Summation of the An-harmonic part of the Propagator

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In this work, we present the summation of the infinite series function represented in the form of the operator function in our previous article.We show, that the final function is finite.
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I. TABLE 1

Let us discuss the an-harmonic part of the propagator function defined by the help of the operators $\hat{\mathcal{I}}_\kappa$ and $\hat{A}_\kappa$:

$$\frac{1}{1-\frac{1}{1-\sum_{\kappa=0}^{4}\ h_\kappa\hat{\mathcal{I}}_\kappa}\cdot\sum_{\kappa_0=0}^{4}\hat{A}_{\kappa_0}\hat{\mathcal{I}}_{\kappa_0}}\ \exp\left(h_\rho I_\rho\right)=\sum_{\mu=0}^{\infty}\left(\frac{1}{1-\sum_{\kappa=0}^{4}\ h_\kappa\hat{\mathcal{I}}_\kappa}\cdot\sum_{\kappa_0=0}^{4}\hat{A}_{\kappa_0}\hat{\mathcal{I}}_{\kappa_0}\right)^{\mu}\ \exp\left(h_\rho I_\rho\right) \tag{1}$$

The $\mu-th$ term of the infinite series for the an-harmonics part of propagator can be written as:

$$\left(\sum_{m_1=0}^{\infty}\left(h_\kappa\hat{\mathcal{I}}_\kappa\right)^{m_1}\cdot\hat{A}_{\kappa_0}\hat{\mathcal{I}}_{\kappa_0}\right)\cdots\cdots\left(\sum_{m_\mu=0}^{\infty}\left(h_\kappa\hat{\mathcal{I}}_\kappa\right)^{m_\mu}\cdot\hat{A}_{\kappa_0}\hat{\mathcal{I}}_{\kappa_0}\right)\ \exp\left(h_\rho I_\rho\right) \tag{2}$$

We introduce the notion of the "single term" $\mathcal{A}_i$, defined by:

$$\mathcal{A}_i\equiv A_i(\kappa_1,\cdots.\kappa_{i+1})I_{\kappa_1,\cdots,\kappa_{i+1}}=(\hat{\mathcal{I}}_{\kappa_1}\hat{A}_{\kappa_1})\cdots(\hat{\mathcal{I}}_{\kappa_i}\hat{A}_{\kappa_i})\ h_{\kappa_{i+1}}I_{\kappa_{i+1}} \tag{3}$$

where

$$A_i(\kappa_1,\cdots.\kappa_{i+1})=\hat{A}_{\kappa_1}.\cdots.\hat{A}_{\kappa_i}.\ h_{\kappa_{i+1}}=(\partial_y h_{\kappa_1})(\partial_x(\partial_y h_{\kappa_2})(\partial_x\cdots(\partial_y h_{\kappa_i})(\partial_x h_{\kappa_{i+1}})\cdots))$$

In the $Tab.1$ we show the first few R_μ results, in the columns are the products of the single terms for the same μ, in the rows are the terms possesses the same numbers of the single terms in the product.

Tab. 1: The results for R_μ. The $\{\mu,i\}$ cell contain the products of i single terms $\mathcal{A}_{\nu_1}.\cdots.\mathcal{A}_{\nu_i}$, satisfying the condition $\nu_1+\cdots+\nu_i=\mu$.

