Propagating-wave approximation and the transfer matrix in two-dimensional potential scattering

Ali Mostafazadeh
Koç Univesity

TÜBİTAK
Project 120F061

Joint Work with
Farhang Loran
(Isfahan Univ. Tech.)

arXiv: 2204.05153
(Phys Rev A, to appear)
& arXiv: 2207.10054

Outline:

- Potential scattering in 1D & the transfer matrix
- Potential scattering in 2D & its dynamical formulation
- Evanescent waves in 2D & the Propagating-wave approximation (PWA)
- Nonlocal potentials *V*_k achieving PWA
- Potentials for which PWA is exact
- Existence of the transfer matrix for V_k
- Concluding remarks

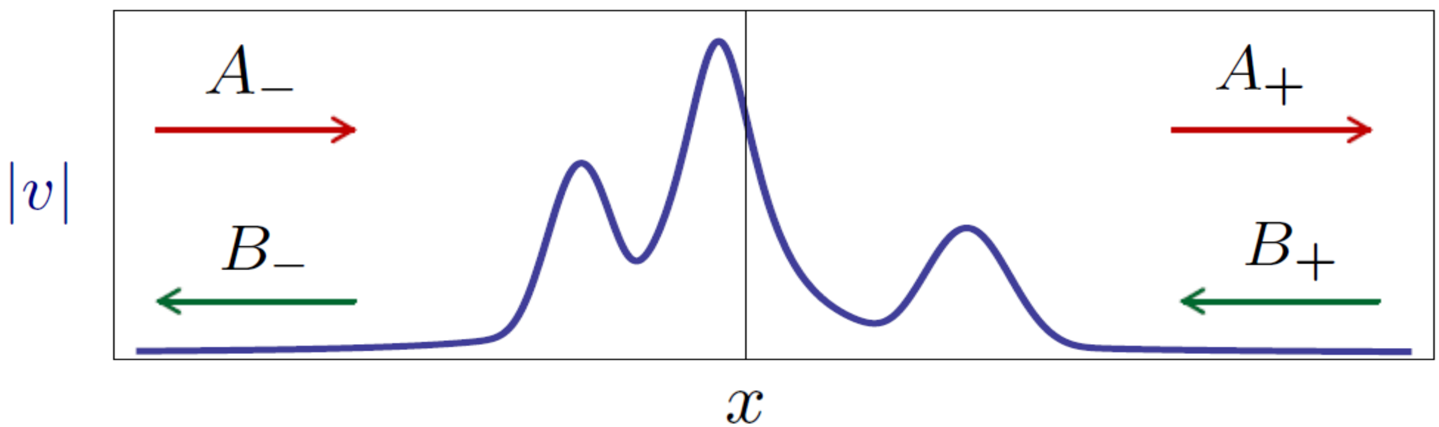
Scattering in 1D $\quad \Psi(x,t) = e^{-i\omega t}\psi(x)$

- Time-Indep. Schrödinger Eq.: $-\psi(x)'' + v(x)\psi(x) = k^2\psi(x)$
- $v : \mathbb{R} \to \mathbb{C}$ is a possibly k-dependent **short-range potential**.

Scattering in 1D $\quad \Psi(x,t) = e^{-i\omega t}\psi(x)$

- Time-Indep. Schrödinger Eq.: $-\psi(x)'' + v(x)\psi(x) = k^2\psi(x)$
- $v : \mathbb{R} \to \mathbb{C}$ is a possibly k-dependent **short-range potential**.

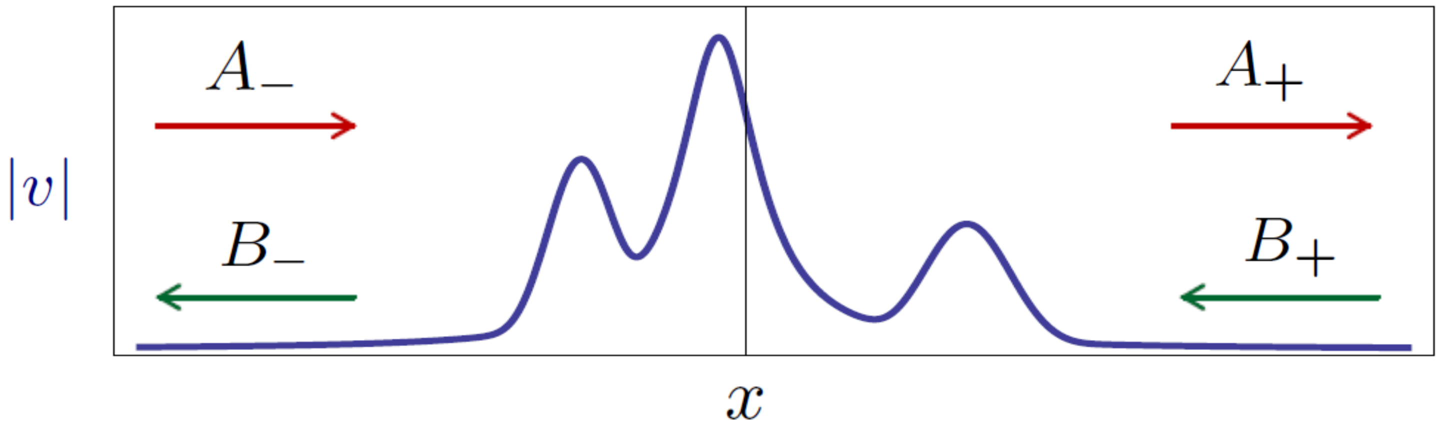
$$\psi(x) \to \begin{cases} A_- e^{ikx} + B_- e^{-ikx} & \text{for } x \to -\infty \\ A_+ e^{ikx} + B_+ e^{-ikx} & \text{for } x \to +\infty \end{cases}$$



Scattering in 1D $\quad \Psi(x,t) = e^{-i\omega t}\psi(x)$

- Time-Indep. Schrödinger Eq.: $-\psi(x)'' + v(x)\psi(x) = k^2\psi(x)$
- $v : \mathbb{R} \to \mathbb{C}$ is a possibly k-dependent **short-range potential**.

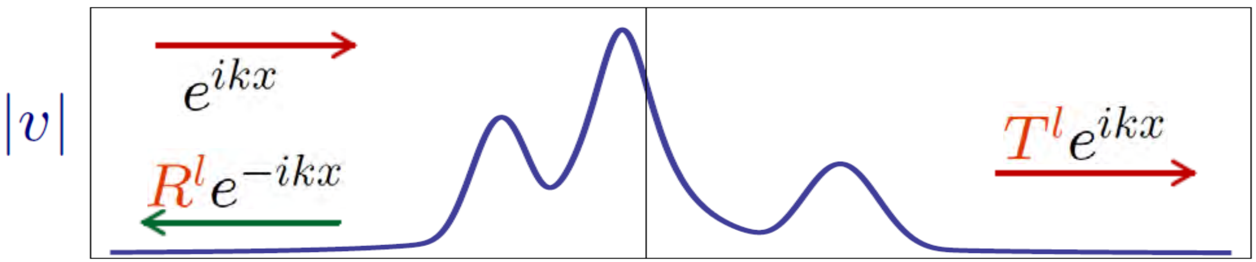
$$\psi(x) \to \begin{cases} A_- e^{ikx} + B_- e^{-ikx} & \text{for } x \to -\infty \\ A_+ e^{ikx} + B_+ e^{-ikx} & \text{for } x \to +\infty \end{cases}$$



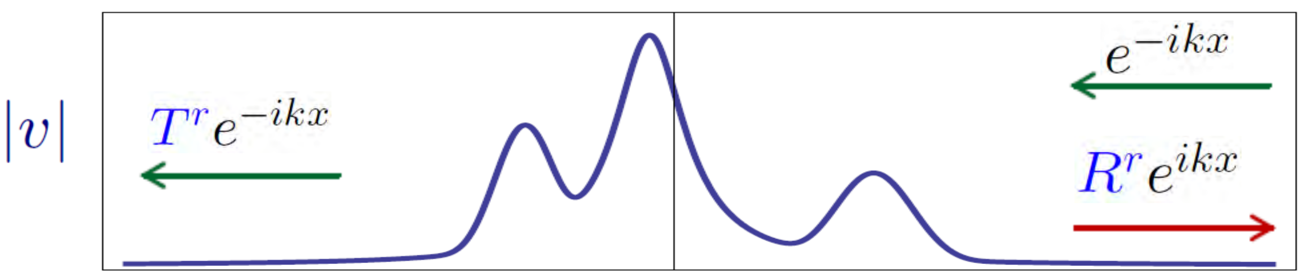
- Transfer matrix: $\begin{bmatrix} A_+ \\ B_+ \end{bmatrix} = \mathbf{M} \begin{bmatrix} A_- \\ B_- \end{bmatrix}$.

