Computing Eigenvalues of Sturm–Liouville Operators with PT-Symmetric Trigonometric Polynomial Potentials

Cemile NUR

Yalova University

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We consider the operators $S_t(q)$, for $t = 0, 1$, generated in $L_2[0, \pi]$ by the differential expression

$$-y''(x) + q(x)y(x) \tag{1}$$

and the boundary conditions

$$y(\pi) = e^{i\pi t}y(0), \qquad y'(\pi) = e^{i\pi t}y'(0) \tag{2}$$

that is, periodic and antiperiodic boundary conditions, where q is the trigonometric polynomial potential of the form

$$q(x) = q_{-m}e^{-i2mx} + q_m e^{i2mx}, \qquad m \geq 1, \tag{3}$$

$(q_{-m}q_m) \in \mathbb{R}$ and $m \in \mathbb{Z}$. Note that, in the case $m = 1$, potential (3) can be considered as the optical potential

$$q\left(x\right) = \left(1 + 2V\right)e^{i2x} + \left(1 - 2V\right)e^{-i2x}, \qquad V \geq 0, \tag{4}$$

with $q_{-1} = 1 - 2V$, $q_1 = 1 + 2V$, $V \geq 0$. In our work, we investigate the case $m = 1$ for the optical potential (4).

It was proved by Veliev [17, O.A. Veliev, 2013 (see Theorem 1 and (26))] that, if $ab = cd$, where a, b, c, and d are arbitrary complex numbers, then the Hill operators $S(q)$ and $S(p)$ generated in $L_2(-\infty, \infty)$ by the expression $-y'' + q(x)y$ with the potentials $q(x) = ae^{-i2x} + be^{i2x}$ and $p(x) = ce^{-i2x} + de^{i2x}$, have the same Hill discriminant, and hence the same Bloch eigenvalues and spectrum. Therefore, the investigations of the operators $S_t(q)$, for $t = 0, 1$, can be reduced to the investigations of the operators generated in $L_2[0, \pi]$ by the differential expression $-y'' + q(x)y$ and the boundary conditions (2) with the potential

$$p(x) = re^{-i2mx} + re^{i2mx} = 2r_m \cos(2mx), \tag{5}$$

where

$$r_m = \sqrt{q_{-m}q_m}.$$

In particular, $r_1 = \sqrt{q_{-1}q_1} = \sqrt{1 - 4V^2}$.

It is well known that the spectra of the operators $S_0(q)$ and $S_1(q)$ are discrete and for large enough n, there are two periodic (if n is even) or antiperiodic (if n is odd) eigenvalues (counted with multiplicity) in the neighborhood of n^2. See the basic and detailed classical results in the works of Brown et al. [3, B.M. Brown, M.S.P. Eastham and K.M. Schmidt, 2013], Levy and Keller [8, D.M. Levy and J.B. Keller, 1963], Magnus, and Winkler [9, W. Magnus, and S. Winkler, 1969], Marchenko [12, V. Marchenko, 1986] and references therein.

Some physically interesting results have been obtained by considering the optical potential (4). The detailed investigations of the periodic optical potentials were illustrated on (4) in the papers [10, 11, K.G. Makris, R. El-Ganainy, D.N. Christodoulides and Z.H. Musslimani, 2010, 2011]. For the first time, the mathematical explanations of the nonreality of the spectrum of the Hill operator $S(q)$, generated in $L_2(-\infty,\infty)$ by the differential expression $-y'' + q(x)y$ with potential (4), for $V > 0.5$ and finding the threshold 0.5 (the first critical point V_1) were given by Makris et al. [10, 11, K.G. Makris, R. El-Ganainy, D.N. Christodoulides and Z.H. Musslimani, 2010, 2011]. Moreover, using numerical methods they sketched the real and imaginary parts of the first two bands for $V = 0.85$. Midya et al. [13, B. Midya, B. Roy and R. Roychoudhury, 2010] reduced the operator $S(q)$ to the Mathieu operator and using the tabular values, they established that there is a second critical point $V_2 \sim 0.888437$ after which no parts of the first and second bands remain real.

Some of the most valuable results were given by Veliev [20, 21, O.A. Veliev, 2018, 2020]. In [20, O.A. Veliev, 2018], he gave a complete description, along with a mathematical proof, of the shape of the spectrum of the Hill operator $S(q)$ with potential (4), when V changes from $1/2$ to $\sqrt{5}/2$. Then, he extended his results for all $V > 1/2$ in [21, O.A. Veliev, 2020].

Note that, the trigonometric polynomial potential (3) is a PT-symmetric potential if $q_{-m}, q_m \in \mathbb{R}$. For the properties of the general PT-symmetric potentials, see [1, F. Bagarello, J.P. Gazeau, F.H. Szafraniec and M. Znojil, 2015], [14, A. Mostafazadeh, 2010], [19, 22, O.A. Veliev, 2017, 2021] and references therein. Here, we only note that, the investigations of PT-symmetric periodic potentials were initiated by Bender et al. [2, C.M. Bender, G.V. Dunne and P.N. Meisinger, 1999].

The eigenvalues of the operators $S_0(0)$ and $S_1(0)$ are $(2n)^2$ and $(2n+1)^2$, for $n \in \mathbb{Z}$, respectively and all eigenvalues of $S_0(0)$ and $S_1(0)$, except 0, are double. The eigenvalues of $S_0(q)$ and $S_1(q)$ are called the periodic and antiperiodic eigenvalues and they are denoted by $\lambda_n(q)$, for $n \in \mathbb{Z}$ and $\mu_n(q)$, for $n \in \mathbb{Z} - \{0\}$, respectively.

