

# SOLITON GENERATION IN OPTICAL FIBER NETWORKS

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# Outline

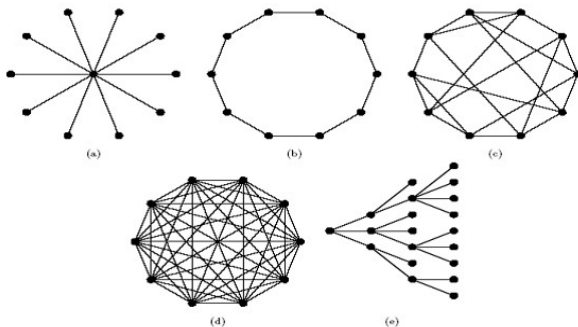
- 1 Motivation
- 2 Graphs and their topology
- 3 Evolution equations on graphs
- 4 Nonlinear Schrödinger equation on graphs
- 5 Soliton generation on graphs as initial value problem
- 6 Nonlocal NLS equation and soliton generation on graphs
- 7 Conclusion

# Motivation

## Why networks, why solitons in networks, why PT-symmetry?

- Most of the optical, electronic and opto-electronic devices have branched structure;
- Energy, signal and information transfer through the networks is a key process in modern technologies;
- Controlling, manipulating particle/wave transport in networks by tuning network architecture (topology) and nodes transfer properties provides effective tool for functional optimization of the devices and materials;
- Solitons are the signal, energy and charge carriers in many optical, electronic, optoelectronic, condensed matter and even biological structures which often have branched architectures;
- PT-symmetric nonlinear structures are the new forms of materials having challenging practical applications in newly emerging technologies.

# Graphs and their topology



- (a) star graph ( $B = 10$ ,  $V = 11$ ),
- (b) ring graph ( $B = 10$ ,  $V = 10$ ),
- (c) v-regular graph with  $v = 4$  ( $B = 20$ ,  $V = 10$ ),
- (d) complete (or well-connected) graph ( $B = 45$ ,  $V = 10$ ),
- (e) tree graph ( $B = 19$ ,  $V = 20$ ).

# Graphs and their topology

The topology of the graph, that is, the way the vertices and bonds are connected is given in terms of the  $V \times V$  connectivity matrix  $C_{i,j}$  (sometimes referred to as the adjacency matrix) which is defined as:

$$C_{i,j} = C_{j,i} = \begin{cases} 1 & \text{if } i,j \text{ are connected} \\ 0 & \text{otherwise} \end{cases}, \quad i,j = 1, \dots, V$$

## Metric graphs

A graph with the bonds which can be assigned length,

$$0 < l_b \leq D$$

is called metric graph.

# Evolution equations on graphs

- Metric graphs are the effective tool for modeling networks and branched domains;
- They greatly simplify modeling wave dynamics in branched structures;
- The problem of tunable wave transport;
- Dimensional reduction;
- Opening closing the nodes, directed motion.

# Nonlinear evolution equations on networks

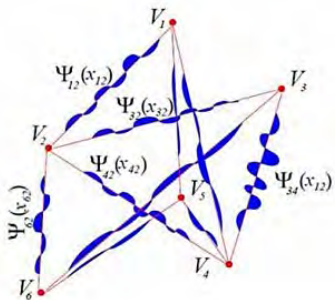
- ① Nonlinear Schrödinger equation;
- ② Sine-Gordon equation;
- ③ Korteweg-de Vries equation;
- ④ Nonlinear Dirac equation;
- ⑤ Burgers equation;
- ⑥ And other nonlinear wave equations.