i	$\mu=1$	$\mu=2$	$\mu=3$	$\mu=4$	$\mu=5$	$\mu=6$	$\mu=7$
1	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$	$\mathcal{A}_5$	$\mathcal{A}_6$	$\mathcal{A}_7$
2		$\frac{1}{2}\mathcal{A}_1^2$	$\mathcal{A}_1\mathcal{A}_2$	$\mathcal{A}_1\mathcal{A}_3+\frac{1}{2}\mathcal{A}_2^2$	$\mathcal{A}_1\mathcal{A}_4+\mathcal{A}_2\mathcal{A}_3$	$\mathcal{A}_1\mathcal{A}_5+\mathcal{A}_2\mathcal{A}_4+\frac{1}{2}\mathcal{A}_3^2$	$\mathcal{A}_1\mathcal{A}_6+\mathcal{A}_2\mathcal{A}_5+\mathcal{A}_3\mathcal{A}_4$
3			$\frac{1}{3!}\mathcal{A}_1^3$	$\frac{1}{2!}\mathcal{A}_1^2\mathcal{A}_2$	$\frac{1}{2!}\mathcal{A}_1^2\mathcal{A}_3+\frac{1}{2!}\mathcal{A}_2^2\mathcal{A}_1$	$\frac{1}{2!}\mathcal{A}_1^2\mathcal{A}_4+\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3+\frac{1}{3!}\mathcal{A}_2^3$	$\frac{1}{2!}\mathcal{A}_1^2\mathcal{A}_5+\mathcal{A}_1\mathcal{A}_2\mathcal{A}_4+$ $\frac{1}{2!}\mathcal{A}_2^2\mathcal{A}_3+\frac{1}{2!}\mathcal{A}_3^2\mathcal{A}_1$
4				$\frac{1}{4!}\mathcal{A}_1^4$	$\frac{1}{3!}\mathcal{A}_1^3\mathcal{A}_2$	$\frac{1}{3!}\mathcal{A}_1^3\mathcal{A}_3+\frac{1}{2!}\mathcal{A}_1^2\ \frac{1}{2!}\mathcal{A}_2^2$	$\frac{1}{3!}\mathcal{A}_1^3\mathcal{A}_4+\frac{1}{3!}\mathcal{A}_2^3\mathcal{A}_1+$ $\frac{1}{2!}\mathcal{A}_1^2\mathcal{A}_2\mathcal{A}_3$
5					$\frac{1}{5!}\mathcal{A}_1^5$	$\frac{1}{4!}\mathcal{A}_1^4\mathcal{A}_2$	$\frac{1}{4!}\mathcal{A}_1^4\mathcal{A}_3+\frac{1}{3!}\mathcal{A}_1^3\ \frac{1}{2!}\mathcal{A}_2^2$
6						$\frac{1}{6!}\mathcal{A}_1^6$	$\frac{1}{5!}\mathcal{A}_1^5\mathcal{A}_2$

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We find the rules for the evaluations of the Eq.(1) in terms of results corresponding to (2):

1. The evaluation of Eq.(2) for μ fixed gives the sum of the products of the single terms in R_μ multiplied by the factor $\exp(h_\kappa I_\kappa)$.

2. The application of the operator $\hat{A}_{\kappa 0}\hat{I}_{\kappa 0}$ on the products of the single terms do not results to products of the single terms. The operator $\hat{A}_{\kappa_0}\hat{I}_{\kappa_0}$ create from the product of the n single terms and exponential function, i.e. $\exp(h_\nu I_\nu)$ $\mathcal{A}_{i_1}.\cdots.\mathcal{A}_{i_n}\exp(h_\nu I_\nu)$, for μ fixed:

2.a Rule 1:

When applied on the products of $\mathcal{A}_i$, the sum of n terms, each of them is products of the n terms:

$$\begin{aligned} &\hat{A}_{\kappa 0}\hat{I}_{\kappa 0}[\mathcal{A}_{i_1}(\rho_1,\cdots,\rho_{i_1+1})\cdots\mathcal{A}_{i_n}(\nu_1,\cdots,\nu_{i_n+1})\exp(h_\nu I_\nu)] = \\ =\ &\hat{I}_{\kappa_0}\left[\overline{\mathcal{A}}_{i_1+1}(\kappa_0,\rho_1,\cdots,\rho_{i_1+1})\mathcal{A}_{i_2}\mathcal{A}_{i_3}\cdots\mathcal{A}_{i_n} + \mathcal{A}_{i_1}\overline{\mathcal{A}}_{i_2+1}(\kappa_0,\nu_1,\cdots,\nu_{i_1+1})A_{i_3}\cdots A_{i_n} + \right.\\ +\ &\left.\cdots + A_{i_1}.A_{i_2}.A_{i_3}.\cdots.\overline{A}_{i_n+1}(\kappa_0,\nu_1,\cdots,\nu_{i_n+1})\right]\exp(h_\nu I_\nu) \end{aligned} \tag{4}$$