It is well known that (see [5, MSP Eastham, 1974], [9, W. Magnus and S. Winkler, 1966], [12, V. Marchenko, 1986]), if r_m is a real nonzero number, then all eigenvalues of the operator $H_t(r_m)$, generated in $L_2[0,\pi]$ by expression $-y'' + q(x)y$ and the boundary conditions (2) with potential (5), are real, for all $t \in (-1,1]$, and the spectrum $\sigma(H(r_m))$ of the Hill operator $H(r_m)$, generated in $L_2(-\infty,\infty)$ by expression (1) with potential (5), consists of the real intervals

$$\begin{aligned} \Gamma_1 \quad &: \quad = [\lambda_0(r_m), \mu_{-1}(r_m)], \quad \Gamma_2 := [\mu_{+1}(r_m), \lambda_{-1}(r_m)], \\ \Gamma_3 \quad &: \quad = [\lambda_{+1}(r_m), \mu_{-2}(r_m)], \quad \Gamma_4 := [\mu_{+2}(r_m), \lambda_{-2}(r_m)], \ldots, \end{aligned}$$

where $\lambda_0(r_m)$, $\lambda_{-n}(r_m)$, $\lambda_{+n}(r_m)$, for $n = 1, 2, \ldots$ are the eigenvalues of $H_0(r_m)$ and $\mu_{-n}(r_m)$, $\mu_{+n}(r_m)$, for $n = 1, 2 \ldots$ are the eigenvalues of $H_1(r_m)$ and the following inequalities hold:

$$\begin{aligned} &\lambda_0(r_m) < \mu_{-1}(r_m) \leq \mu_{+1}(r_m) < \lambda_{-1}(r_m) \leq \lambda_{+1}(r_m) < \mu_{-2}(r_m) \\ &\qquad \leq \mu_{+2}(r_m) < \lambda_{-2}(r_m) \leq \lambda_{+2}(r_m) < \cdots . \end{aligned}$$

The bands $\Gamma_1, \Gamma_2, \ldots$ of the spectrum $\sigma(H(r_m))$ of $H(r_m)$ are separated by the gaps

$$\Delta_1 := (\mu_{-1}(r_m), \mu_{+1}(r_m)), \quad \Delta_2 := (\lambda_{-1}(r_m), \lambda_{+1}(r_m)),$$
$$\Delta_3 := (\mu_{-2}(r_m), \mu_{+2}(r_m)), \ldots$$

if and only if the eigenvalues at the endpoints of the intervals are simple. In other notation, $\Gamma_n = \{\gamma_n(t) : t \in [0,1]\}$, where $\gamma_1(t), \gamma_2(t), \ldots$ are the eigenvalues of $H_t(r_m)$, called as Bloch eigenvalues corresponding to the quasimomentum t. The Bloch eigenvalue $\gamma_n(t)$, continuously depends on t and $\gamma_n(-t) = \gamma_n(t)$. These statements continue to hold for $S_t(q)$ and $S(q)$ if $q_{-m}q_m > 0$.

Obviously, $\lambda_{-n}(r_m)$ and $\lambda_{+n}(r_m)$, for $n = 1, 2, \ldots$ are the $(2n)$th and $(2n+1)$th periodic eigenvalues; $\mu_{-n}(r_m)$ and $\mu_{+n}(r_m)$, for $n = 1, 2, \ldots$ are the $(2n-1)$th and $(2n)$th antiperiodic eigenvalues, respectively.
If one of the numbers q_{-m} and q_m is zero and the other is real in (3), then all eigenvalues of the operator $S_0(q)$, except 0, are double and they are equal to $(2n)^2$. This fact was proved for the first time in [6, M.G. Gasymov, 1980]. This case was investigated also in [7, N.B. Kerimov, 2013], [15, C. Nur, 2021], [18, O.A. Veliev, 2015]. In [15, C. Nur, 2021], we investigated the operators $S_t(q)$, for $t = 0, 1$, with potential (3), when the periodic and antiperiodic eigenvalues are real.

In this work, we give estimates for the eigenvalues of $S_0(q)$ and $S_1(q)$, when $(q_{-m}q_m) \in \mathbb{R}$. We even approximate complex eigenvalues by the roots of some polynomials derived from some iteration formulas. Finally, we give numerical examples with error analysis using Rouche's theorem.

It is well known that [16, J. Poschel and E. Trubowitz, 1987], [21, 22, O. A. Veliev, 2020, 2021]

$$|\lambda_{\pm n}(q) - \lambda_{\pm n}(0)| \leq \sup_{x\in[0,\pi]} |p(x)| = 2|r_m|,$$
$$|\mu_{\pm n}(q) - \mu_{\pm n}(0)| \leq \sup_{x\in[0,\pi]} |p(x)| = 2|r_m|,$$

for $n = 1, , 2 \ldots$, where $\lambda_{\pm n}(0) = (2n)^2$, $\mu_{\pm n}(0) = (2n-1)^2$ and $r_m = \sqrt{q_{-m}q_m}$. Moreover, for $n = 0$, $|\lambda_0(q)| \leq 2|r_m|$ holds. Therefore, we have

$$(2n)^2 - 2|r_m| \leq |\lambda_n| \leq (2n)^2 + 2|r_m|$$

and

$$|\lambda_n - (2k)^2| \geq |(2n)^2 - (2k)^2| - 2|r_m| = 4|n-k||n+k| - 2|r_m| \\ \geq 4|2n-1| - 2|r_m|,$$

for $n \in \mathbb{Z}$ and $k \neq \pm n$.

In particular, if $n = 1$, we have $|\lambda_{\pm 1}| \leq 4 + 2|r_m|$ and

$$|\lambda_{\pm 1} - (2k)^2| \geq ||\lambda_{\pm 1}| - (2k)^2| \geq 16 - |\lambda_{\pm 1}| \geq 12 - 2|r_m|,$$

for $k \geq 2$. Besides, if $|n| \geq 2$, we have $|\lambda_n| \geq |\lambda_{-2}| \geq 16 - 2|r_m|$ and

$$|\lambda_n - (2k)^2| \geq ||\lambda_{-2}| - (2k)^2| \geq |\lambda_{-2}| - 4 \geq 12 - 2|r_m|,$$

for $k \neq \pm n$. The analogous inequalities can be written for the antiperiodic eigenvalues from

$$(2n-1)^2 - 2|r_m| \leq |\mu_{\pm n}| \leq (2n-1)^2 + 2|r_m|, \tag{6}$$

for $n = 1, 2, \ldots$.

First, we consider the operator $S_0(q)$ which is associated with the periodic boundary conditions. From now on, when we use the notation λ_n, we mean the $(2n)$th and $(2n+1)$th periodic eigenvalues λ_{-n} and λ_{+n}, for $n = 1, 2, \ldots$. We begin with the equations

$$(\lambda_N - (2n)^2)(\Psi_N, e^{i2nx}) = (q\Psi_N, e^{i2nx}), \tag{7}$$
$$(\lambda_N - (2n)^2)(\Psi_N, e^{-i2nx}) = (q\Psi_N, e^{-i2nx}) \tag{8}$$

which are obtained from

$$-\Psi_N''(x) + q(x)\Psi_N(x) = \lambda_N \Psi_N(x),$$

by multiplying both sides of the equality by e^{i2nx} and e^{-i2nx}, respectively, where $\Psi_N(x)$ is the eigenfunction corresponding to the eigenvalue λ_N.