# Evolution equation on graphs

$$i \frac{\partial \vec{\psi}}{\partial t} = H \vec{\psi}$$

where  $H$  is the Schrödinger, Dirac, or other differential operator. Wave function  $\vec{\psi}$  is a  $B$ -component vector

$$\vec{\psi} = (\psi_{b_1}(x_{b_1}), \psi_{b_2}(x_{b_2}), \psi_{b_3}(x_{b_3}), \dots, \psi_{b_B}(x_{b_B}))^T.$$

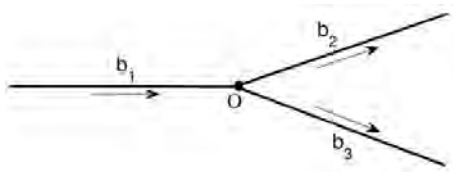




# Nonlinear evolution equations on metric graphs: Our approach

- 1 Write a given PDE on a graph;
- 2 Find/define its solution on a (infinite) line;
- 3 Define conserving quantities for the PDE;
- 4 Derive vertex boundary conditions from the conservation laws;
- 5 Require fulfilling vertex boundary conditions by the solution on a line and derive constraints ensuring such fulfilling;
- 6 Numerical solution of the problem, when the above constraints are not fulfilled;
- 7 Simulation of the soliton dynamics on a graph.

# Nonlinear Schrödinger equation on graphs



Defining of the bonds:

$$b_1 \sim (-\infty, 0], \quad b_{2,3} \sim [0, +\infty)$$

NLS equation on the each bond of the star graph<sup>1</sup>:

$$i \frac{\partial \psi_k}{\partial t} + \frac{\partial^2 \psi_k}{\partial x^2} + \beta_k |\psi_k|^2 \psi_k = 0, \quad x_k \in b_k, \quad k = 1, 2, 3$$

<sup>1</sup>Z. Sobirov, D. Matrasulov, K. Sabirov, S. Sawada, and K. Nakamura, Phys. Rev. E 81, 066602 (2010).

# Soliton solution of NLSE on a line

NLS equation on a line:

$$i\frac{\partial\psi}{\partial t} + \frac{\partial^2\psi}{\partial x^2} + 2|\psi|^2\psi = 0$$

soliton solution:

$$\psi(x, t) = \eta \operatorname{sech}[\eta(x + x_0 - ct)] e^{-i[2cx - (c^2 - 4\eta^2)t]/4}$$

$\eta$ —is the amplitude,  $x_0$ —is the center and  $c$ —is the velocity of the soliton.

# Conservative quantities

The norm:

$$N = \sum_{b_k} \int_{b_k} |\psi_k|^2 dx.$$

Energy:

$$E = \sum_{b_k} \int_{b_k} \left( \left| \frac{\partial \psi_k}{\partial x} \right|^2 - \frac{\beta_k}{2} |\psi_k|^4 \right) dx$$

# Conservative quantities

Norm – const:

$$\operatorname{Im} \left[ \psi_1^* \frac{\partial \psi_1}{\partial x} \right] \Big|_{x=0} = \operatorname{Im} \left[ \psi_2^* \frac{\partial \psi_2}{\partial x} \right] \Big|_{x=0} + \operatorname{Im} \left[ \psi_3^* \frac{\partial \psi_3}{\partial x} \right] \Big|_{x=0}$$

Energy – const:

$$\operatorname{Re} \left[ \frac{\partial \psi_1^*}{\partial x} \frac{\partial \psi_1}{\partial t} \right] \Big|_{x=0} = \operatorname{Re} \left[ \frac{\partial \psi_2^*}{\partial x} \frac{\partial \psi_2}{\partial t} \right] \Big|_{x=0} + \operatorname{Re} \left[ \frac{\partial \psi_3^*}{\partial x} \frac{\partial \psi_3}{\partial t} \right] \Big|_{x=0}.$$

# Vertex boundary conditions

- The first choice:

$$\alpha_1 \psi_1|_{x=0} = \alpha_2 \psi_2|_{x=0} = \alpha_3 \psi_3|_{x=0}$$

$$\frac{1}{\alpha_1} \frac{\partial \psi_1}{\partial x} \Big|_{x=0} = \frac{1}{\alpha_2} \frac{\partial \psi_2}{\partial x} \Big|_{x=0} + \frac{1}{\alpha_3} \frac{\partial \psi_3}{\partial x} \Big|_{x=0}$$

- The second choice:

$$\frac{1}{\alpha_1} \psi_1|_{x=0} = \frac{1}{\alpha_2} \psi_2|_{x=0} + \frac{1}{\alpha_3} \psi_3|_{x=0}$$

$$\alpha_1 \frac{\partial \psi_1}{\partial x} \Big|_{x=0} = \alpha_2 \frac{\partial \psi_2}{\partial x} \Big|_{x=0} = \alpha_3 \frac{\partial \psi_3}{\partial x} \Big|_{x=0}$$

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<sup>1</sup>Z. Sobirov, D. Matrasulov, K. Sabirov, S. Sawada, and K. Nakamura, Phys. Rev. E 81, 066602 (2010).