2.b Rule 2:

When applied on $\frac{(h_\nu I_\nu)^m}{m!}$ from decompositions of $\exp(h_\nu I_\nu)$, the result is product of $n+1$ terms:

$$\hat{I}_{\kappa_0}\left[\mathcal{A}_{i_1+1}(\rho_1,\cdots,\rho_{i_1+1})\cdots\mathcal{A}_{i_n}(\nu_1,\cdots,\nu_{i_n+1})\overline{\mathcal{A}}_1(\kappa_0,\nu)\right]\exp(h_\nu I_\nu)$$

This is not product of the single terms also. Only combination of the results of both methods forms the products of the single terms. Let $R_{\mu,i}$ is the sum of the products of i single terms for μ fixed in Eq.(2), see Tab.1. Then

$$R_{\mu+1,i} = \hat{I}_{\kappa_0}[\hat{A}_{\kappa_0}R_{\mu,i}] + \hat{I}_{\kappa_0}[R_{\mu,i-1}\overline{\mathcal{A}}_1(\kappa_0,\nu)]$$

can be written as sum of the products of the single terms. In Appendix D we show this phenomenon in detail for construction of the column for R_μ from the column for $R_{\mu-1}$.

3. The index i of the single term $\mathcal{A}_i$ represent the number of the application of the operator $\hat{A}_\kappa$ on the function h_ν. Therefore, the sum of the indices of a products of the single terms in $R_{i,\mu}$ is equals to μ.

4. Each $n-$power of the single term is accompanied by the coefficient $\frac{1}{n!}$.

5. The infinite sum in Eq.(1) expressed in term of the functions $R_{\mu,i}$ can be read:

$$\sum_{\mu=0}^{\infty}\left(\frac{1}{1-\sum_{\kappa=0}^{4}\ h_\kappa\hat{\mathcal{I}}_\kappa}\cdot\hat{A}_{\kappa_0}\hat{\mathcal{I}}_{\kappa_0}\right)^{\mu}\exp(h_\kappa I_\kappa) = \left(1+\sum_{\mu=1}^{\infty}\sum_{i=1}^{\mu}\ R_{\mu,i}\right)\exp(h_\kappa I_\kappa)$$

We can change the order of the summations in the above equation:

$$\left(1+\sum_{\mu=1}^{\infty}\sum_{i=1}^{\mu}\ R_{\mu,i}\right)\exp(h_\kappa I_\kappa) = \left(1+\sum_{i=1}^{\infty}\sum_{\mu=i}^{\infty}\ R_{\mu,i}\right)\exp(h_\kappa I_\kappa)$$

The sum $\sum_{\mu=1}^{\infty}\ R_{\mu,1} = \sum_{i=1}^{\infty}\mathcal{A}_i$ is the sum of the single terms. The sum $\sum_{\mu=2}^{\infty}\ R_{\mu,2}$ is the sum of the products of two single terms, which can be written as

$$\frac{1}{2}\left(\sum_{i=1}^{\infty}\mathcal{A}_i\right)^2$$

In the limit $n\to\infty$, all terms of the infinite sum of $R_{\mu,i}$ can be find in the expansion of

$$\frac{1}{i!}\left(\sum_{k=1}^{\infty}\mathcal{A}_k\right)^i.$$

Therefore, the Eq.(1) in this simpler example is equal to

$$\exp(h_\kappa I_\kappa + \sum_{k=1}^{\infty}\mathcal{A}_k) = \exp\left(\frac{1}{1-\hat{A}_{\kappa_0}\hat{I}_{\kappa_0}}h_\nu I_\nu\right) \tag{5}$$

Tab. 2: The few first sheet items.