- Scattering from the left and right:

$$\psi^{\text{left}}(x) = \begin{cases} e^{ikx} + R^l e^{-ikx} & \text{for} \quad x \to -\infty \\ T^l e^{ikx} & \text{for} \quad x \to +\infty \end{cases}$$

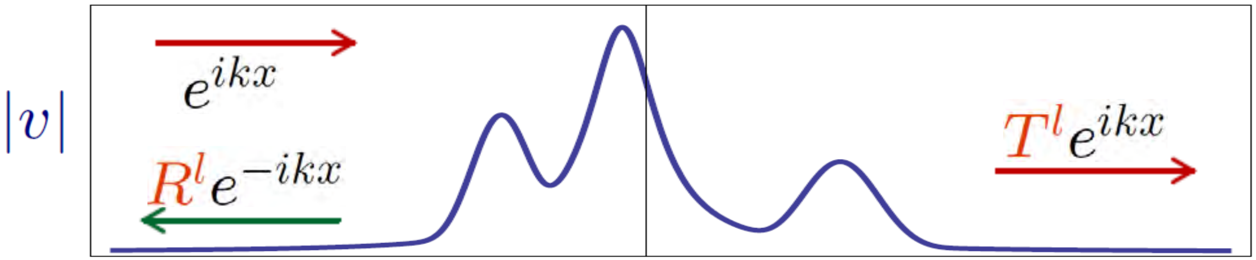


$$\psi^{\text{right}}(x) = \begin{cases} T^r e^{-ikx} & \text{for} \quad x \to -\infty \\ e^{-ikx} + R^r e^{ikx} & \text{for} \quad x \to +\infty \end{cases}$$

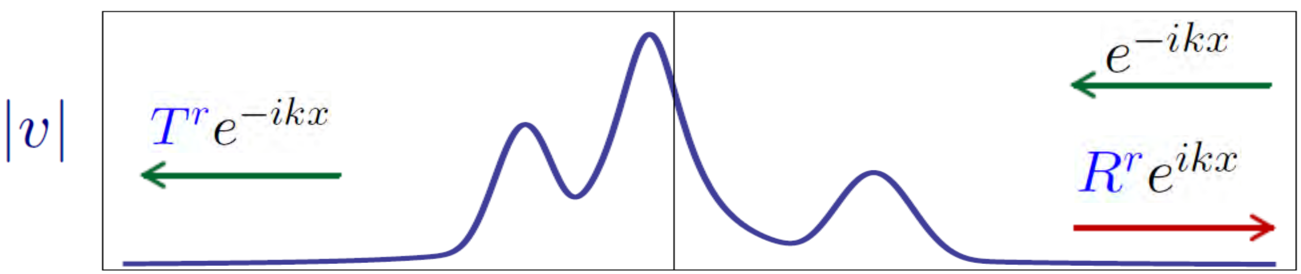


- Scattering from the left and right:

$$\psi^{\text{left}}(x) = \begin{cases} e^{ikx} + R^l e^{-ikx} & \text{for} \quad x \to -\infty \\ T^l e^{ikx} & \text{for} \quad x \to +\infty \end{cases}$$



$$\psi^{\text{right}}(x) = \begin{cases} T^r e^{-ikx} & \text{for} \quad x \to -\infty \\ e^{-ikx} + R^r e^{ikx} & \text{for} \quad x \to +\infty \end{cases}$$

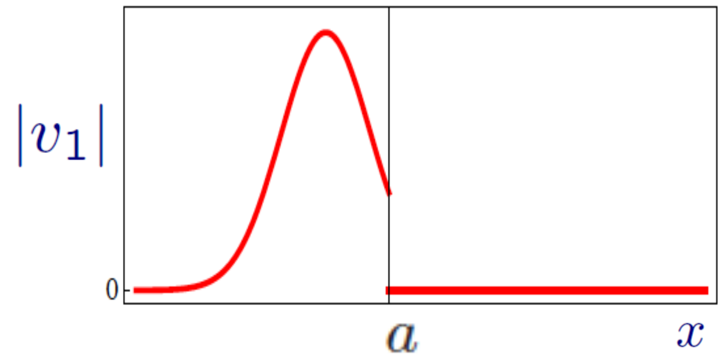


$$R^l = -\frac{M_{21}}{M_{22}}, \qquad R^r = \frac{M_{12}}{M_{22}}, \qquad T^l = T^r =: T = \frac{1}{M_{22}}.$$

Composition Property of $\mathbf{M}$

Let v_1 and v_2 be scattering potentials such that

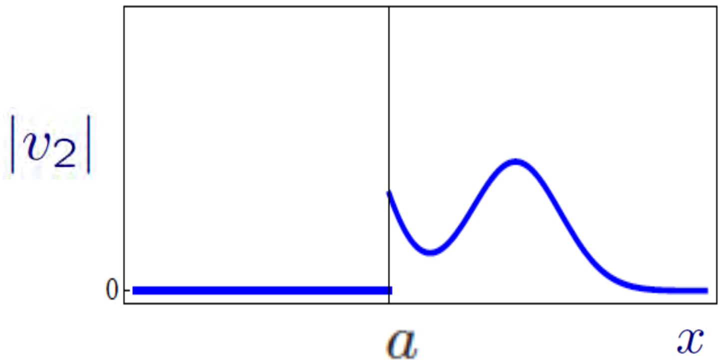
$$v_1(x) = 0 \quad \text{for} \quad x > a,$$
$$v_2(x) = 0 \quad \text{for} \quad x < a$$
$$v(x) = v_1(x) + v_2(x).$$

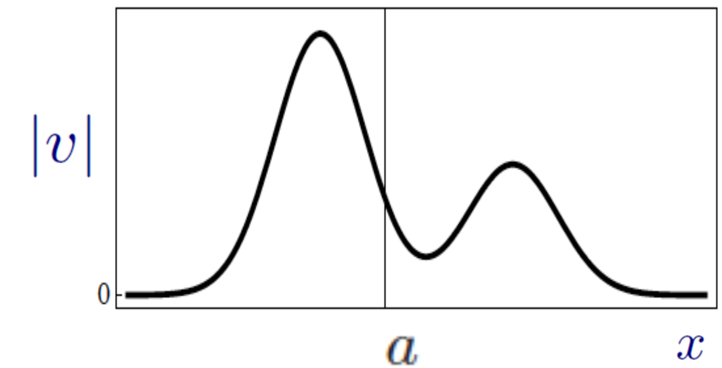


$\mathbf{M}_1$: Transfer matrix of v_1

$\mathbf{M}_2$: Transfer matrix of v_2

$\mathbf{M}$: Transfer matrix of $v = v_1 + v_2$

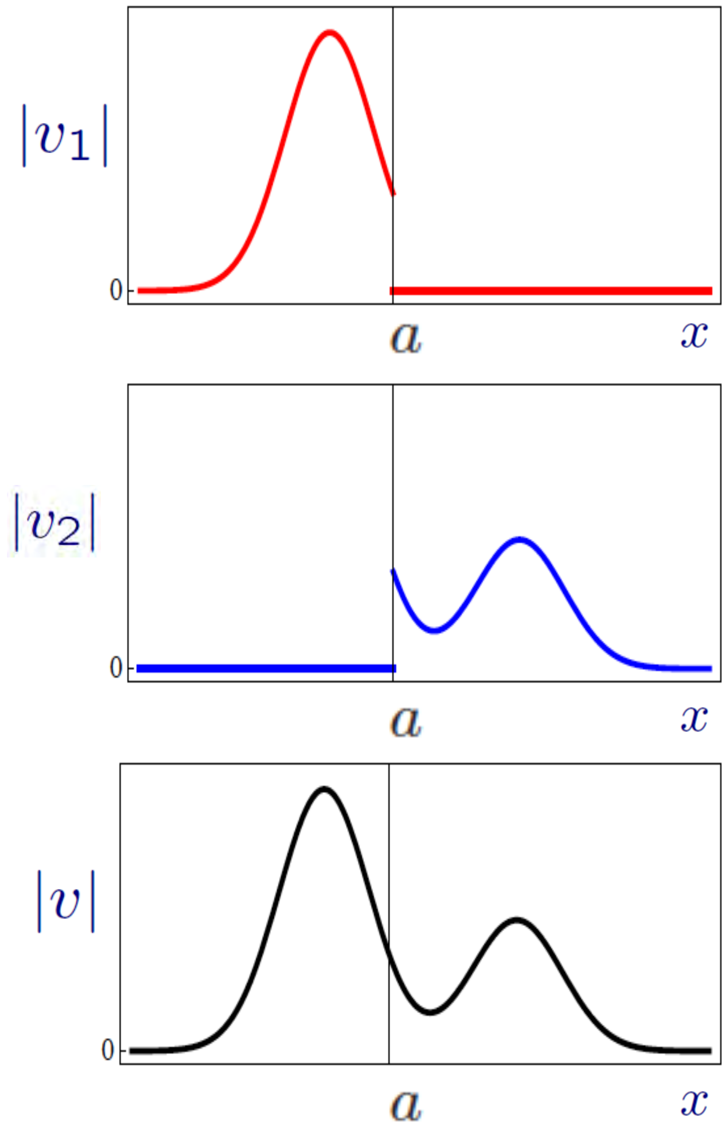




Composition Property of $\mathbf{M}$

Let v_1 and v_2 be scattering potentials such that

$$v_1(x) = 0 \quad \text{for} \quad x > a,$$
$$v_2(x) = 0 \quad \text{for} \quad x < a$$
$$v(x) = v_1(x) + v_2(x).$$



$\mathbf{M}_1$: Transfer matrix of v_1

$\mathbf{M}_2$: Transfer matrix of v_2

$\mathbf{M}$: Transfer matrix of $v = v_1 + v_2$

Then $\mathbf{M} = \mathbf{M}_2\,\mathbf{M}_1$.