# Soliton solution and the constraint

In particular at  $\alpha_k = \sqrt{\beta_k}$  the wave function can be found in the form

$$\psi_k(x, t) = i\sqrt{\frac{2}{\beta_k}}\psi(x, t)$$

Where  $q(x, t)$  is the solution of NLSE (on a line).  
Boundary conditions are satisfied if

$$\frac{1}{\beta_1} = \frac{1}{\beta_2} + \frac{1}{\beta_3}$$

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<sup>1</sup>Z. Sobirov, D. Matrasulov, K. Sabirov, S. Sawada, and K. Nakamura, Phys. Rev. E 81, 066602 (2010).

# Soliton dynamics



# Soliton generation on the line

$$i \frac{\partial \psi}{\partial t} + \frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} + |\psi|^2 \psi = 0$$

The number of generated solitons<sup>2</sup>

$$N = \left\langle \frac{1}{2} + \frac{F}{\pi} \right\rangle$$

where  $F = \int_{-\infty}^{+\infty} |\psi(x, 0)| dx$ .

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<sup>2</sup>N.-C. Panoiu, I. V. Mel'nikov, D. Mihalache, C. Etrich, and F. Lederer, Phys. Rev. E **60**, 4868 (1999).

# Example for soliton generation

The initial pulse profile<sup>3</sup>

$$\psi(x, 0) = -iq(x) \quad q(x) = \begin{cases} 0 & \text{for } |x| > a/2 \\ b & \text{for } |x| \leq a/2 \end{cases}$$

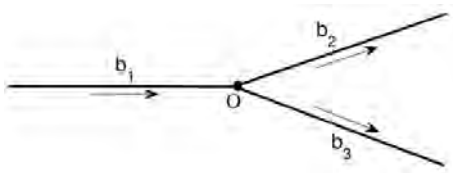
$$N = \left\langle \frac{1}{2} + \frac{ab}{\pi} \right\rangle$$

<sup>3</sup>J. Burzlaff, J. Phys. A: Math. Gen. **21** (1988), 561-566.

# Nonlinear Schrödinger equation on a star graph

$$i\frac{\partial\psi_j}{\partial t} + \frac{\partial^2\psi_j}{\partial x^2} + \beta_j|\psi_j|^2\psi_j = 0$$

where  $j = 1, 2, 3$ .



# Zakharov–Shabat problem on the star graph

$$\begin{aligned}\frac{\partial v_j^{(1)}}{\partial x} &= i\lambda v_j^{(1)} + i\psi_j(x, 0)v_j^{(2)}, \\ \frac{\partial v_j^{(2)}}{\partial x} &= -i\lambda v_j^{(2)} + i\psi_j^*(x, 0)v_j^{(1)},\end{aligned}$$

Let us consider the special family of the initial potentials

$$\psi_j(x, 0) = \Psi_j(x, 0)e^{i\delta_j},$$

and eigenfunction transformations

$$v_j^{(1)} \rightarrow V_j^{(1)} e^{i\gamma_j}, \quad v_j^{(2)} \rightarrow V_j^{(2)} e^{i(\gamma_j - \delta_j)}$$

## Zakharov–Shabat problem on the star graph

$$\begin{aligned}\frac{\partial V_j^{(1)}}{\partial x} &= i\lambda V_j^{(1)} + i\Psi_j(x, 0)V_j^{(2)}, \\ \frac{\partial V_j^{(2)}}{\partial x} &= -i\lambda V_j^{(2)} + i\Psi_j(x, 0)V_j^{(1)}.\end{aligned}$$

We will define the number of the zeros of the Jost coefficients  $a_j(\lambda)$  at  $\lambda = 0$ .