i	$\mu=1$	$\mu=2$	$\mu=3$	$\mu=4$	$\mu=5$	$\mu=6$	$\mu=7$
1	$\mathcal{Z}_1$	$\mathcal{Z}_2$	$\mathcal{Z}_3$	$\mathcal{Z}_4$	$\mathcal{Z}_5$	$\mathcal{Z}_6$	$\mathcal{Z}_7$
2		$\frac{1}{2}\mathcal{Z}_1^2$	$\mathcal{Z}_1\mathcal{Z}_2$	$\mathcal{Z}_1\mathcal{Z}_3+\frac{1}{2}\mathcal{Z}_2^2$	$\mathcal{Z}_1\mathcal{Z}_4+\mathcal{Z}_2\mathcal{Z}_3$	$\mathcal{Z}_1\mathcal{Z}_5+\mathcal{Z}_2\mathcal{Z}_4+\frac{1}{2}\mathcal{Z}_3^2$	$\mathcal{Z}_1\mathcal{Z}_6+\mathcal{Z}_2\mathcal{Z}_5+\mathcal{Z}_3\mathcal{Z}_4$
3			$\frac{1}{3!}\mathcal{Z}_1^3$	$\frac{1}{2!}\mathcal{Z}_1^2\mathcal{Z}_2$	$\frac{1}{2!}\mathcal{Z}_1^2\mathcal{Z}_3+\frac{1}{2!}\mathcal{Z}_2^2\mathcal{Z}_1$	$\frac{1}{2!}\mathcal{Z}_1^2\mathcal{Z}_4+\mathcal{Z}_1\mathcal{Z}_2\mathcal{Z}_3+\frac{1}{3!}\mathcal{Z}_2^3$	$\frac{1}{2!}\mathcal{Z}_1^2\mathcal{Z}_5+\mathcal{Z}_1\mathcal{Z}_2\mathcal{Z}_4+$ $\frac{1}{2!}\mathcal{Z}_2^2\mathcal{Z}_3+\frac{1}{2!}\mathcal{Z}_3^2\mathcal{Z}_1$
4				$\frac{1}{4!}\mathcal{Z}_1^4$	$\frac{1}{3!}\mathcal{Z}_1^3\mathcal{Z}_2$	$\frac{1}{3!}\mathcal{Z}_1^3\mathcal{Z}_3+\frac{1}{2!}\mathcal{Z}_1^2\ \frac{1}{2!}\mathcal{Z}_2^2$	$\frac{1}{3!}\mathcal{Z}_1^3\mathcal{Z}_4+\frac{1}{3!}\mathcal{Z}_2^3\mathcal{Z}_1+$ $\frac{1}{2!}\mathcal{Z}_1^2\mathcal{Z}_2\mathcal{Z}_3$
5					$\frac{1}{5!}\mathcal{Z}_1^5$	$\frac{1}{4!}\mathcal{Z}_1^4\mathcal{Z}_2$	$\frac{1}{4!}\mathcal{Z}_1^4\mathcal{Z}_3+\frac{1}{3!}\mathcal{Z}_1^3\ \frac{1}{2!}\mathcal{Z}_2^2$
6						$\frac{1}{6!}\mathcal{Z}_1^6$	$\frac{1}{5!}\mathcal{Z}_1^5\mathcal{Z}_2$
7							$\frac{1}{7!}\mathcal{Z}_1^7$

The $\{\mu, i\}$ cell contain the sum of the products of i single terms $\mathcal{Z}_{\nu_1}.\cdots.\mathcal{Z}_{\nu_i}$, satisfying the condition $\nu_1+\cdots+\nu_i=\mu$. Each m power of the single term $\mathcal{Z}_{\nu_i}^m$ is accompanied by the numerical factor $\frac{1}{m!}$. This imply, that the items on $i-th$ line are composed as $i-th$ power of the first, $i=1$ line, multiplied by numerical factor $\frac{1}{i!}$.

Tab. 3: The first few second sheet items obtained by above procedure.