Theorem: $\mathbf{M} = \mathbf{U}(+\infty, -\infty)$ where $\mathbf{U}(x, x_0)$ is the evolution operator for

$$\mathbf{H}(x) := \frac{v(x)}{2k}\begin{bmatrix} 1 & e^{-2ikx} \\ -e^{2ikx} & -1 \end{bmatrix}.$$

---

x plays the role of "time".

[Ann. Phys. (NY), **341**, 77 (2014)]

Theorem: $\mathbf{M} = \mathbf{U}(+\infty, -\infty)$ where $\mathbf{U}(x, x_0)$ is the evolution operator for

$$\mathbf{H}(x) := \frac{v(x)}{2k}\begin{bmatrix} 1 & e^{-2ikx} \\ -e^{2ikx} & -1 \end{bmatrix}.$$

---

x plays the role of "time".

$$i\partial_x \mathbf{U}(x, x_0) = \mathbf{H}(x)\mathbf{U}(x, x_0),$$

$$\mathbf{U}(x_0, x_0) = \mathbf{I}.$$

[Ann. Phys. (NY), **341**, 77 (2014)]

Theorem: $\mathbf{M} = \mathbf{U}(+\infty, -\infty)$ where $\mathbf{U}(x, x_0)$ is the evolution operator for

$$\mathbf{H}(x) := \frac{v(x)}{2k} \begin{bmatrix} 1 & e^{-2ikx} \\ -e^{2ikx} & -1 \end{bmatrix},$$

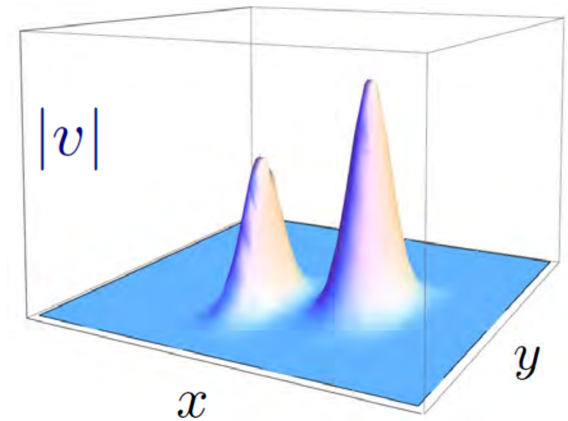
i.e.,

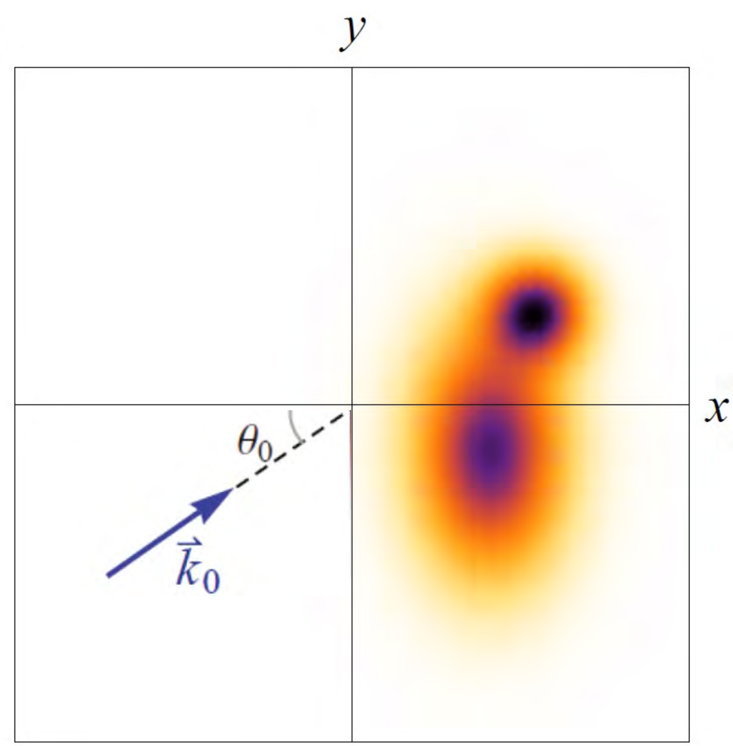
$$\begin{aligned}
\mathbf{M} =& \mathscr{T} \exp \int_{-\infty}^{\infty} -i\mathbf{H}(x)dx \\
=& \mathbf{I} - i \int_{-\infty}^{\infty} dx_1 \mathbf{H}(x_1) \\
&+ (-i)^2 \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{x_2} dx_1 \mathbf{H}(x_2)\mathbf{H}(x_1) + \cdots
\end{aligned}$$

[Ann. Phys. (NY), **341**, 77 (2014)]

Scattering in 2D

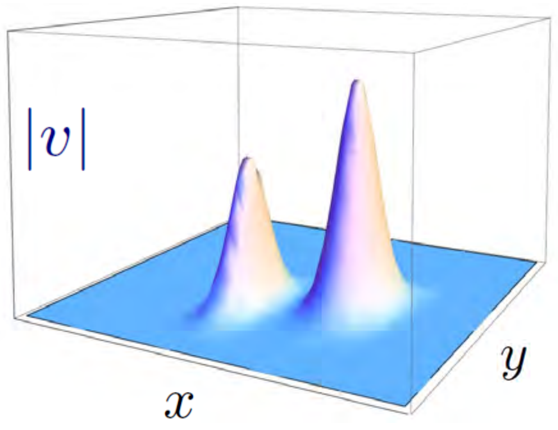
$$\left[-\partial_x^2 - \partial_y^2 + v(x,y)\right]\psi(x,y) = k^2\psi(x,y)$$





Scattering in 2D

$$\left[-\partial_x^2 - \partial_y^2 + v(x,y)\right]\psi(x,y) = k^2\psi(x,y)$$



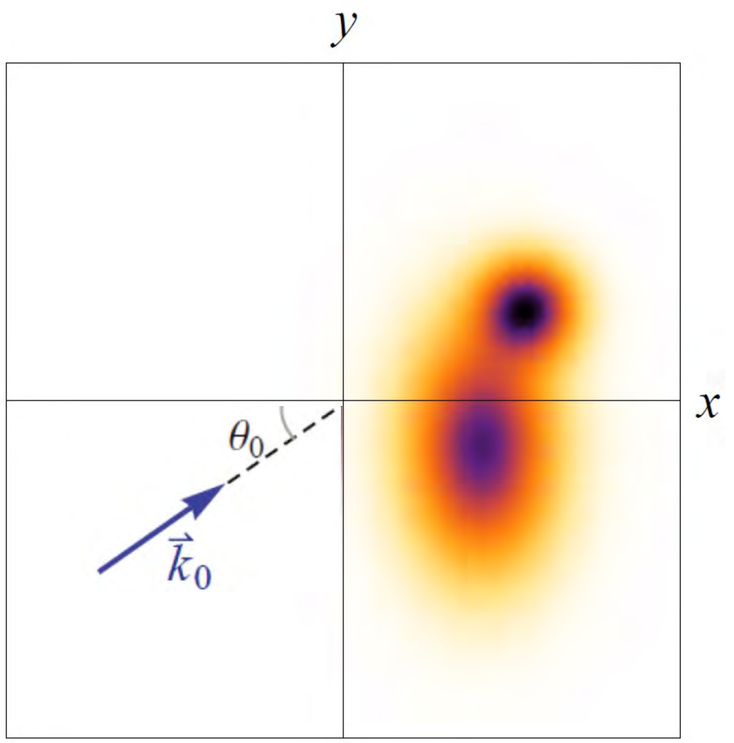
$$\psi(\mathbf{r}) \rightarrow \frac{e^{i\mathbf{k}_0\cdot\mathbf{r}}}{2\pi} + \sqrt{\frac{i}{kr}}\frac{e^{ikr} f(\theta)}{2\pi} \quad \text{as} \quad r \to \infty$$