$$\begin{aligned}\frac{\partial V_j^{(1)}}{\partial x} &= i\Psi_j(x, 0)V_j^{(2)}, \\ \frac{\partial V_j^{(2)}}{\partial x} &= i\Psi_j(x, 0)V_j^{(1)}.\end{aligned}$$

# Explicit form of the eigenfunctions

$$V_1^{(1)}(x, 0) = \exp(-iS_1(x)) \left( C_1^{(1)} \int_{-\infty}^x \psi_1(x', 0) \exp(2iS_1(x')) dx' + C_1^{(2)} \right),$$

$$V_1^{(2)}(x, 0) = -iC_1^{(1)} \exp(iS_1(x)) - V_1^{(1)},$$

$$V_{2,3}^{(1)}(x, 0) = \exp(-iS_{2,3}(x)) \left( C_{2,3}^{(1)} \int_0^x \psi_{2,3}(x', 0) \exp(2iS_{2,3}(x')) dx' + C_{2,3}^{(2)} \right),$$

$$V_{2,3}^{(2)}(x, 0) = -iC_{2,3}^{(1)} \exp(iS_{2,3}(x)) - V_{2,3}^{(1)},$$

where  $S_1(x) = \int_{-\infty}^x \psi_1(x', 0) dx'$ , and  $S_{2,3}(x) = \int_0^x \psi_{2,3}(x', 0) dx'$ .

# Deriving the soliton number

If one chooses  $V_1^{(1)}(x, 0) \rightarrow 0$  for  $x \rightarrow -\infty$  and  $V_{2,3}^{(1)}(x, 0) \rightarrow 0$  for  $x \rightarrow -0$ , then  $C_j^{(2)} = 0$ , and we have

$$\begin{aligned} a_j(0) &= \lim_{x \rightarrow \pm 0} V_j^{(2)}(x, 0) \\ &= -iC_j^{(1)} \left( \exp(iS_{0j}) - i \exp(-iS_{0j}) \int_{b_j} \psi_j(x, 0) \exp(2iS_j(x)) dx \right) \\ &= -iC_j^{(1)} \cos S_{0j}, \end{aligned}$$

where  $S_{0j} = \int_{b_j} \psi_j(x, 0) dx$ .

# Deriving the soliton number

The soliton number on the star graph<sup>4</sup>

$$N = \left\langle \frac{3}{2} + \frac{S_{01} + S_{02} + S_{03}}{\pi} \right\rangle.$$

Noting that for the initial pulses for any  $x$  and with  $\Psi_j(x, 0) > 0$

$$S_{0j} \equiv \int_{b_j} \Psi_j(x, 0) dx = \int_{b_j} |\psi_j(x, 0)| dx = F_j.$$

we have

$$N = \left\langle \frac{3}{2} + \frac{F_1 + F_2 + F_3}{\pi} \right\rangle.$$

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<sup>4</sup>K.K Sabirov, M.E Akramov, R.Sh Otajonov, D.U Matrasulov, Chaos, Solitons & Fractals **133**, 109636 (2020).



# The initial pulse profile on the star graph

$$\psi_j(x, 0) = -i\sqrt{\frac{2}{\beta_j}}q_j(x)$$

$$q_1(x) = \begin{cases} 0, & x < -\frac{1}{2}a \\ b, & -\frac{1}{2}a \leq x \leq 0 \end{cases}$$

$$q_{2,3}(x) = \begin{cases} 0, & x > \frac{1}{2}a \\ b, & 0 \leq x \leq \frac{1}{2}a \end{cases}$$

The soliton number for the above initial condition

$$N = \left\langle \frac{3}{2} + \frac{F}{\pi} \right\rangle, \quad F = \sum_{j=1}^3 \int_{b_j} |\psi_j(x, 0)| dx = \frac{ab}{2} \left[ \sqrt{\frac{2}{\beta_1}} + \sqrt{\frac{2}{\beta_2}} + \sqrt{\frac{2}{\beta_3}} \right].$$