In our notation, the line over the symbol signals, that $\overline{\mathcal{B}_1(\kappa_0,\mathcal{Z}_i,\mathcal{Z}_j)}$ is the non-complete single term. This situation takes place, when the operator $\hat{B}_1(\kappa_0)\hat{I}_{\kappa_0}$ acts on the product at least three single terms. The operator $\hat{B}_1(\kappa_0,\cdot,\cdot)$ acts on the non-integral parts of the single terms. The operator $\hat{I}_{\kappa_0}$ acts on the whole products of the integrals, which can be mined from the original products the single term in question. For the simplicity in the next table $\overline{\mathcal{B}_1(\kappa_0,\mathcal{Z}_1,\mathcal{Z}_2)\mathcal{Z}_1}$ means $\hat{I}_{\kappa_0}\left[\overline{\mathcal{B}_1(\kappa_0,\mathcal{Z}_1,\mathcal{Z}_2)\mathcal{Z}_1}\right]$, etc.

i	1	2	$\mu=3$	$\mu=4$	$\mu=5$	$\mu=6$	$\mu=7$
1			$\frac{1}{2}\mathcal{B}_1(\mathcal{Z}_1,\mathcal{Z}_1)$	$\mathcal{B}_1(\mathcal{Z}_1,\mathcal{Z}_2)$	$\mathcal{B}_1(\mathcal{Z}_1,\mathcal{Z}_3)+$ $\frac{1}{2}\mathcal{B}_1(\mathcal{Z}_2,\mathcal{Z}_2)$	$\mathcal{B}_1(\mathcal{Z}_1,\mathcal{Z}_4)+$ $\mathcal{B}_1(\mathcal{Z}_2,\mathcal{Z}_3)$	$\mathcal{B}_1(\mathcal{Z}_1,\mathcal{Z}_5)+$ $\mathcal{B}_1(\mathcal{Z}_2,\mathcal{Z}_4)+$ $\frac{1}{2}\mathcal{B}_1(\mathcal{Z}_3,\mathcal{Z}_3)$
2				$\frac{1}{2}\overline{\mathcal{B}_1(\kappa_0,\mathcal{Z}_1,\mathcal{Z}_1)}\mathcal{Z}_1$	$\frac{1}{2}\overline{\mathcal{B}_1(\kappa_0,\mathcal{Z}_1,\mathcal{Z}_1)}\mathcal{Z}_2+$ $\overline{\mathcal{B}_1(\kappa_0,\mathcal{Z}_1,\mathcal{Z}_2)}\mathcal{Z}_1$	$\frac{1}{2}\overline{\mathcal{B}_1(\kappa_0,\mathcal{Z}_1,\mathcal{Z}_1)}\mathcal{Z}_3+$ $\overline{\mathcal{B}_1(\kappa_0,\mathcal{Z}_1,\mathcal{Z}_3)}\mathcal{Z}_1+$ $\frac{1}{2}\overline{\mathcal{B}_1(\kappa_0,\mathcal{Z}_2,\mathcal{Z}_2)}\mathcal{Z}_1+$ $\overline{\mathcal{B}_1(\kappa_0,\mathcal{Z}_2,\mathcal{Z}_1)}\mathcal{Z}_2$	$\frac{1}{2}\overline{\mathcal{B}_1(\kappa_0,\mathcal{Z}_1,\mathcal{Z}_1)}\mathcal{Z}_4+$ $\overline{\mathcal{B}_1(\kappa_0,\mathcal{Z}_1,\mathcal{Z}_4)}\mathcal{Z}_1+$ $\frac{1}{2}\overline{\mathcal{B}_1(\kappa_0,\mathcal{Z}_2,\mathcal{Z}_2)}\mathcal{Z}_2+$ $\overline{\mathcal{B}_1(\kappa_0,\mathcal{Z}_1,\mathcal{Z}_2)}\mathcal{Z}_3+$ $\overline{\mathcal{B}_1(\kappa_0,\mathcal{Z}_1,\mathcal{Z}_3)}\mathcal{Z}_2+$ $\overline{\mathcal{B}_1(\kappa_0,\mathcal{Z}_2,\mathcal{Z}_3)}\mathcal{Z}_1$
3					$\frac{1}{2}\overline{\mathcal{B}_1(\kappa_0,\mathcal{Z}_1,\mathcal{Z}_1)}\ \frac{1}{2}\mathcal{Z}_1^2$	$\frac{1}{2}\overline{\mathcal{B}_1(\kappa_0,\mathcal{Z}_1,\mathcal{Z}_1)}\ \mathcal{Z}_1\mathcal{Z}_2+$ $\frac{1}{2!}\mathcal{Z}_1^2\ \overline{\mathcal{B}_1(\kappa_0,\mathcal{Z}_1,\mathcal{Z}_2)}$	$\frac{1}{2!}\overline{\mathcal{B}_1(\kappa_0,\mathcal{Z}_1,\mathcal{Z}_1)}\ \mathcal{Z}_1\mathcal{Z}_3+$ $\frac{1}{2!}\mathcal{Z}_1^2\ \overline{\mathcal{B}_1(\kappa_0,\mathcal{Z}_1,\mathcal{Z}_3)}+$ $\frac{1}{4}\mathcal{Z}_1^2\ \overline{\mathcal{B}_1(\kappa_0,\mathcal{Z}_2,\mathcal{Z}_2)}+$ $\frac{1}{4}\mathcal{Z}_2^2\ \overline{\mathcal{B}_1(\kappa_0,\mathcal{Z}_1,\mathcal{Z}_1)}+$ $\mathcal{Z}_1\mathcal{Z}_2\ \overline{\mathcal{B}_1(\kappa_0,\mathcal{Z}_1,\mathcal{Z}_2)}$
4						$\frac{1}{3!}\mathcal{Z}_1^3\ \frac{1}{2!}\overline{\mathcal{B}_1(\kappa_0,\mathcal{Z}_1,\mathcal{Z}_1)}$	$\frac{1}{3!}\mathcal{Z}_1^3\ \overline{\mathcal{B}_1(\kappa_0,\mathcal{Z}_1,\mathcal{Z}_2)}+$ $\frac{1}{2}\mathcal{Z}_1^2\mathcal{Z}_2\frac{1}{2}\overline{\mathcal{B}_1(\kappa_0,\mathcal{Z}_1,\mathcal{Z}_1)}$
5							$\frac{1}{4!}\mathcal{Z}_1^4\ \frac{1}{2}\overline{\mathcal{B}_1(\kappa_0,\mathcal{Z}_1,\mathcal{Z}_1)}$