Iterating equation (7) k times for $N = n$, the way it was done in the paper [4, N. Dernek and O.A. Veliev , 2005], we obtain

$$\left(\lambda_n - (2n)^2 - \sum_{j=1}^{k} \alpha_j(\lambda_n)\right)(\Psi_n, e^{i2nx}) -$$
$$\left(q_{2n} + \sum_{j=1}^{k} \beta_j(\lambda_n)\right)(\Psi_n, e^{-i2nx}) = \rho_k(\lambda_n), \quad (9)$$

where

$$\alpha_j(\lambda_n) = \sum_{n_1, n_2, \dots, n_j} \frac{q_{n_1} q_{n_2} \cdots q_{n_j} q_{-n_1-n_2-\cdots-n_j}}{[\lambda_n - (2(n-n_1))^2] \cdots [\lambda_n - (2(n-n_1-\cdots-n_j))^2]},$$

$$\beta_j(\lambda_n) = \sum_{n_1, n_2, \dots, n_j} \frac{q_{n_1} q_{n_2} \cdots q_{n_j} q_{2n-n_1-n_2-\cdots-n_j}}{[\lambda_n - (2(n-n_1))^2] \cdots [\lambda_n - (2(n-n_1-\cdots-n_j))^2]},$$

$$\rho_k(\lambda_n) = \sum_{n_1, n_2, \dots, n_{k+1}} \frac{q_{n_1} q_{n_2} \cdots q_{n_k} q_{n_{k+1}} (q\Psi_n, e^{i2(n-n_1-\cdots-n_{k+1})x})}{[\lambda_n - (2(n-n_1))^2] \cdots [\lambda_n - (2(n-n_1-\cdots-n_{k+1}))^2}$$

Here, the sums are taken under the conditions $n_l = \pm m$, $\sum_{i=1}^{l} n_i \neq 0, 2n$ for $l = 1, 2, ..., k + 1$. Note that, for the trigonometric polynomial potential of the form (3), we have $q_i = 0$ for $i \neq \pm m$.
Similarly, iterating equation (8) k times for $N = n$, we obtain

$$\Big(\lambda_n - (2n)^2 - \sum_{j=1}^{k} \alpha_j^*(\lambda_n)\Big)(\Psi_n, e^{-i2nx}) -$$

$$\Big(q_{-2n} + \sum_{j=1}^{k} \beta_j^*(\lambda_n)\Big)(\Psi_n, e^{i2nx}) = \rho_k^*(\lambda_n), \qquad (10)$$

where

$$\alpha_j^*(\lambda_n) = \sum_{n_1, n_2, \ldots, n_j} \frac{q_{n_1} q_{n_2} \cdots q_{n_j} q_{-n_1-n_2-\cdots-n_j}}{[\lambda_n - (2(n+n_1))^2] \cdots [\lambda_n - (2(n+n_1+\cdots+n_j))^2]},$$

$$\beta_j^*(\lambda_n) = \sum_{n_1, n_2, \ldots, n_j} \frac{q_{n_1} q_{n_2} \cdots q_{n_j} q_{-2n-n_1-n_2-\cdots-n_j}}{[\lambda_n - (2(n+n_1))^2] \cdots [\lambda_n - (2(n+n_1+\cdots+n_j))^2]},$$

$$\rho_k^*(\lambda_n) = \sum_{n_1, n_2, \ldots, n_{k+1}} \frac{q_{n_1} q_{n_2} \cdots q_{n_k} q_{n_{k+1}} (q\Psi_n, e^{-i2(n+n_1+\cdots+n_{k+1})x})}{[\lambda_n - (2(n+n_1))^2] \cdots [\lambda_n - (2(n+n_1+\cdots+n_{k+1}))^2]}$$

Here, the sums are taken under the conditions $n_l = \pm m$, $\sum_{i=1}^{l} n_i \neq 0, -2n$ for $l = 1, 2, ..., k+1$.

Since the potential q is the trigonometric polynomial potential of the form (3), we have the followings, after some calculations (see [15, C. Nur, 2021]):

$$\begin{aligned} \alpha^*_{2j-1}(\lambda_n) &= \alpha_{2j-1}(\lambda_n), \qquad \alpha^*_{2j}(\lambda_n) = \alpha_{2j}(\lambda_n) = 0, \\ \beta^*_j(\lambda_n) &= \left(\frac{q_{-m}}{q_m}\right)^{2n/m} \beta_j(\lambda_n), \end{aligned} \tag{11}$$

for $j = 1, 2, \ldots$.

In order to give the main results, we need the following lemma. Without loss of generality, we assume that $\Psi_n(x)$ is the normalized eigenfunction corresponding to the eigenvalue λ_n.

Lemma

The statements

(a) $\lim_{k\to\infty} \rho_k(\lambda_n) = 0, \qquad \lim_{k\to\infty} \rho_k^*(\lambda_n) = 0,$

(b) $|u_n|^2 + |v_n|^2 > 0$, *where* $u_n = (\Psi_n, e^{i2nx})$ *and* $v_n = (\Psi_n, e^{-i2nx})$,

are valid in the following cases:

(i) *if* $|r_1| = \left|\sqrt{1-4V^2}\right| < 3$, *for* $n \geq 1$ *and* $m = 1$,

(ii) *if* $|q_{-2}| + |q_2| \leq 29/10$, *for* $n = 1$ *and* $m = 2$,

(iii) *if* $|q_{-m}| + |q_m| \leq 7/2$, *for* $n = 1$ *and* $m \geq 3$,

(iv) *if* $|r_m| < 2s - 1$, *for* $n \geq s$, $s = 2, 3, \ldots$ *and* $m \geq 2$, *where* $r_m = \sqrt{q_{-m}q_m}$.

Now, we consider the statements of Lemma 1 for the case $n = 0$:

Lemma

The statements **(a)** $\lim_{k\to\infty} \rho_k(\lambda_0) = 0$ *and* **(b)** $|(\Psi_0, 1)| > 0$ *hold in the following cases:*
(i) *if* $|q_{-2}| + |q_2| \leq 2$, *for* $m = 2$,
(ii) *if* $|q_{-m}| + |q_m| \leq 3$, *for* $m \geq 3$.