# The Gaussian input pulse

$$\psi_j(x, 0) = \sqrt{\frac{2}{\beta_j}} A \exp \left[ -\frac{1}{2} (1 - i\alpha) \left( \frac{x}{\sigma} \right)^{2m} \right]$$

The soliton number for Gaussian input pulse

$$N = \left\langle \frac{3}{2} + \frac{F}{\pi} \right\rangle,$$

$$F = \sum_{j=1}^3 \int_{b_j} |\psi_j(x, 0)| dx = \frac{2^{\frac{1}{2m}} A \sigma}{2m} \Gamma \left( \frac{1}{2m} \right) \left[ \sqrt{\frac{2}{\beta_1}} + \sqrt{\frac{2}{\beta_2}} + \sqrt{\frac{2}{\beta_3}} \right].$$

# Nonlocal nonlinear Schrödinger equation on a line

Nonlocal NLS equation<sup>5</sup>

$$i\frac{\partial}{\partial t}q(x, t) = \frac{\partial^2}{\partial x^2}q(x, t) + 2q^2(x, t)q^*(-x, t).$$

Nonlocal NLS equation can be rewritten as

$$i\frac{\partial}{\partial t}q(x, t) = \frac{\partial^2}{\partial x^2}q(x, t) + V(x, t)q(x, t),$$

where  $V = 2q(x, t)q^*(-x, t)$  is the PT-symmetric self-induced potential.

<sup>5</sup>Mark J. Ablowitz and Ziad H. Musslimani, Phys. Rev. Lett. **110**, 064105 (2013).

# Soliton generation for nonlocal NLS equation

The soliton number<sup>6</sup>

$$N = \left\langle \frac{1}{2} + \frac{F}{\pi} \right\rangle, \quad \text{where} \quad F = \int_{-\infty}^{+\infty} |q(x, 0)| dx$$

Initial pulse profile is given as follows

$$q(x, 0) = \begin{cases} 0, & \text{for } |x| > \frac{1}{2}a \\ b, & \text{for } |x| \leq \frac{1}{2}a \end{cases} \quad b > 0.$$

The soliton number

$$N = \left\langle \frac{1}{2} + \frac{ab}{\pi} \right\rangle, \quad F = \int_{-\infty}^{+\infty} |q(x, 0)| dx = ab.$$

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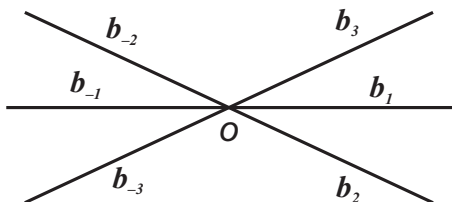
<sup>6</sup>M. Akramov, K. Sabirov, D. Matrasulov, H. Susanto, S. Usanov, and O. Karpova, Phys. Rev. E **105**, 054205 (2022).

# Numerical simulation of soliton generation

Two soliton generation for rectangular initial condition with the parameters:  $a = 3$ ,  $b = 2$ , i.e.,  $N = \left\langle \frac{1}{2} + \frac{ab}{\pi} \right\rangle = 2$ .

# Nonlocal NLS equation on the star graph

$$i \frac{\partial}{\partial t} q_{\pm j}(x, t) = \frac{\partial^2}{\partial x^2} q_{\pm j}(x, t) + \sqrt{\beta_j \beta_{-j}} q_{\pm j}^2(x, t) q_{\mp j}^*(-x, t),$$



<sup>6</sup>M. Akramov, K. Sabirov, D. Matrasulov, H. Susanto, S. Usanov, and O. Karpova, Phys. Rev. E **105**, 054205 (2022).