$\mathbf{k}_0 :=$ Incident wave vector

$\mathbf{r} := (x,y)$

$(r,\theta) :=$ polar coordinates

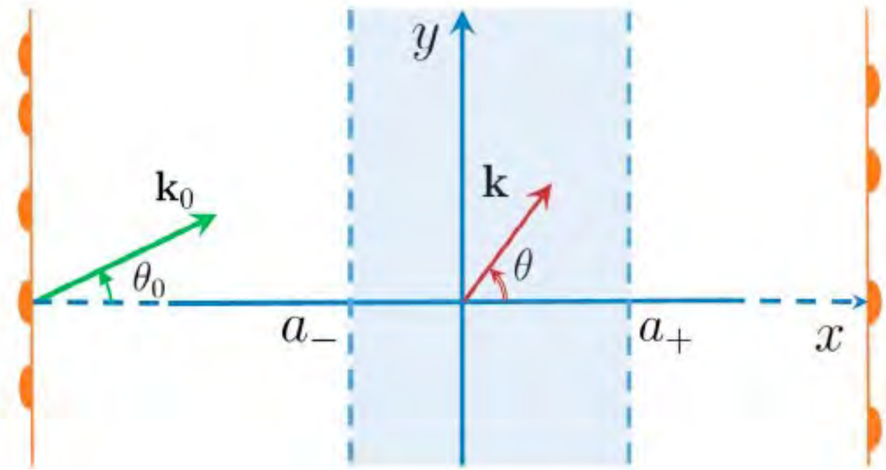
$f(\theta)$: Scattering amplitude



Suppose that $v(x,y) = 0$ for $x \notin [a_-, a_+]$

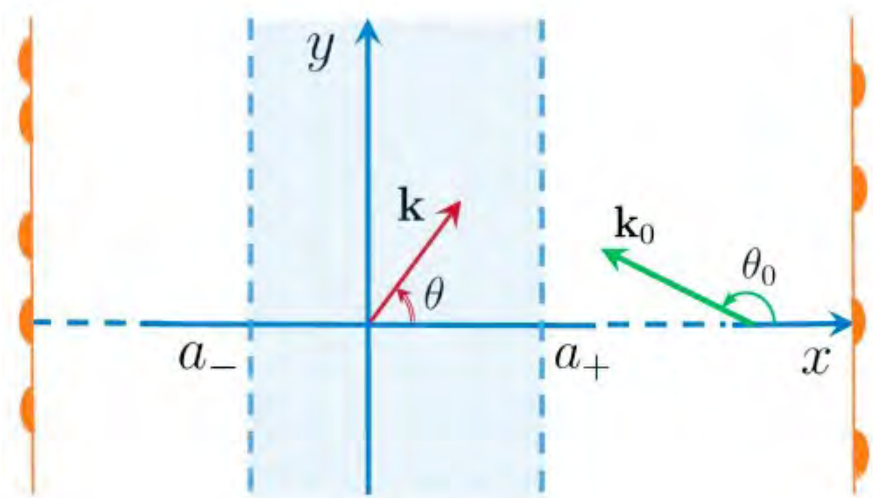
Suppose that $v(x,y)=0$ for $x \notin [a_-, a_+]$

Left-incident wave



$$\mathbf{k} := \frac{k\,\mathbf{r}}{r}: \text{ scattered wave vector}$$

Right-incident wave



In regions where $v(x, y) = 0$,

$$(-\nabla^2 + v)\psi = k^2\psi \quad \rightarrow \quad (\nabla^2 + k^2)\psi = 0$$

In regions where $v(x,y) = 0$,

$$(-\nabla^2 + v)\psi = k^2\psi \quad \rightarrow \quad (\nabla^2 + k^2)\psi = 0$$

Plane-wave solutions:

$$e^{\pm i\sqrt{k^2-p^2}\,x} e^{ipy} \quad \text{with} \quad p \in (-k,k)$$

Evanescent-wave solutions:

$$e^{\pm\sqrt{p^2-k^2}\,x} e^{ipy} \quad \text{with} \quad p \notin (-k,k)$$

In regions where $v(x,y) = 0$,

$$(-\nabla^2 + v)\psi = k^2\psi \quad \rightarrow \quad (\nabla^2 + k^2)\psi = 0$$

Plane-wave solutions:

$$e^{\pm i\sqrt{k^2-p^2}\,x} e^{ipy} \quad \text{with} \quad p \in (-k, k)$$

Evanescent-wave solutions:

$$e^{\pm\sqrt{p^2-k^2}\,x} e^{ipy} \quad \text{with} \quad p \notin (-k, k)$$

$$e^{\pm i\varpi(p)\,x} e^{ipy} \quad \text{with} \quad p \in \mathbb{R},$$

$$\varpi(p) := \begin{cases} \sqrt{k^2 - p^2} & \text{for} \quad |p| < k, \\ i\sqrt{p^2 - k^2} & \text{for} \quad |p| \geq k. \end{cases}$$

Bounded solutions: $\psi = \psi_{\text{os}} + \psi_{\text{ev}}$

$$\psi_{\text{os}}(x,y) = \begin{cases} \int_{-k}^{k} \frac{dp\, e^{ipy}}{4\pi^2\varpi(p)} [A_-(p) e^{i\varpi(p)x} + B_-(p) e^{-i\varpi(p)x}] & \text{for} \quad x \le a_-, \\ \int_{-k}^{k} \frac{dp\, e^{ipy}}{4\pi^2\varpi(p)} [A_+(p) e^{i\varpi(p)x} + B_+(p) e^{-i\varpi(p)x}] & \text{for} \quad x \ge a_+, \end{cases}$$

$$\psi_{\text{ev}}(x,y) = \begin{cases} \int_{|p|\ge k} \frac{dp\, e^{ipy}}{4\pi^2\varpi(p)} C_-(p) e^{|\varpi(p)|x} & \text{for} \quad x \le a_-, \\ \int_{|p|\ge k} \frac{dp\, e^{ipy}}{4\pi^2\varpi(p)} C_+(p) e^{-|\varpi(p)|x} & \text{for} \quad x \ge a_+, \end{cases}$$

Bounded solutions: $\psi = \psi_{\text{os}} + \psi_{\text{ev}}$

$$\psi_{\text{os}}(x,y) = \begin{cases} \int_{-k}^{k} \frac{dp\, e^{ipy}}{4\pi^2 \varpi(p)} [A_-(p) e^{i\varpi(p)x} + B_-(p) e^{-i\varpi(p)x}] & \text{for} \quad x \leq a_-, \\ \int_{-k}^{k} \frac{dp\, e^{ipy}}{4\pi^2 \varpi(p)} [A_+(p) e^{i\varpi(p)x} + B_+(p) e^{-i\varpi(p)x}] & \text{for} \quad x \geq a_+, \end{cases}$$

$$\psi_{\text{ev}}(x,y) = \begin{cases} \int_{|p|\geq k} \frac{dp\, e^{ipy}}{4\pi^2 \varpi(p)} C_-(p) e^{|\varpi(p)|x} & \text{for} \quad x \leq a_-, \\ \int_{|p|\geq k} \frac{dp\, e^{ipy}}{4\pi^2 \varpi(p)} C_+(p) e^{-|\varpi(p)|x} & \text{for} \quad x \geq a_+, \end{cases}$$

$$A_\pm(p) = B_\pm(p) = 0 \quad \text{for} \quad |p| \geq k, \qquad C_\pm(p) = 0 \quad \text{for} \quad |p| < k.$$

Bounded solutions: $\psi = \psi_{\text{os}} + \psi_{\text{ev}}$

$$\psi_{\text{os}}(x,y) = \begin{cases} \int_{-k}^{k} \frac{dp\, e^{ipy}}{4\pi^2\varpi(p)}[A_-(p)e^{i\varpi(p)x} + B_-(p)e^{-i\varpi(p)x}] & \text{for} \quad x \le a_-, \\ \int_{-k}^{k} \frac{dp\, e^{ipy}}{4\pi^2\varpi(p)}[A_+(p)e^{i\varpi(p)x} + B_+(p)e^{-i\varpi(p)x}] & \text{for} \quad x \ge a_+, \end{cases}$$

$$\psi_{\text{ev}}(x,y) = \begin{cases} \int_{|p|\ge k} \frac{dp\, e^{ipy}}{4\pi^2\varpi(p)} C_-(p)e^{|\varpi(p)|x} & \text{for} \quad x \le a_-, \\ \int_{|p|\ge k} \frac{dp\, e^{ipy}}{4\pi^2\varpi(p)} C_+(p)e^{-|\varpi(p)|x} & \text{for} \quad x \ge a_+, \end{cases}$$

$$A_\pm(p) = B_\pm(p) = 0 \quad \text{for} \quad |p| \ge k, \qquad C_\pm(p) = 0 \quad \text{for} \quad |p| < k.$$