Once the terms on the second sheet are evaluated for μ^{th} recurrence step by Rule 3 (i.e. we fill μ^{th} column in Tab. 3, starting from $(\mu-1)^{th}$ column of Tab. 2), we complete the evaluation of the items on the second sheet by application of Rule 1 and Rule 2 on the column $\mu-1$ on the second sheet. We find the contributing items to the cell $\{\mu, i\}$ on the second sheet from the cell $\{\mu-1, i-1\}$ of the second sheet by Rule 1, and from the cell $\{\mu-1, i\}$ of the second sheet by Rule 2. It is clear, that the new contributions to the cell $\{\mu, i\}$ must be combined with the item in the cell $\{\mu, i\}$ evaluated by Rule 3.

Tab. 4: The first few second sheet items.

i	1	2	$\mu=3$	$\mu=4$	$\mu=5$	$\mu=6$
1			$\frac{1}{2}\mathcal{B}_1(\mathcal{Z}_1,\mathcal{Z}_1)$	$\mathcal{B}_1(\mathcal{Z}_1,\mathcal{Z}_2)+$ $\mathcal{O}_1(\frac{1}{2}\mathcal{B}_1(\mathcal{Z}_1,\mathcal{Z}_1))$	$\mathcal{B}_1(\mathcal{Z}_1,\mathcal{Z}_3)+\frac{1}{2}\mathcal{B}_1(\mathcal{Z}_2,\mathcal{Z}_2)$ $\mathcal{O}_1(\mathcal{B}_1(\mathcal{Z}_1,\mathcal{Z}_2))$ $\mathcal{O}_2(\frac{1}{2}\mathcal{B}_1(\mathcal{Z}_1,\mathcal{Z}_1))$	$\mathcal{B}_1(\mathcal{Z}_1,\mathcal{Z}_4)+\mathcal{B}_1(\mathcal{Z}_2,\mathcal{Z}_3)$ $\mathcal{O}_1(\mathcal{B}_1(\mathcal{Z}_1,\mathcal{Z}_3))+\mathcal{O}_1(\frac{1}{2}\mathcal{B}_1(\mathcal{Z}_2,\mathcal{Z}_2))$ $\mathcal{O}_2(\mathcal{B}_1(\mathcal{Z}_1,\mathcal{Z}_2))$ $\mathcal{O}_3(\frac{1}{2}\mathcal{B}_1(\mathcal{Z}_1,\mathcal{Z}_1))$
2				$\frac{1}{2}\mathcal{B}_1(\mathcal{Z}_1,\mathcal{Z}_1)\mathcal{Z}_1$	$\frac{1}{2}\mathcal{B}_1(\mathcal{Z}_1,\mathcal{Z}_1)\mathcal{Z}_2+\mathcal{B}_1(\mathcal{Z}_1,\mathcal{Z}_2)\mathcal{Z}_1$ $\mathcal{O}_1(\frac{1}{2}\mathcal{B}_1(\mathcal{Z}_1,\mathcal{Z}_1))\ \mathcal{Z}_1$	$\frac{1}{2}\mathcal{B}_1(\mathcal{Z}_1,\mathcal{Z}_1)\mathcal{Z}_3+\mathcal{B}_1(\mathcal{Z}_2,\mathcal{Z}_1)\mathcal{Z}_2$ $\mathcal{B}_1(\mathcal{Z}_1,\mathcal{Z}_3)\mathcal{Z}_1+\frac{1}{2}\mathcal{B}_1(\mathcal{Z}_2,\mathcal{Z}_2)\mathcal{Z}_1$ $\mathcal{O}_1(\mathcal{B}_1(\mathcal{Z}_1,\mathcal{Z}_2))\ \mathcal{Z}_1+\mathcal{O}_1(\frac{1}{2}\mathcal{B}_1(\mathcal{Z}_1,\mathcal{Z}_1))\ \mathcal{Z}_2$ $\mathcal{O}_2(\frac{1}{2}\mathcal{B}_1(\mathcal{Z}_1,\mathcal{Z}_1))\ \mathcal{Z}_1$
3					$\frac{1}{2}\mathcal{B}_1(\mathcal{Z}_1,\mathcal{Z}_1)\ \frac{1}{2}\mathcal{Z}_1^2$	$\frac{1}{2}\mathcal{B}_1(\mathcal{Z}_1,\mathcal{Z}_1)\ \mathcal{Z}_1\mathcal{Z}_2+\frac{1}{2!}\mathcal{Z}_1^2\ \mathcal{B}_1(\mathcal{Z}_1,\mathcal{Z}_2)$ $\mathcal{O}_1(\frac{1}{2}\mathcal{B}_1(\mathcal{Z}_1,\mathcal{Z}_1))\ \frac{1}{2}\mathcal{Z}_1^2$
4						$\frac{1}{3!}\mathcal{Z}_1^3\ \frac{1}{2!}\mathcal{B}_1(\mathcal{Z}_1,\mathcal{Z}_1)$

In this Table the single functions of the second sheet $\mathcal{O}_i(\mathcal{B}_1)$ are defined as in Eq. (**??**), by applications of the operator

$$\hat{I}_{\nu_0}\hat{\mathcal{O}}_{\nu_0}(\cdot)=\sum_{n=1}^{2}(\partial_y^n h_{\nu_0})(\partial_x^n\cdot)\ \hat{I}_{\nu_0}[\cdot]+2(\partial_y^2 h_{\nu_0})(\partial_x h_\mu)(\partial_x\cdot)\ \hat{I}_{\nu_0}[I_\mu\cdot]$$

on the function $\hat{\mathcal{O}}_{i-1}(\mathcal{B}_1)$.