# Vertex boundary conditions

$$\begin{aligned}\alpha_1 q_1(x, t)|_{x=0} &= \alpha_{-1} q_{-1}(x, t)|_{x=0} = \alpha_2 q_2(x, t)|_{x=0} \\ &= \alpha_{-2} q_{-2}(x, t)|_{x=0} = \alpha_3 q_3(x, t)|_{x=0} = \alpha_{-3} q_{-3}(x, t)|_{x=0},\end{aligned}$$

$$\begin{aligned}& \frac{1}{\alpha_1} \frac{\partial}{\partial x} q_1(x, t) \Big|_{x=0} + \frac{1}{\alpha_2} \frac{\partial}{\partial x} q_2(x, t) \Big|_{x=0} + \frac{1}{\alpha_3} \frac{\partial}{\partial x} q_3(x, t) \Big|_{x=0} \\ &= \frac{1}{\alpha_{-1}} \frac{\partial}{\partial x} q_{-1}(x, t) \Big|_{x=0} + \frac{1}{\alpha_{-2}} \frac{\partial}{\partial x} q_{-2}(x, t) \Big|_{x=0} + \frac{1}{\alpha_{-3}} \frac{\partial}{\partial x} q_{-3}(x, t) \Big|_{x=0}.\end{aligned}$$

Sum rule

$$\frac{\alpha_{\pm j}}{\alpha_1} = \sqrt{\frac{\beta_{\pm j}}{\beta_1}}, \quad \frac{1}{\beta_1} + \frac{1}{\beta_2} + \frac{1}{\beta_3} = \frac{1}{\beta_{-1}} + \frac{1}{\beta_{-2}} + \frac{1}{\beta_{-3}}.$$

# Solitons solution on the star graph

Breathing soliton solution:

$$q_{\pm j}(x, t) = -\sqrt{\frac{2}{\beta_{\pm j}}} \frac{2(\eta_1 + \bar{\eta}_1) e^{i\bar{\theta}_1} e^{-4i\bar{\eta}_1^2 t} e^{-2\bar{\eta}_1 x}}{1 + e^{i(\theta_1 + \bar{\theta}_1)} e^{4i(\eta_1^2 - \bar{\eta}_1^2) t} e^{-2(\eta_1 + \bar{\eta}_1) x}}$$

Static soliton solution:

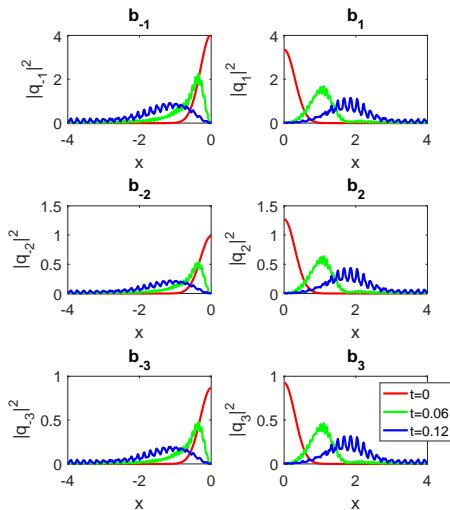
$$q_{\pm j}(x, t) = -\sqrt{\frac{2}{\beta_{\pm j}}} \frac{4\eta e^{i\bar{\varphi}} e^{-4i\eta^2 t} e^{-2\eta x}}{1 + e^{i(\varphi + \bar{\varphi})} e^{-4\eta x}}.$$

$\varphi, \bar{\varphi}, \eta$  are arbitrary complex constants.



# Radiation of the input pulse

Gaussian input pulse



## Zakharov–Shabat problem on the star graph

$$\begin{aligned}\frac{\partial v_{\pm j}^{(1)}}{\partial x} &= -ikv_{\pm j}^{(1)} + \sqrt{\frac{\beta_{\pm j}}{2}} q_{\pm j}(x, 0) v_{\pm j}^{(2)}, \\ \frac{\partial v_{\pm j}^{(2)}}{\partial x} &= ikv_{\pm j}^{(2)} - \sqrt{\frac{\beta_{\mp j}}{2}} q_{\mp j}^*(-x, 0) v_{\pm j}^{(1)},\end{aligned}$$

Note that, the initial condition must be symmetric to  $x = 0$  point,  
 $\sqrt{\beta_{\pm j}} q_{\pm j}(x, 0) = \sqrt{\beta_{\mp j}} q_{\mp j}^*(-x, 0).$

# The number of generated solitons

$$N_{\pm j} = \left\langle \frac{1}{2} + \frac{F_{\pm j}}{\pi} \right\rangle, \quad N = \sum_{j=1}^3 (N_{-j} + N_j).$$

where

$$F_{\pm j} = \sqrt{\frac{\beta_{\pm j}}{2}} \int_{b_{\pm j}} |q_{\pm j}(x, 0)| dx.$$

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<sup>6</sup>M. Akramov, K. Sabirov, D. Matrasulov, H. Susanto, S. Usanov, and O. Karpova, Phys. Rev. E **105**, 054205 (2022).