$\mathscr{F} :=$ Set of complex-valued functions of p,

$\mathscr{F}_k := \{\phi \in \mathscr{F} \mid \phi(p) = 0 \text{ for } p \notin (-k,k)\}$

$$A_\pm, B_\pm \in \mathscr{F}_k$$

Bounded solutions: $\psi = \psi_{\rm os} + \psi_{\rm ev}$

$$\psi_{\rm os}(x,y) = \begin{cases} \int_{-k}^{k} \frac{dp\, e^{ipy}}{4\pi^2\varpi(p)}[A_-(p)e^{i\varpi(p)x} + B_-(p)e^{-i\varpi(p)x}] & \text{for} \quad x \le a_-, \\ \int_{-k}^{k} \frac{dp\, e^{ipy}}{4\pi^2\varpi(p)}[A_+(p)e^{i\varpi(p)x} + B_+(p)e^{-i\varpi(p)x}] & \text{for} \quad x \ge a_+, \end{cases}$$

$$\psi_{\rm ev}(x,y) = \begin{cases} \int_{|p|\ge k} \frac{dp\, e^{ipy}}{4\pi^2\varpi(p)}\, C_-(p)e^{|\varpi(p)|x} & \text{for} \quad x \le a_-, \\ \int_{|p|\ge k} \frac{dp\, e^{ipy}}{4\pi^2\varpi(p)}\, C_+(p)e^{-|\varpi(p)|x} & \text{for} \quad x \ge a_+, \end{cases}$$

$$A_\pm, B_\pm \in \mathscr{F}_k := \left\{\phi \in \mathscr{F} \;\middle|\; \phi(p) = 0 \text{ for } p \notin (-k,k)\right\}$$

$$\psi(x,y) \to \int_{-k}^{k} \frac{dp\, e^{ipy}}{4\pi^2\varpi(p)}\left[A_\pm(p)e^{i\varpi(p)x} + B_\pm(p)e^{-i\varpi(p)x}\right] \text{ for } x \to \pm\infty$$

Bounded solutions: $\psi = \psi_{\text{os}} + \psi_{\text{ev}}$

$$\psi_{\text{os}}(x,y) = \begin{cases} \int_{-k}^{k} \frac{dp\, e^{ipy}}{4\pi^2\varpi(p)}[A_-(p)e^{i\varpi(p)x} + B_-(p)e^{-i\varpi(p)x}] & \text{for} \quad x \le a_-, \\ \int_{-k}^{k} \frac{dp\, e^{ipy}}{4\pi^2\varpi(p)}[A_+(p)e^{i\varpi(p)x} + B_+(p)e^{-i\varpi(p)x}] & \text{for} \quad x \ge a_+, \end{cases}$$

$$\psi_{\text{ev}}(x,y) = \begin{cases} \int_{|p|\ge k} \frac{dp\, e^{ipy}}{4\pi^2\varpi(p)}\, C_-(p)e^{|\varpi(p)|x} & \text{for} \quad x \le a_-, \\ \int_{|p|\ge k} \frac{dp\, e^{ipy}}{4\pi^2\varpi(p)}\, C_+(p)e^{-|\varpi(p)|x} & \text{for} \quad x \ge a_+, \end{cases}$$

$$A_\pm, B_\pm \in \mathscr{F}_k := \{\phi \in \mathscr{F} \mid \phi(p) = 0 \text{ for } p \notin (-k,k)\}$$

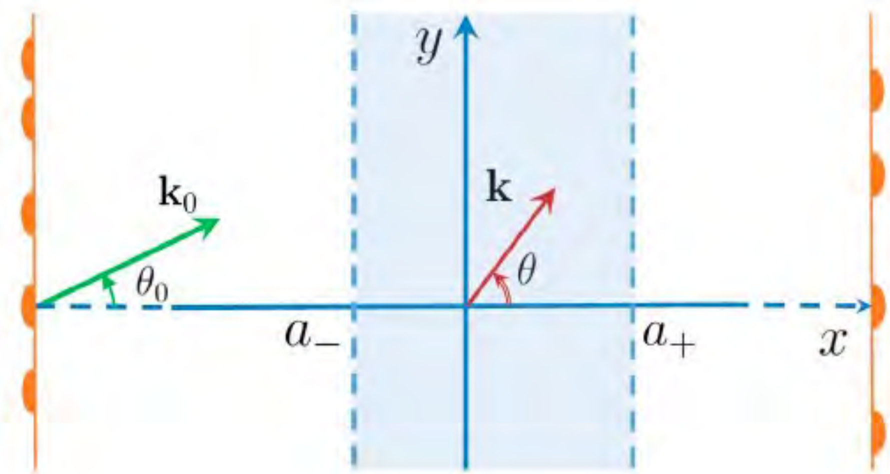
$$\psi(x,y) \to \int_{-k}^{k} \frac{dp\, e^{ipy}}{4\pi^2\varpi(p)}\left[A_\pm(p)e^{i\varpi(p)x} + B_\pm(p)e^{-i\varpi(p)x}\right] \text{ for } x \to \pm\infty$$

Fundamental transfer matrix: $\widehat{\mathrm{M}} : \mathscr{F}_k^2 \to \mathscr{F}_k^2$,

$$\widehat{\mathrm{M}} \begin{bmatrix} A_- \\ B_- \end{bmatrix} = \begin{bmatrix} A_+ \\ B_+ \end{bmatrix}$$

$$\psi(x,y) \to \int_{-k}^{k} \frac{dp\, e^{ipy}}{4\pi^2 \varpi(p)} \left[A_\pm(p) e^{i\varpi(p)x} + B_\pm(p) e^{-i\varpi(p)x} \right] \text{ for } x \to \pm\infty$$

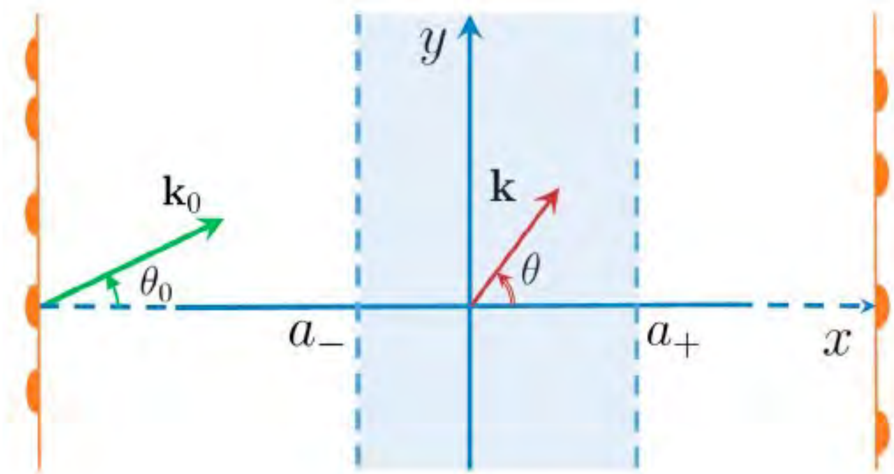
For left-incident waves: $\mathbf{k}_0 = (\varpi(p_0), p_0), \quad p_0 \in (-k, k)$



$$\psi(x,y) \to \int_{-k}^{k} \frac{dp\, e^{ipy}}{4\pi^2 \varpi(p)} \left[A_\pm(p) e^{i\varpi(p)x} + B_\pm(p) e^{-i\varpi(p)x}\right] \quad \text{for} \quad x \to \pm\infty$$

For left-incident waves: $\mathbf{k}_0 = (\varpi(p_0), p_0), \quad p_0 \in (-k, k)$

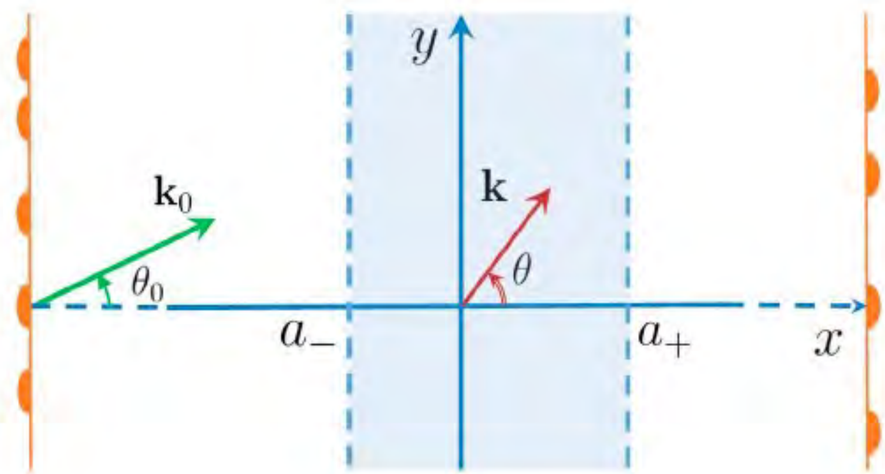
$$\begin{aligned} A_-(p) &= 2\pi\, \varpi(p_0)\, \delta(p - p_0) \\ &= 2\pi \delta(\theta - \theta_0), \\ B_+(p) &= 0. \end{aligned}$$



$$\psi(x,y) \to \int_{-k}^{k} \frac{dp\, e^{ipy}}{4\pi^2\varpi(p)} \left[A_\pm(p)e^{i\varpi(p)x} + B_\pm(p)e^{-i\varpi(p)x}\right] \quad \text{for} \quad x \to \pm\infty$$