Letting k tend to infinity in the equations (9) and (10), we obtain the following results. First, we consider the case $n \geq 2$ for $m = 1$.

Theorem

If $|r_1| = \left|\sqrt{1-4V^2}\right| < 3$ *and* $n \geq 2$, *then* λ *is an eigenvalue of* $S_0(q)$ *if and only if it is either the root of the equation*

$$\lambda - (2n)^2 - \sum_{k=1}^{\infty} \alpha_{2k-1}(\lambda) - \left(\frac{q_{-1}}{q_1}\right)^n \sum_{k=2}^{\infty} \beta_{2k-1}(\lambda) = 0 \qquad (12)$$

or the root of

$$\lambda - (2n)^2 - \sum_{k=1}^{\infty} \alpha_{2k-1}(\lambda) + \left(\frac{q_{-1}}{q_1}\right)^n \sum_{k=2}^{\infty} \beta_{2k-1}(\lambda) = 0 \qquad (13)$$

lying inside the circle $C_n := \{\lambda \in \mathbb{C} : |\lambda - (2n)^2| = 2|r_1|\}$ *and each of the series in these equations converges uniformly to an analytic function on the disk* $D_n := \{\lambda \in \mathbb{C} : |\lambda - (2n)^2| \leq 2|r_1|\}$. *Moreover, the roots of* (12) *and* (13) *lying in* D_n, *coincide with the* $(2n)$*th and* $(2n+1)$*th periodic eigenvalues* λ_{-n} *and* λ_{+n} *of* S_0.

Now we consider the case $n \geq 2$ for $m \geq 2$.

Theorem

Suppose that $|r_m| < 2s-1$, *for* $n \geq s$, $s = 2, 3, \ldots$ *and* $m \geq 2$, *where* $r_m = \sqrt{q_{-m}q_m}$.

(a) *If* m *is even and* $n = m/2$, *then* λ *is an eigenvalue of* $S_0(q)$ *if and only if it is either the root of the equation*

$$\lambda - (2n)^2 - r_m - \sum_{j=1}^{\infty} \alpha_{2j-1}(\lambda) = 0 \tag{14}$$

or the root of

$$\lambda - (2n)^2 + r_m - \sum_{j=1}^{\infty} \alpha_{2j-1}(\lambda) = 0, \tag{15}$$

(b) *If* $n = m$, *then* λ *is an eigenvalue of* $S_0(q)$ *if and only if it is either the root of*

$$\lambda - (2n)^2 - \frac{2r_m^2}{\lambda} - \frac{{r_m}^2}{\lambda - 16n^2} - \sum_{j=2}^{\infty} \alpha_{2j-1}(\lambda) = 0 \tag{16}$$

or the root of

$$\lambda - (2n)^2 - \frac{{r_m}^2}{\lambda - 16n^2} - \sum_{j=2}^{\infty} \alpha_{2j-1}(\lambda) = 0, \tag{17}$$

(c) If $n \neq m$ and $n \neq m/2$, then λ is an eigenvalue of $S_0(q)$ if and only if it is either the root of

$$\lambda - (2n)^2 - \sum_{j=1}^{\infty} \alpha_{2j-1}(\lambda) - \left(\frac{q_{-m}}{q_m}\right)^{n/m} \sum_{j=1}^{\infty} \beta_j(\lambda) = 0 \tag{18}$$

or the root of

$$\lambda - (2n)^2 - \sum_{j=1}^{\infty} \alpha_{2j-1}(\lambda) + \left(\frac{q_{-m}}{q_m}\right)^{n/m} \sum_{j=1}^{\infty} \beta_j(\lambda) = 0 \tag{19}$$

lying inside the circle $C_n := \{\lambda \in \mathbb{C} : |\lambda - (2n)^2| = 2|r_m|\}$ and each of the series in these equations converges uniformly to an analytic function on the disk $D_n := \{\lambda \in \mathbb{C} : |\lambda - (2n)^2| \leq 2|r_m|\}$.

Now, for the case $n = 1$ and $m = 1$ we have:

Theorem

If $|r_1| = \left|\sqrt{1-4V^2}\right| < 3$, *then:*

(a) *the first periodic eigenvalues* λ_0 *and* λ_{-1} *are the roots of the equation*

$$\lambda^2 - 4\lambda - 2r_1^2 - \frac{r_1^2\lambda}{\lambda-16} - \sum_{k=2}^{\infty} \lambda\alpha_{2k-1}(\lambda) = 0 \tag{20}$$

lying in the disk $D_1 := \{\lambda \in \mathbb{C} : |\lambda| \leq 2|r_1| + 4\}$ *and the series* $\sum_{k=2}^{\infty} \alpha_{2k-1}(\lambda)$ *converges uniformly to an analytic function on the disk* D_1. *Moreover,* (20) *has exactly two roots (counting with multiplicities) inside the circle* $C_1 := \{\lambda \in \mathbb{C} : |\lambda| = 2|r_1| + 4\}$ *and these roots coincide with the first two eigenvalues* λ_0 *and* λ_{-1} *of* S_0.

(b) *the third periodic eigenvalue* λ_{+1} *is the root of*

$$\lambda - 4 - \frac{r_1^2}{\lambda-16} - \sum_{k=2}^{\infty} \alpha_{2k-1}(\lambda) = 0 \tag{21}$$

lying in the disk D_1.

Moreover, (21) has exactly one root (counting with multiplicity) inside the circle C_1 and this root coincide with the third eigenvalue λ_{+1} of S_0.

Here we note that, the proof of (a) was given by Veliev [20, O.A. Veliev, 2018] for $|r_1| < 2$. Besides, he gave the spectral analysis of the operators $S_t(q)$, for $t = 0, 1$, and $S(q)$. We have derived the same equation by another method of him. In our work, we have proved that the statements in (a) are still true for $|r_1| < 3$.

Now, we have the following results for the case $n = 1$ and $m \geq 2$ to estimate the periodic eigenvalues λ_{-1} and λ_1.