# The initial pulse profile and the number of generated solitons on the star graph

$$q_{\pm j}(x, 0) = \sqrt{\frac{2}{\beta_{\pm j}}} \psi_{\pm j}(x):$$

$$\psi_{-j}(x) = \begin{cases} 0, & \text{for } x < -\frac{1}{2}a \\ b, & \text{for } -\frac{1}{2}a \leq x \leq 0 \end{cases},$$

$$\psi_j(x) = \begin{cases} 0, & \text{for } x > \frac{1}{2}a \\ b, & \text{for } 0 \leq x \leq \frac{1}{2}a \end{cases},$$

where  $b > 0$ .

The number of generated solitons can be written as

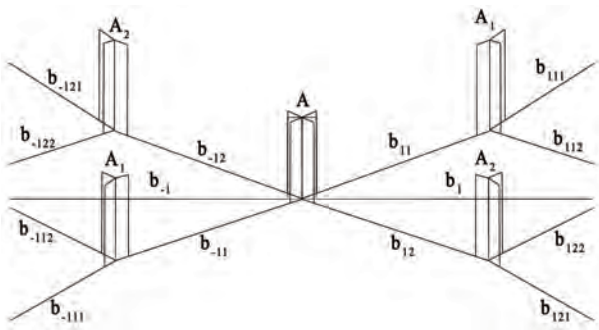
$$N = 6 \left\langle \frac{1}{2} + \frac{ab}{2\pi} \right\rangle, \quad F_{\pm j} = \sqrt{\frac{\beta_{\pm j}}{2}} \int_{b_{\pm j}} |q_{\pm j}(x, 0)| dx = \frac{ab}{2}$$

# Extending for a tree graph

Nonlocal NLS equation on tree graph

$$i \frac{\partial}{\partial t} q_{\pm e}(x, t) = \frac{\partial^2}{\partial x^2} q_{\pm e}(x, t) + \sqrt{\beta_e \beta_{-e}} q_{\pm e}^2(x, t) q_{\mp e}^*(-x, t),$$

where  $e = \{1, 1m, 1mn\}$ ,  $n, m = 1, 2$ .



# Soliton generation on tree graph

The number of solitons

$$N_e = \left\langle \frac{1}{2} + \frac{F_e}{\pi} \right\rangle, \quad N = \sum_{e \in \Omega} N_e,$$

where  $\Omega = \{\pm 1; \pm 1m; \pm 1mn\}$  and

$$F_{\pm 1} = \sqrt{\frac{\beta_{\pm 1}}{2}} \int_{b_{\pm 1}} |q_{\pm 1}(x, 0)| dx = \frac{aA}{2},$$

$$F_{\pm 1m} = \sqrt{\frac{\beta_{\pm 1m}}{2}} \int_{b_{\pm 1m}} |q_{\pm 1m}(x, 0)| dx = \frac{a}{2}(A + A_m),$$

$$F_{\pm 1mn} = \sqrt{\frac{\beta_{\pm 1mn}}{2}} \int_{b_{\pm 1mn}} |q_{\pm 1mn}(x, 0)| dx = \frac{aA_m}{2}.$$

# Conslusions

- Nonlinear Schrödinger equation and soliton dynamics on graphs are studied;
- We studied the problem of soliton generation in optical fiber networks using a model based ordinary and nonlocal NLS equation on metric graphs;
- Initial value (Cauchy) problem for ordinary and nonlocal NLS equation on metric graphs is solved for different graph topologies, such as star and tree;
- Analytic expression of number of generated solitons is proved by numerically;
- It was shown that the number of generated solitons depends not on the shape of the initial condition, but on its initial area;
- It was observed that the soliton profile is radiated when the sum rule is not fulfilled.

# References

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