For left-incident waves: $\mathbf{k}_0 = (\varpi(p_0), p_0), \quad p_0 \in (-k, k)$

$$\mathfrak{f}(\theta) = -\frac{i}{\sqrt{2\pi}} \times \begin{cases} A_+(k\sin\theta) - 2\pi\delta(\theta - \theta_0) & \text{for} \quad \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}), \\ B_-(k\sin\theta) & \text{for} \quad \theta \in (\frac{\pi}{2}, \frac{3\pi}{2}), \end{cases}$$



[PRA **93**, 042707 (2016) & **104**, 032222 (2021)]

$$\psi(x,y)\to\int_{-k}^{k}\frac{dp\,e^{ipy}}{4\pi^2\varpi(p)}\Big[A_\pm(p)e^{i\varpi(p)x}+B_\pm(p)e^{-i\varpi(p)x}\Big]\quad\text{for}\quad x\to\pm\infty$$

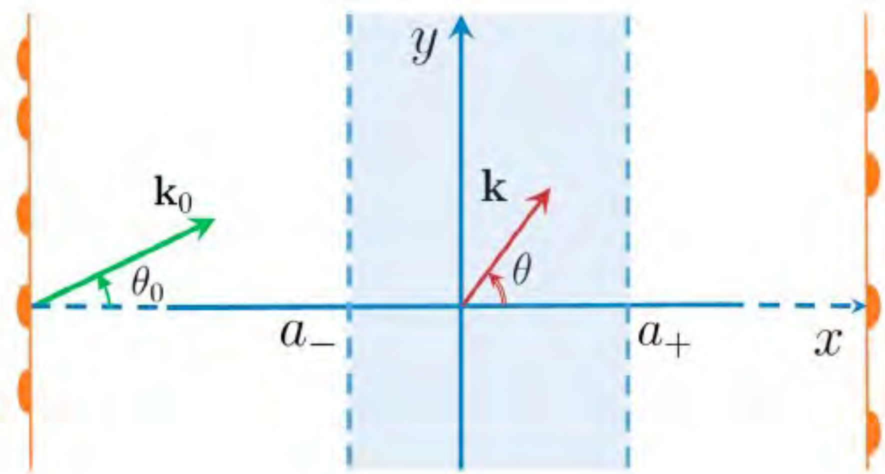
For left-incident waves: $\mathbf{k}_0=(\varpi(p_0),p_0),\quad p_0\in(-k,k)$

$$\mathfrak{f}(\theta)=-\frac{i}{\sqrt{2\pi}}\times\begin{cases}A_+(k\sin\theta)-2\pi\delta(\theta-\theta_0) & \text{for}\quad \theta\in(-\frac{\pi}{2},\frac{\pi}{2}),\\ B_-(k\sin\theta) & \text{for}\quad \theta\in(\frac{\pi}{2},\frac{3\pi}{2}),\end{cases}$$

$$\widehat{M}_{22}\,B_-=-2\pi\widehat{M}_{21}\delta_{p_0},$$

$$A_+=2\pi\widehat{M}_{11}\,\delta_{p_0}+\widehat{M}_{12}B_-$$

$$\delta_{p_0}(p):=\delta(p-p_0)$$



[PRA **93**, 042707 (2016) & **104**, 032222 (2021)]

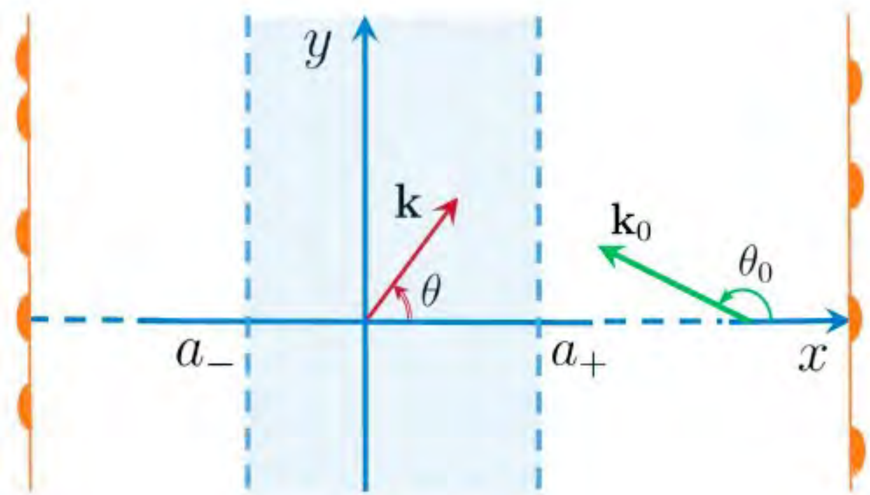
$$\psi(x,y) \to \int_{-k}^{k} \frac{dp\, e^{ipy}}{4\pi^2 \varpi(p)} \left[A_\pm(p) e^{i\varpi(p)x} + B_\pm(p) e^{-i\varpi(p)x}\right] \quad \text{for} \quad x \to \pm\infty$$

For right-incident waves: $\mathbf{k}_0 = (-\varpi(p_0), p_0), \quad p_0 \in (-k, k)$

$$\mathfrak{f}(\theta) = \frac{i}{\sqrt{2\pi}} \times \begin{cases} A_+(k\sin\theta) & \text{for} \quad \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}), \\ B_-(k\sin\theta) - 2\pi\delta(\theta - \theta_0) & \text{for} \quad \theta \in (\frac{\pi}{2}, \frac{3\pi}{2}), \end{cases}$$

$$\widehat{M}_{22} B_- = 2\pi \delta_{p_0},$$

$$A_+ = \widehat{M}_{12} B_-$$



Projection onto $\mathscr{F}_k^2$: $(\widehat{\mathbf{\Pi}}_k F)(p) := \begin{cases} F(p) & \text{for} \quad p \in (-k,k) \\ 0 & \text{for} \quad p \notin (-k,k) \end{cases}$

Projection onto $\mathscr{F}_k^2$: $(\widehat{\mathbf{\Pi}}_k F)(p) := \left\{ \begin{array}{ccc} F(p) & \text{for} & p \in (-k,k) \\ 0 & \text{for} & p \notin (-k,k) \end{array} \right.$

Theorem: $\widehat{\mathbf{M}} = \widehat{\mathbf{\Pi}}_k \, \widehat{\mathfrak{M}} \, \widehat{\mathbf{\Pi}}_k$, where $\widehat{\mathfrak{M}} := \widehat{\boldsymbol{\mathcal{U}}}(-\infty,\infty)$,

$$\widehat{\boldsymbol{\mathcal{U}}}(x,x_0) := \mathscr{T} \exp\left\{-i \int_{x_0}^{x} dx' \widehat{\boldsymbol{\mathcal{H}}}(x')\right\},$$

[PRA **104**, 03222 (2021)]

Projection onto $\mathscr{F}_k^2$: $(\widehat{\mathbf{\Pi}}_k F)(p) := \left\{ \begin{array}{ccc} F(p) & \text{for} & p \in (-k, k) \\ 0 & \text{for} & p \notin (-k, k) \end{array} \right.$

Theorem: $\widehat{\mathbf{M}} = \widehat{\mathbf{\Pi}}_k \, \widehat{\mathfrak{M}} \, \widehat{\mathbf{\Pi}}_k$, where $\widehat{\mathfrak{M}} := \widehat{\boldsymbol{\mathcal{U}}}(-\infty, \infty)$,

$$\widehat{\boldsymbol{\mathcal{U}}}(x, x_0) := \mathscr{T} \exp\left\{-i \int_{x_0}^{x} dx' \widehat{\boldsymbol{\mathcal{H}}}(x')\right\},$$

$$\widehat{\boldsymbol{\mathcal{H}}}(x) := \frac{1}{2} e^{-i\widehat{\varpi} x \boldsymbol{\sigma}_3} \, v(x, \widehat{y}) \, \widehat{\varpi}^{-1} \boldsymbol{\mathcal{K}} \, e^{i\widehat{\varpi} x \boldsymbol{\sigma}_3},$$

$$(\widehat{\varpi} f)(p) := \varpi(p) \, f(p), \qquad \boldsymbol{\sigma}_3 := \left[\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right],$$

$$\varpi(p) := \left\{ \begin{array}{ccc} \sqrt{k^2 - p^2} & \text{for} & |p| < k, \\ i\sqrt{p^2 - k^2} & \text{for} & |p| \geq k, \end{array} \right.$$

$$(\widehat{y} f)(p) := i\partial_p f(p), \qquad \boldsymbol{\mathcal{K}} := \left[\begin{array}{cc} 1 & 1 \\ -1 & -1 \end{array}\right].$$

[PRA **104**, 03222 (2021)]

Projection onto $\mathscr{F}^2_k$: $(\widehat{\mathbf{\Pi}}_k F)(p) := \begin{cases} F(p) & \text{for} \quad p \in (-k,k) \\ 0 & \text{for} \quad p \notin (-k,k) \end{cases}$