Theorem

(a) *If* $|q_{-2}|+|q_2|\leq 29/10$, *for* $n=1$ *and* $m=2$, *then* λ *is an eigenvalue of* $S_0(q)$ *if and only if it is either the root of the equation*

$$\lambda-4-r_2-\sum_{j=1}^{\infty}\alpha_{2j-1}(\lambda)=0 \tag{22}$$

or the root of

$$\lambda-4+r_2-\sum_{j=1}^{\infty}\alpha_{2j-1}(\lambda)=0, \tag{23}$$

(b) *If* $|q_{-m}|+|q_m|\leq 7/2$, *for* $n=1$ *and* $m\geq 3$, *then* λ *is a double eigenvalue of* $S_0(q)$ *if and only if it is the double root of the equation*

$$\left(\lambda-4-\sum_{j=1}^{\infty}\alpha_{2j-1}(\lambda)\right)^2=0 \tag{24}$$

lying inside the circle $C_1:=\{\lambda\in\mathbb{C}:|\lambda|\leq 2|r_m|+4\}$ *and the series* $\sum_{j=1}^{\infty}\alpha_{2j-1}(\lambda)$ *converges uniformly to an analytic function on the disk*

$D_1 := \{\lambda \in \mathbb{C} : |\lambda| \leq 2|r_m| + 4\}$ in each case.

Finally, in order to estimate the first periodic eigenvalue λ_0 for $m \geq 2$, we consider the case $n = 0$ and $m \geq 2$. By Lemma 2, we have:

Theorem

(a) *If* $|q_{-2}| + |q_2| \leq 2$, *for* $n = 0$ *and* $m = 2$,
(b) *If* $|q_{-m}| + |q_m| \leq 3$, *for* $n = 0$ *and* $m \geq 3$,
then λ *is an eigenvalue of* $S_0(q)$ *if and only if it is the root of the equation*

$$\lambda - \sum_{j=1}^{\infty} \alpha_{2j-1}(\lambda) = 0 \tag{25}$$

lying inside the circle $C_0 := \{\lambda \in \mathbb{C} : |\lambda| \leq 2|r_m|\}$ *and the series* $\sum_{j=1}^{\infty} \alpha_{2j-1}(\lambda)$ *converges uniformly to an analytic function on the disk* $D_0 := \{\lambda \in \mathbb{C} : |\lambda| \leq 2|r_m|\}$.

In order to estimate eigenvalues numerically, we take finite summations instead of the infinite series in the equations (12)-(25). When we say the $(2k-1)$th approximations, we mean the equations containing $\sum_{j=1}^{k} \alpha_{2j-1}(\lambda)$ and $\sum_{j=1}^{k} \beta_{2j-1}(\lambda)$ instead of $\sum_{j=1}^{\infty} \alpha_{2j-1}(\lambda)$ and $\sum_{j=1}^{\infty} \beta_{2j-1}(\lambda)$. For instance, in the case $m = 2$, the $(2k-1)$th approximations of the above equations are

$$\lambda - \sum_{j=1}^{k} \alpha_{2j-1}(\lambda) = 0, \tag{26}$$

for $n = 0$;

$$\lambda - 4 - r_2 - \sum_{j=1}^{k} \alpha_{2j-1}(\lambda) = 0, \tag{27}$$

$$\lambda - 4 + r_2 - \sum_{j=1}^{k} \alpha_{2j-1}(\lambda) = 0, \tag{28}$$

for $n = 1$; and

$$\lambda - 16 - \frac{2{r_2}^2}{\lambda} - \frac{{r_2}^2}{\lambda - 64} - \sum_{j=2}^{k} \alpha_{2j-1}(\lambda) = 0, \tag{29}$$

$$\lambda - 16 - \frac{{r_2}^2}{\lambda - 64} - \sum_{j=2}^{k} \alpha_{2j-1}(\lambda) = 0, \tag{30}$$

for $n = 2$.

In particular, we obtain practical equations to calculate the small eigenvalues for the case $m = 1$, namely for the optical potential. If we consider the $(2k-1)$th approximation

$$\lambda^2 - 4\lambda - 2r_1^2 - \frac{r_1^2\lambda}{\lambda - 16} - \sum_{j=2}^{k} \lambda\alpha_{2j-1}(\lambda) = 0 \tag{31}$$

for the first periodic eigenvalues λ_0 and λ_{-1}, the $(2k-1)$th approximation

$$\lambda - 4 - \frac{r_1^2}{\lambda - 16} - \sum_{j=2}^{k} \alpha_{2j-1}(\lambda) = 0 \tag{32}$$

for the third periodic eigenvalue λ_{+1}, and the $(2k-1)$th approximation

$$\lambda - (2n)^2 - \sum_{j=1}^{k} \alpha_{2j-1}(\lambda) \pm \left(\frac{q_{-1}}{q_1}\right)^n \sum_{j=2}^{k} \beta_{2j-1}(\lambda) = 0 \tag{33}$$

for the other eigenvalues λ_{-n} and λ_{+n} of L_0, then we have the following estimates for the remaining terms:

$$\Big|\sum_{j=k+1}^{\infty} A_{2j-1}(\lambda_1)\Big| < 312\Big(\frac{9}{104}\Big)^k$$

and

$$\Big|\sum_{j=k+1}^{\infty} A_{2j-1}(\lambda_n) \pm \Big(\frac{q_{-1}}{q_1}\Big)^n \sum_{j=k+1}^{\infty} B_{2j-1}(\lambda_n)\Big| < \frac{45}{14}\Big(\frac{3}{10}\Big)^k,$$

for $|r_1| < 3$ and $n \geq 1$. Obviously, we will have better approximations as k grows. Besides, for a fixed k, this method gives better approximations as n grows.

Now, we approach the eigenvalues by the roots of the polynomials derived from the $(2k-1)$th approximations (31), (32), and (33), the way it was done by Veliev in [20, O.A. Veliev, 2018]. For example, for $m=1$, $n=1$ and $k=3$, we have the fifth approximations

$$Q_1(\lambda) := \lambda^2 - 4\lambda - 2c^2 - \frac{c^2\lambda}{\lambda-16} - \frac{c^4\lambda}{(\lambda-16)^2(\lambda-36)} - \frac{c^6\lambda}{(\lambda-16)^2(\lambda-36)^2(\lambda-64)} - \frac{c^6\lambda}{(\lambda-16)^3(\lambda-36)^2} = 0,$$

and

$$Q_{-1}(\lambda) := \lambda - 4 - \frac{c^2}{\lambda-16} - \frac{c^4}{(\lambda-16)^2(\lambda-36)} - \frac{c^6}{(\lambda-16)^2(\lambda-36)^2(\lambda-64)} - \frac{c^6}{(\lambda-16)^3(\lambda-36)^2} = 0.$$

Then,

$$P_1(\lambda) := (\lambda - 16)^3(\lambda - 36)^2(\lambda - 64)Q_1(\lambda)$$

and

$$P_{-1}(\lambda) := (\lambda - 16)^3(\lambda - 36)^2(\lambda - 64)Q_{-1}(\lambda) \tag{34}$$

are polynomials of degree 8 and 7, respectively. By the same token, we can derive polynomials to approximate the periodic eigenvalues, for $n \geq 2$.