The single terms on the second sheet in the cells $\{\mu,i=1\}$ of the Tab. 4 are defined by the rule:

$$R_{\mu,1}=\sum_{i_\mu=0}^{\mu}\sum_{j_\mu=1}^{\mu}\sum_{k_\mu=1}^{\mu}\left(\hat{I}_{\nu_0}\hat{\mathcal{O}}_{\nu_0}\right)^{i_\mu}\left(\frac{1}{2!}\mathcal{B}_1(\mathcal{Z}_{j_\mu},\mathcal{Z}_{k_\mu})\right)$$

The terms belonging to above sum fulfils the conditions:

$$\mathcal{B}_1(\mathcal{Z}_{j_\mu},\mathcal{Z}_{k_\mu})=\mathcal{B}_1(\mathcal{Z}_{k_\mu},\mathcal{Z}_{j_\mu}),\quad i_\mu+j_\mu+k_\mu+1=\mu.$$

Tab.5:

In the $\{\mu,i\}$ cell we have the sum of the all products of the $s=\mu-i+1$ single terms on the $i-th$ sheet. We can see, that on the main diagonal are stored the corresponding sum of the single terms on the all sheets. on $i-th$ lateral diagonal, we can find $i-th$ power of the sum of the main diagonal elements, divided by the factor $i!$. The sum of all elements of this table, representing the non-perturbative contribution to the propagator, together with the indestructible $h_\kappa I_\kappa$ is $\exp\left(h_\kappa I_\kappa+S_I+S_{II}+\cdots+S_\infty\right)$.

i	$\mu=1$	$\mu=2$	$\mu=3$	$\mu=4$	$\mu=5$	$\mu=6$
1	$\mathcal{S}_I=\sum_{j=1}^{\infty}\mathcal{Z}_j$	$\frac{1}{2}\mathcal{S}_I^2$	$\frac{1}{3!}\mathcal{S}_I^3$	$\frac{1}{4!}\mathcal{S}_I^4$	$\frac{1}{5!}\mathcal{S}_I^5$	$\frac{1}{6!}\mathcal{S}_I^6$
2		$\mathcal{S}_{II}$	$\mathcal{S}_I\mathcal{S}_{II}$	$\frac{1}{2}\mathcal{S}_I^2\mathcal{S}_{II}$	$\frac{1}{3!}\mathcal{S}_I^3\mathcal{S}_{II}$	$\frac{1}{4!}\mathcal{S}_I^4\mathcal{S}_{II}$
3			$\mathcal{S}_{III}$	$\frac{1}{2!}\mathcal{S}_{II}^2+$ $\mathcal{S}_I\mathcal{S}_{III}$	$\mathcal{S}_I\frac{1}{2!}\mathcal{S}_{II}^2+$ $\frac{1}{2!}\mathcal{S}_I^2\mathcal{S}_{III}$	$\frac{1}{2!}\mathcal{S}_I^2\frac{1}{2!}\mathcal{S}_{II}^2$ $\frac{1}{3!}\mathcal{S}_I^3\mathcal{S}_{III}$
4				$\mathcal{S}_{IV}$	$\mathcal{S}_I\mathcal{S}_{IV}+$ $\mathcal{S}_{II}\mathcal{S}_{III}$	$\frac{1}{3!}\mathcal{S}_{II}^3+$ $\mathcal{S}_I\mathcal{S}_{II}\mathcal{S}_{III}+$ $\frac{1}{2!}\mathcal{S}_I^2\mathcal{S}_{IV}+$
5					$\mathcal{S}_V$	$\frac{1}{2!}\mathcal{S}_{III}^2+$ $\mathcal{S}_I\mathcal{S}_V+$ $\mathcal{S}_{II}\mathcal{S}_{IV}$
6						$\mathcal{S}_{VI}$

[1] J. Boháčik and P. Prešnajder and P. Augustín, The Possibility of the Non-perturbative An-harmonic Corrections to Mehler's Formula for Propagator of the Harmonic Oscillator, arXiv:1306.1694.
[2] J. Boháčik and P. Prešnajder and P. Augustín, JMP