gamma}_1 < \tilde{\gamma}_2 < \dots < \tilde{\gamma}_m < \Lambda.$$

Then $\delta > 0$ there exist $\varepsilon > 0$ and $\tilde{d}_k \in (\mathcal{F}_k(\tilde{\gamma}_k) - \delta, \mathcal{F}_k(\tilde{\gamma}_k) + \delta)$, $k \in \{1, \dots, m\}$ such that

$$\sigma(\mathcal{A}_\varepsilon^{\tilde{d}_1, \dots, \tilde{d}_m}) \cap [0, \Lambda) = \bigcup_{k=1}^m \{\tilde{\gamma}_k\}.$$

The eigenvalues $\tilde{\gamma}_k$ are simple.

Thank you for your attention!