Similarly, for $n = 0$ and $m = 2$, the fifth approximation is

$$K_0(\lambda) := \lambda - \frac{2{r_2}^2}{\lambda - 16} - \frac{2{r_2}^4}{(\lambda - 16)^2(\lambda - 64)} - \frac{2{r_2}^6}{(\lambda - 16)^2(\lambda - 64)^2(\lambda - 144)} - \frac{2{r_2}^6}{(\lambda - 16)^3(\lambda - 64)^2} = 0,$$

for $n = 1$ and $m = 2$, the fifth approximations are

$$K_{-1}(\lambda) := \lambda - 4 + r_2 - \frac{{r_2}^2}{\lambda - 36} - \frac{{r_2}^4}{(\lambda - 36)^2(\lambda - 100)} - \frac{{r_2}^6}{(\lambda - 36)^2(\lambda - 100)^2(\lambda - 196)} - \frac{{r_2}^6}{(\lambda - 36)^3(\lambda - 100)^2} = 0,$$

and

$$K_1(\lambda) := \lambda - 4 - r_2 - \frac{{r_2}^2}{\lambda - 36} - \frac{{r_2}^4}{(\lambda - 36)^2(\lambda - 100)} - \frac{{r_2}^6}{(\lambda - 36)^2(\lambda - 100)^2(\lambda - 196)} - \frac{{r_2}^6}{(\lambda - 36)^3(\lambda - 100)^2} = 0,$$

for $n = 2$ and $m = 2$, the fifth approximations are

$$K_{-2}(\lambda) := \lambda - 16 - \frac{2{r_2}^2}{\lambda} - \frac{{r_2}^2}{\lambda - 64} - \frac{{r_2}^4}{(\lambda - 64)^2(\lambda - 144)} - \frac{{r_2}^6}{(\lambda - 64)^2(\lambda - 144)^2(\lambda - 256)} - \frac{{r_2}^6}{(\lambda - 64)^3(\lambda - 144)^2} = 0,$$

and

$$K_{2}(\lambda) := \lambda - 16 - \frac{{r_2}^2}{\lambda - 64} - \frac{{r_2}^4}{(\lambda - 64)^2(\lambda - 144)} - \frac{{r_2}^6}{(\lambda - 64)^2(\lambda - 144)^2(\lambda - 256)} - \frac{{r_2}^6}{(\lambda - 64)^3(\lambda - 144)^2} = 0.$$

Then, the corresponding polynomials are

$$R_0(\lambda) := (\lambda - 16)^3(\lambda - 64)^2(\lambda - 144)K_0(\lambda), \tag{35}$$

$$R_{-1}(\lambda) := (\lambda - 36)^3(\lambda - 100)^2(\lambda - 196)K_{-1}(\lambda), \tag{36}$$

$$R_1(\lambda) := (\lambda - 36)^3(\lambda - 100)^2(\lambda - 196)K_1(\lambda), \tag{37}$$

$$R_{-2}(\lambda) := (\lambda - 64)^3(\lambda - 144)^2(\lambda - 256)K_{-2}(\lambda) \tag{38}$$

and

$$R_2(\lambda) := (\lambda - 64)^3(\lambda - 144)^2(\lambda - 256)K_2(\lambda), \tag{39}$$

respectively. By the same token, we can derive polynomials to approximate other periodic eigenvalues, as well.

Now, we state the analogous theorems to Theorem 3 and Theorem 5 for the operator $S_1(q)$ associated with the antiperiodic boundary conditions.

Theorem

(a) *If* $|r_1| = \left|\sqrt{1-4V^2}\right| < 3$ *and* $n \geq 3$, *then* μ *is an eigenvalue of* S_1 *if and only if it is either the root of the equation*

$$\mu - (2n-1)^2 - \sum_{j=1}^{\infty} a_{2j-1}(\mu) - \left(\frac{q_{-1}}{q_1}\right)^{n-1/2} \sum_{j=2}^{\infty} b_{2j}(\mu) = 0 \tag{40}$$

or the root of

$$\mu - (2n-1)^2 - \sum_{j=1}^{\infty} a_{2j-1}(\mu) + \left(\frac{q_{-1}}{q_1}\right)^{n-1/2} \sum_{j=2}^{\infty} b_{2j}(\mu) = 0 \tag{41}$$

lying inside the circle $c_n := \{\mu \in \mathbb{C} : |\mu - (2n-1)^2| = 2|r_1|\}$, where

$$a_j(\mu) = \sum_{n_1, n_2, \ldots, n_j} \frac{q_{n_1} q_{n_2} \cdots q_{n_j} q_{-n_1-n_2-\cdots-n_j}}{[\mu - (2(n-n_1)-1)^2] \cdots [\mu - (2(n-n_1-\cdots-n_j)-1)^2}$$

$$b_j(\mu) = \sum_{n_1, n_2, \ldots, n_j} \frac{q_{n_1} q_{n_2} \cdots q_{n_j} q_{2n-1-n_1-n_2-\cdots-n_j}}{[\mu - (2(n-n_1)-1)^2] \cdots [\mu - (2(n-n_1-\cdots-n_j)-1)^2}$$

and each of the series in these equations converges uniformly to an analytic function on the disk $d_n := \{\mu \in \mathbb{C} : |\mu - (2n-1)^2| \leq 2|r_1|\}$. Moreover, the roots of (40) and (41) lying in d_n, coincide with the eigenvalues μ_{-n} and μ_{+n} of S_1.

(b) In the case $n = 2$, the statements in (a) are valid for $|r_1| < 2$.