Theorem: $\widehat{\mathbf{M}} = \widehat{\mathbf{\Pi}}_k \, \widehat{\mathfrak{M}} \, \widehat{\mathbf{\Pi}}_k$, where $\widehat{\mathfrak{M}} := \widehat{\boldsymbol{\mathcal{U}}}(-\infty, \infty)$,

$$\widehat{\boldsymbol{\mathcal{U}}}(x, x_0) := \mathscr{T} \exp\left\{ -i \int_{x_0}^{x} dx' \widehat{\boldsymbol{\mathcal{H}}}(x') \right\},$$

$$\widehat{\boldsymbol{\mathcal{H}}}(x) := \frac{1}{2} e^{-i\widehat{\varpi} x \boldsymbol{\sigma}_3} \, v(x, \widehat{y}) \, \widehat{\varpi}^{-1} \boldsymbol{\mathcal{K}} \, e^{i\widehat{\varpi} x \boldsymbol{\sigma}_3}.$$

$$\big(v(x, \widehat{y}) f\big)(p) := \frac{1}{2\pi} \int_{-\infty}^{\infty} dq \, \tilde{v}(x, p-q) f(q),$$

$$\tilde{v}(x, p) := \int_{-\infty}^{\infty} dy \, e^{-ipy} v(x, y).$$

$$\widehat{\mathbf{M}} := \widehat{\mathbf{\Pi}}_k \mathscr{T} \exp\left\{-i \int_{-\infty}^{\infty} dx \widehat{\boldsymbol{\mathcal{H}}}(x)\right\} \widehat{\mathbf{\Pi}}_k$$
$$= \widehat{\mathbf{I}} - i \int_{-\infty}^{\infty} dx_1 \widehat{\mathbf{\Pi}}_k \widehat{\boldsymbol{\mathcal{H}}}(x_1) \widehat{\mathbf{\Pi}}_k$$
$$- \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{x_2} dx_1 \widehat{\mathbf{\Pi}}_k \widehat{\boldsymbol{\mathcal{H}}}(x_2) \widehat{\boldsymbol{\mathcal{H}}}(x_1) \widehat{\mathbf{\Pi}}_k + \cdots$$

$$(\widehat{\mathbf{\Pi}}_k F)(p) := \begin{cases} F(p) & \text{for} \quad p \in (-k, k) \\ 0 & \text{for} \quad p \notin (-k, k) \end{cases}$$

$$\widehat{\boldsymbol{\mathcal{H}}}(x) := \frac{1}{2} e^{-i\widehat{\varpi} x \boldsymbol{\sigma}_3} v(x, \widehat{y}) \, \widehat{\varpi}^{-1} \boldsymbol{\mathcal{K}} \, e^{i\widehat{\varpi} x \boldsymbol{\sigma}_3},$$

$$\boldsymbol{\mathcal{K}} := \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

[PRA **104**, 03222 (2021)]

Hybrid Rep:

For each $x \in \mathbb{R}$, $|\psi(x)\rangle : \mathbb{R} \to \mathbb{C}$ is the function:

$$\langle y|\psi(x)\rangle := \psi(x, y)$$

$$[-\partial_x^2 + \widehat{p}^2 + v(x, \widehat{y})]|\psi(x)\rangle = k^2|\psi(x)\rangle$$

$$\langle y|\widehat{y}|\phi\rangle = y\langle y|\phi\rangle, \qquad \langle y|\widehat{p}|\phi\rangle := -i\partial_y\langle y|\phi\rangle$$

$$\widehat{\Pi}_k := \int_{-k}^{k} dp\, |p\rangle\langle p|, \qquad \langle y|p\rangle = \frac{e^{ipy}}{\sqrt{2\pi}}$$

$$\langle p|\widehat{\Pi}_k|\phi\rangle = \begin{cases} \langle p|\phi\rangle & \text{for} \quad p \in (-k, k) \\ 0 & \text{for} \quad p \notin (-k, k) \end{cases}$$

$$\widehat{\Pi}_k := \int_{-k}^{k} dp\,|p\rangle\langle p|, \qquad \langle p|\widehat{\Pi}_k|\phi\rangle = \begin{cases} \langle p|\phi\rangle & \text{for} \quad p \in (-k,k) \\ 0 & \text{for} \quad p \notin (-k,k) \end{cases}$$

$$\psi = \psi_{\text{os}} + \psi_{\text{ev}}$$

$$\psi_{\text{os}}(x,y) = \begin{cases} \int_{-k}^{k} \frac{dp\, e^{ipy}}{4\pi^2\varpi(p)}[A_-(p)e^{i\varpi(p)x} + B_-(p)e^{-i\varpi(p)x}] & \text{for} \quad x \le a_-, \\ \int_{-k}^{k} \frac{dp\, e^{ipy}}{4\pi^2\varpi(p)}[A_+(p)e^{i\varpi(p)x} + B_+(p)e^{-i\varpi(p)x}] & \text{for} \quad x \ge a_+, \end{cases}$$

$$\psi_{\text{ev}}(x,y) = \begin{cases} \int_{|p|\ge k} \frac{dp\, e^{ipy}}{4\pi^2\varpi(p)}\, C_-(p)e^{|\varpi(p)|x} & \text{for} \quad x \le a_-, \\ \int_{|p|\ge k} \frac{dp\, e^{ipy}}{4\pi^2\varpi(p)}\, C_+(p)e^{-|\varpi(p)|x} & \text{for} \quad x \ge a_+, \end{cases}$$

$$A_\pm(p) = B_\pm(p) = 0 \quad \text{for} \quad |p| \ge k, \qquad C_\pm(p) = 0 \quad \text{for} \quad |p| < k.$$

$$\widehat{\Pi}_k|A_\pm\rangle = |A_\pm\rangle, \qquad \widehat{\Pi}_k|B_\pm\rangle = |B_\pm\rangle, \qquad \widehat{\Pi}_k|C_\pm\rangle = 0.$$

$$\widehat{\Pi}_k := \int_{-k}^{k} dp\,|p\rangle\langle p|, \qquad \langle p|\widehat{\Pi}_k|\phi\rangle = \begin{cases} \langle p|\phi\rangle & \text{for} \quad p \in (-k,k) \\ 0 & \text{for} \quad p \notin (-k,k) \end{cases}$$

$$\psi = \psi_{\rm os} + \psi_{\rm ev}$$

$$\psi_{\rm os}(x,y) = \begin{cases} \int_{-k}^{k} \frac{dp\,e^{ipy}}{4\pi^2\varpi(p)}[A_-(p)e^{i\varpi(p)x} + B_-(p)e^{-i\varpi(p)x}] & \text{for} \quad x \le a_-, \\ \int_{-k}^{k} \frac{dp\,e^{ipy}}{4\pi^2\varpi(p)}[A_+(p)e^{i\varpi(p)x} + B_+(p)e^{-i\varpi(p)x}] & \text{for} \quad x \ge a_+, \end{cases}$$

$$\psi_{\rm ev}(x,y) = \begin{cases} \int_{|p|\ge k} \frac{dp\,e^{ipy}}{4\pi^2\varpi(p)}\, C_-(p)e^{|\varpi(p)|x} & \text{for} \quad x \le a_-, \\ \int_{|p|\ge k} \frac{dp\,e^{ipy}}{4\pi^2\varpi(p)}\, C_+(p)e^{-|\varpi(p)|x} & \text{for} \quad x \ge a_+, \end{cases}$$

$$A_\pm(p) = B_\pm(p) = 0 \quad \text{for} \quad |p| \ge k, \qquad C_\pm(p) = 0 \quad \text{for} \quad |p| < k.$$

$$\widehat{\Pi}_k|A_\pm\rangle = |A_\pm\rangle, \qquad \widehat{\Pi}_k|B_\pm\rangle = |B_\pm\rangle, \qquad \widehat{\Pi}_k|C_\pm\rangle = 0.$$

$$|\psi_{\rm os}(x)\rangle := \widehat{\Pi}_k|\psi(x)\rangle \qquad |\psi_{\rm ev}(x)\rangle := |\psi(x)\rangle - |\psi_{\rm os}(x)\rangle$$

$$\widehat{\Pi}_k := \int_{-k}^{k} dp\,|p\rangle\langle p|, \qquad \psi = \psi_{\text{os}} + \psi_{\text{ev}}$$

$$|\psi_{\text{os}}(x)\rangle := \widehat{\Pi}_k|\psi(x)\rangle \qquad |\psi_{\text{ev}}(x)\rangle := |\psi(x)\rangle - |\psi_{\text{os}}(x)\rangle$$

Propagating-wave approximation (PWA): $\psi \approx \psi_{\text{os}}$

PWA neglects the contribution of the evanescent waves to the scattering amplitude.