Now, for $n = 1$, we have:

Theorem

If $|r_1| < 2$, then μ is an eigenvalue of S_1 if and only if it is either the root of the equation

$$\mu - 1 - a - \sum_{j=1}^{\infty} a_{2j-1}(\mu) = 0 \tag{42}$$

or the root of

$$\mu - 1 + a - \sum_{j=1}^{\infty} a_{2j-1}(\mu) = 0 \tag{43}$$

lying inside the circle $c_1 := \{\mu \in \mathbb{C} : |\mu| = 2|r_1| + 1\}$ and each of the series in these equations converges uniformly to an analytic function on the disk $d_1 := \{\mu \in \mathbb{C} : |\mu| \leq 2|r_1| + 1\}$. Moreover, the roots of (42) and (43) lying in d_1, coincide with the first antiperiodic eigenvalues μ_{-1} and μ_{+1}.

Now, let us approach the antiperiodic eigenvalues by the polynomials derived from the $(2k-1)$th approximations of (40)-(43). For $n=1$, $k=3$, and $j=1,2$, we have

$$H_j(\mu) := \mu - 1 + (-1)^j a - \frac{c^2}{\mu-9} - \frac{c^4}{(\mu-9)(\mu-25)} - \frac{c^6}{(\mu-9)^2(\mu-25)^2(\mu-49)} - \frac{c^6}{(\mu-9)^3(\mu-25)^2} = 0.$$

Then,

$$S_j(\mu) := (\mu-9)^3(\mu-25)^2(\mu-49)H_j(\mu) \tag{44}$$

is a polynomial of degree 7. By the same token, we can derive polynomials to approximate the antiperiodic eigenvalues, for $n \geq 2$.
Now, we present a numerical example for the case $m=1$:

Example

For $k = 3$ and $r_1^2 = 1 - 4V^2 = -2.157281295$, Veliev [20, O.A. Veliev, 2018] approximated the first periodic eigenvalues λ_0 and λ_{-1} for the optical potential $q(x) = (1 + 2V)e^{i2x} + (1 - 2V)e^{-i2x}$. Now, we have the following approximations for the third periodic eigenvalue λ_{+1} and the first antiperiodic eigenvalues μ_{-1} and μ_{+1}:
First, we show that λ_{+1} is the real eigenvalue lying inside the circle

$$C = \{\lambda \in \mathbb{C} : |\lambda - 4.1814942277| = 1.7 \times 10^{-6}\}.$$

The root of the polynomial $P_{-1}(\lambda)$ defined by (34), lying in the disk $D_1 = \{\lambda \in \mathbb{C} : |\lambda| \leq 2|r_1| + 4\}$, is $p_1 = 4.1814942277$. The other roots of $P_{-1}(\lambda)$ are $p_2 = 15.8535021182$,
$p_3 = (15.9823184944 - 0.119095369803i)$,
$p_4 = (15.9823184944 + 0.119095369803i)$,
$p_5 = (36.000183379 - 0.00333664975667i)$,
$p_6 = (36.000183379 + 0.00333664975667i)$ and $p_7 = 63.9999999074$.
Using the decomposition

$$Q_{-1}(\lambda) = \frac{(\lambda - p_1)(\lambda - p_2)\cdots(\lambda - p_7)}{(\lambda - 16)^3(\lambda - 36)^2(\lambda - 64)},$$

we obtain by direct calculation $|Q_{-1}(\lambda)| > 1.8496 \times 10^{-7}$, for all $\lambda \in C$. On the other hand, again by direct calculations, we have $\sum_{k=4}^{\infty} |\alpha_{2k-1}(\lambda)| < 1.8269 \times 10^{-7}$, for all $\lambda \in C$. Therefore, by Rouche's theorem, equation (21) has only one root inside the circle C. Thus, using Theorem 5 (b) and the spectral analysis of S_0 given by Veliev [20, O.A. Veliev, 2018], we conclude that λ_{+1} is the real eigenvalue lying inside the circle C.

Now, we show that μ_{-1} and μ_{+1} are the complex eigenvalues lying inside the circles

$$\delta_1 = \{\mu \in \mathbb{C} : |\mu - (1.26575008922 - 1.52020432568i)| = 1.4 \times 10^{-5}\}$$

and

$$\delta_2 = \{\mu \in \mathbb{C} : |\mu - (1.26575008922 + 1.52020432568i)| = 1.4 \times 10^{-5}\},$$

respectively.

The roots of the polynomials $S_1(\mu)$ and $S_2(\mu)$ defined by (44), lying in the disk $d_1 = \{\mu \in \mathbb{C} : |\mu| \leq 2|c| + 1\}$ are
$x_1 = (1.26575008922 + 1.52020432568i)$ and
$y_1 = (1.26575008922 - 1.52020432568i)$, respectively. The other roots of $S_1(\mu)$ are $x_2 = (8.96777697119 + 0.142338162679i)$,
$x_3 = (8.79563202223 - 0.0317230792875i)$,
$x_4 = (8.97007606112 - 0.162097407292i)$,
$x_5 = (25.0005579806 - 0.00577397577187i)$,
$x_6 = (25.0002071021 + 0.00582061314113i)$ and
$x_7 = (48.9999997735 - 0.000000000692262634543i)$ and the other roots of $S_2(\mu)$ are $y_2 = (8.96777697119 - 0.142338162679i)$,
$y_3 = (8.79563202223 + 0.0317230792875i)$,
$y_4 = (8.97007606112 + 0.162097407292i)$,
$y_5 = (25.0005579806 + 0.00577397577187i)$,
$y_6 = (25.0002071021 - 0.00582061314113i)$ and
$y_7 = (48.9999997735 + 0.000000000692262634543i)$.

Using the decompositions

$$H_1(\mu) = \frac{(\mu - x_1)(\mu - x_2)\cdots(\mu - x_7)}{(\mu - 9)^3(\mu - 25)^2(\mu - 49)}$$

and

$$H_2(\mu) = \frac{(\mu - y_1)(\mu - y_2)\cdots(\mu - y_7)}{(\mu - 9)^3(\mu - 25)^2(\mu - 49)},$$

by direct calculations, we obtain $|H_1(\mu)| > 4.6113 \times 10^{-6}$, for all $\mu \in \delta_2$ and $|H_2(\mu)| > 4.6113 \times 10^{-6}$, for all $\mu \in \delta_1$. On the other hand, one can easily calculate that $\sum_{k=4}^{\infty} |a_{2k-1}(\lambda)| < 4.4786 \times 10^{-6}$, for all $\mu \in \delta_1 \cup \delta_2$. The proof follows from Rouche's theorem and Theorem 9; each of the equations (37) and (38) has only one root inside the circle δ_2 and δ_1, respectively and μ_{-1} and μ_{+1} are the complex eigenvalues lying inside δ_1 and δ_2, respectively.

Now, we present another numerical example.

Example

Consider the potential $q(x) = e^{i4x} - e^{-i4x} = 2i\sin(4x)$ or $p(x) = ie^{i4x} + ie^{-i4x} = 2i\cos(4x)$. In this case, $m = 2$, $r_2 = \sqrt{-1} = i$ and we have the following approximations for the first periodic eigenvalues λ_0, λ_{-1}, λ_{+1}, λ_{-2} and λ_2:
First, we show that λ_0 is the eigenvalue lying inside the circle

$$c_0 := \{\lambda \in \mathbb{C} : |\lambda - 0.125867010858| = 4.8 \times 10^{-10}\}.$$

The root of the polynomial $R_0(\lambda)$ defined by (35), lying in the disk $D_0 = \{\lambda \in \mathbb{C} : |\lambda| \leq 2|r_2|\}$, is $a_1 = 0.125867010858$. The other roots of $R_0(\lambda)$ are $a_2 = 15.8939999572$,
$a_3 = (15.9900597315 - 0.0204223963085i)$,
$a_4 = (15.9900597315 + 0.0204223963085i)$,
$a_5 = (64.0000067845 - 0.000336043226373i)$,
$a_6 = (64.0000067845 + 0.000336043226373i)$ and $a_7 = 144.0$. Using the decomposition

$$K_0(\lambda) = \frac{(\lambda - a_1)(\lambda - a_2)\cdots(\lambda - a_7)}{(\lambda - 16)^3(\lambda - 64)^2(\lambda - 144)},$$

we obtain by direct calculation $|K_0(\lambda)| > 4.4990 \times 10^{-10}$, for all $\lambda \in c_0$. On the other hand, again by direct calculations, we have $\sum_{j=4}^{\infty} |\alpha_{2j-1}(\lambda)| < 2.6416 \times 10^{-10}$, for all $\lambda \in c_0$. Therefore, by Rouche's theorem, equation (32) has only one root inside the circle c_0. Thus, using Theorem 7 (a), we conclude that λ_0 is the eigenvalue lying inside the circle c_0.

Now, we show that λ_{-1} and λ_1 are the complex eigenvalues lying inside the circles

$$c_{-1} := \{\lambda \in \mathbb{C} : |\lambda - (4.0312397462 - 1.00097772667i)| = 8.8 \times 10^{-12}\}.$$

and

$$c_1 := \{\lambda \in \mathbb{C} : |\lambda - (4.0312397462 + 1.00097772667i)| = 8.8 \times 10^{-12}\}.$$

respectively.

The roots of the polynomials $R_{-1}(\lambda)$ and $R_1(\lambda)$ defined by (36) and (37), lying in the disk $D_1 = \{\lambda \in \mathbb{C} : |\lambda| \leq 4 + 2|r_2|\}$ are $x_1 = (4.0312397462 - 1.00097772667i)$ and $y_1 = (4.0312397462 + 1.00097772667i)$, respectively. The other roots of $R_{-1}(\lambda)$ are $x_2 = (35.9964522039 + 0.0176168557191i)$,
$x_3 = (35.9964154572 - 0.0172437280769i)$,
$x_4 = (35.9758900488 + 0.00060462552073i)$,
$x_5 = (100.00000187 + 0.00011473727234 8i)$,
$x_6 = (100.000000674 - 0.000114763768311i)$,
$x_7 = (196.0 + 1.20513462491e - 13i)$ and the other roots of $R_1(\lambda)$ are
$y_2 = (35.9964522039 - 0.0176168557191i)$,
$y_3 = (35.9964154572 + 0.0172437280769i)$,
$y_4 = (35.9758900488 - 0.00060462552073i)$,
$y_5 = (100.00000187 - 0.000114737272348i)$,
$y_6 = (100.000000674 + 0.000114763768311i)$ and
$y_7 = (196.0 - 1.20513462491e - 13i)$.

Using the decompositions

$$K_{-1}(\lambda) = \frac{(\lambda - x_1)(\lambda - x_2)\cdots(\lambda - x_7)}{(\lambda - 36)^3(\lambda - 100)^2(\lambda - 196)},$$

and

$$K_1(\lambda) = \frac{(\lambda - y_1)(\lambda - y_2)\cdots(\lambda - y_7)}{(\lambda - 36)^3(\lambda - 100)^2(\lambda - 196)},$$

by direct calculations, we obtain $|K_{-1}(\lambda)| > 3.5600 \times 10^{-12}$, for all $\lambda \in c_{-1}$ and $|K_1(\lambda)| > 3.5600 \times 10^{-12}$, for all $\lambda \in c_1$. On the other hand, one can easily calculate that $\sum_{j=4}^{\infty} |\alpha_{2j-1}(\lambda)| < 2.0038 \times 10^{-12}$, for all $\lambda \in c_{-1} \cup c_1$. The proof follows from Rouche's theorem and Theorem 6 (a); each of the equations (22) and (23) has only one root inside the circle c_{-1} and c_1, respectively and λ_{-1} and λ_{+1} are the complex eigenvalues lying inside c_{-1} and c_1, respectively.

Using the equations (38) and (39), Theorem 4 (b) and the estimations $|K_{-2}(\lambda)| > 3.7055 \times 10^{-9}$, for all $\lambda \in c_{-2}$; $|K_2(\lambda)| > 2.3100 \times 10^{-9}$, for all $\lambda \in c_2$ and $\sum_{j=4}^{\infty} |\alpha_{2j-1}(\lambda)| < 1.1464 \times 10^{-9}$, for all $\lambda \in c_{-2} \cup c_2$, one can show in a similar way that λ_{-2} and λ_2 are the eigenvalues lying inside the circles

$$c_{-2} := \{\lambda \in \mathbb{C} : |\lambda - 15.8949584087| = 1.9 \times 10^{-9}\}.$$

and

$$c_2 := \{\lambda \in \mathbb{C} : |\lambda - 16.0208389883| = 1.9 \times 10^{-8}\}.$$

respectively.

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Thank you...

cnur@yalova.edu.tr