[a[arXiv: 2204.05153]

$$\widehat{\Pi}_k := \int_{-k}^{k} dp\, |p\rangle\langle p|, \qquad \psi = \psi_{\rm os} + \psi_{\rm ev}$$

$$|\psi_{\rm os}(x)\rangle := \widehat{\Pi}_k |\psi(x)\rangle \qquad |\psi_{\rm ev}(x)\rangle := |\psi(x)\rangle - |\psi_{\rm os}(x)\rangle$$

Propagating-wave approximation (PWA): $\psi \approx \psi_{\rm os}$

PWA neglects the contribution of the evanescent waves to the scattering amplitude.

$$[-\partial_x^2 + \widehat{p}^{\,2} + v(x, \widehat{y})]|\psi(x)\rangle = k^2|\psi(x)\rangle$$

$$\Rightarrow \left[-\partial_x^2 + \widehat{p}^2 + \widehat{V}_k(x)\right]|\psi_{\rm os}(x)\rangle + \widehat{W}_k(x)|\psi_{\rm ev}(x)\rangle = k^2|\psi_{\rm os}(x)\rangle,$$

$$\widehat{V}_k(x) := \widehat{\Pi}_k v(x, \widehat{y})\widehat{\Pi}_k, \qquad \widehat{W}_k(x) := \widehat{\Pi}_k v(x, \widehat{y})(\widehat{1} - \widehat{\Pi}_k).$$

[arXiv: 2204.05153]

$$\widehat{\Pi}_k := \int_{-k}^{k} dp\,|p\rangle\langle p|, \qquad \psi = \psi_{\rm os} + \psi_{\rm ev}$$

$$|\psi_{\rm os}(x)\rangle := \widehat{\Pi}_k|\psi(x)\rangle \qquad |\psi_{\rm ev}(x)\rangle := |\psi(x)\rangle - |\psi_{\rm os}(x)\rangle$$

Propagating-wave approximation (PWA): $\psi \approx \psi_{\rm os}$

PWA neglects the contribution of the evanescent waves to the scattering amplitude.

$$[-\partial_x^2 + \widehat{p}^{\,2} + v(x,\widehat{y})]|\psi(x)\rangle = k^2|\psi(x)\rangle$$

$$\Rightarrow \left[-\partial_x^2 + \widehat{p}^2 + \widehat{V}_k(x)\right]|\psi_{\rm os}(x)\rangle + \widehat{W}_k(x)|\psi_{\rm ev}(x)\rangle = k^2|\psi_{\rm os}(x)\rangle,$$

$$\widehat{V}_k(x) := \widehat{\Pi}_k v(x,\widehat{y})\widehat{\Pi}_k, \qquad \widehat{W}_k(x) := \widehat{\Pi}_k v(x,\widehat{y})(\widehat{1} - \widehat{\Pi}_k).$$

PWA: $\big[-\partial_x^2 + \widehat{p}^2 + \widehat{V}_k(x)\big]|\psi(x)\rangle = k^2|\psi(x)\rangle$ i.e., $v \longrightarrow \mathscr{V}_k$

$$\big(\mathscr{V}_k\psi\big)(x,y) := \langle y|\widehat{V}_k(x)|\psi(x)\rangle$$

[arXiv: 2204.05153]

$$\left[-\partial_x^2 + \widehat{p}^2 + \widehat{V}_k(x)\right]|\psi_{\mathsf{os}}(x)\rangle + \widehat{W}_k(x)|\psi_{\mathsf{ev}}(x)\rangle = k^2|\psi_{\mathsf{os}}(x)\rangle$$

$$\widehat{V}_k(x) := \widehat{\Pi}_k v(x,\widehat{y})\widehat{\Pi}_k, \qquad \widehat{W}_k(x) := \widehat{\Pi}_k v(x,\widehat{y})(\widehat{1} - \widehat{\Pi}_k).$$

PWA : $-\nabla^2\psi(x,y) + \big(\mathscr{V}_k\psi\big)(x,y) = k^2\psi(x,y)$

$$\big(\mathscr{V}_k\psi\big)(x,y) := \langle y|\widehat{V}_k(x)|\psi(x)\rangle$$

[arXiv: 2204.05153]

$$\left[-\partial_x^2 + \widehat{p}^2 + \widehat{V}_k(x)\right] |\psi_{\rm os}(x)\rangle + \widehat{W}_k(x)|\psi_{\rm ev}(x)\rangle = k^2|\psi_{\rm os}(x)\rangle$$

$$\widehat{V}_k(x) := \widehat{\Pi}_k v(x,\widehat{y})\widehat{\Pi}_k, \qquad \widehat{W}_k(x) := \widehat{\Pi}_k v(x,\widehat{y})(\widehat{1} - \widehat{\Pi}_k).$$

PWA : $-\nabla^2\psi(x,y) + (\mathscr{V}_k\psi)(x,y) = k^2\psi(x,y)$

$$(\mathscr{V}_k\psi)(x,y) := \langle y|\widehat{V}_k(x)|\psi(x)\rangle$$

$$= \frac{1}{4\pi^2}\int_{-k}^{k} dp \int_{-k}^{k} dq\, e^{iyp}\, \tilde{v}(x,p-q)\tilde{\psi}(x,q)$$

$$\tilde{v}(x,p) := \int_{-\infty}^{\infty} dy\, e^{-ipy}\, v(x,y)$$

$\mathscr{V}_k$ is an energy-dependent nonlocal potential.

[arXiv: 2204.05153]

$$\left[-\partial_x^2+\widehat{p}^2+\widehat{V}_k(x)\right]|\psi_{\rm os}(x)\rangle+\widehat{W}_k(x)|\psi_{\rm ev}(x)\rangle=k^2|\psi_{\rm os}(x)\rangle$$

$$\widehat{V}_k(x):=\widehat{\Pi}_k v(x,\widehat{y})\widehat{\Pi}_k,\qquad \widehat{W}_k(x):=\widehat{\Pi}_k v(x,\widehat{y})(\widehat{1}-\widehat{\Pi}_k).$$

PWA : $-\nabla^2\psi(x,y)+(\mathscr{V}_k\psi)(x,y)=k^2\psi(x,y)$

$$(\mathscr{V}_k\psi)(x,y):=\langle y|\widehat{V}_k(x)|\psi(x)\rangle$$

For $k\to\infty$, $\widehat{\Pi}_k:=\int_{-k}^{k}dp\,|p\rangle\langle p|\to\widehat{1}$

$\Rightarrow$ $\widehat{V}_k(x)\to v(x,\widehat{y})$ & $\widehat{W}_k(x)\to\widehat{0}$

$\Rightarrow$ **PWA** is valid at high energies.

[arXiv: 2204.05153]

$$\left[-\partial_x^2 + \hat{p}^2 + \widehat{V}_k(x)\right] |\psi_{\rm os}(x)\rangle + \widehat{W}_k(x)|\psi_{\rm ev}(x)\rangle = k^2|\psi_{\rm os}(x)\rangle$$

$$\widehat{V}_k(x) := \widehat{\Pi}_k v(x,\hat{y})\widehat{\Pi}_k, \qquad \widehat{W}_k(x) := \widehat{\Pi}_k v(x,\hat{y})(\hat{1} - \widehat{\Pi}_k).$$

Theorem: Every short-range potential $v : \mathbb{R}^2 \to \mathbb{C}$ couples to evanescent waves, i.e., $\widehat{W}_k(x) \neq \hat{0}$.

Nevertheless, PWA can produce the exact expression for the scattering amplitudes!

[arXiv: 2204.05153]

Transfer matrix for $\mathscr{V}_k$:

Recall: $\widehat{\mathbf{M}} : \mathscr{F}_k^2 \to \mathscr{F}_k^2$,

$$F \in \mathscr{F}_k^2 \quad \Leftrightarrow \quad F(p) = 0 \ \text{ for } \ p \notin (-k, k)$$

Transfer matrix for $\mathscr{V}_k$:

Recall: $\widehat{\mathrm{M}} : \mathscr{F}_k^2 \to \mathscr{F}_k^2$,

$$\widehat{\mathrm{M}} := \widehat{\Pi}_k \mathscr{T} \exp \left\{ -i \int_{-\infty}^{\infty} dx \widehat{\boldsymbol{\mathcal{H}}}(x) \right\} \widehat{\Pi}_k$$

$$\widehat{\boldsymbol{\mathcal{H}}}(x) := \tfrac{1}{2}\, e^{-i\widehat{\varpi} x \boldsymbol{\sigma}_3}\, v(x, \widehat{y})\, \widehat{\varpi}^{-1} \boldsymbol{\mathcal{K}}\, e^{i\widehat{\varpi} x \boldsymbol{\sigma}_3}$$

Transfer matrix for $\mathscr{V}_k$:

Recall: $\widehat{\mathbf{M}} : \mathscr{F}_k^2 \to \mathscr{F}_k^2$,

$$\widehat{\mathbf{M}} := \widehat{\Pi}_k \mathscr{T} \exp\left\{-i\int_{-\infty}^{\infty} dx \widehat{\boldsymbol{\mathcal{H}}}(x)\right\} \widehat{\Pi}_k$$

$$\widehat{\boldsymbol{\mathcal{H}}}(x) := \tfrac{1}{2}\, e^{-i\widehat{\varpi} x \boldsymbol{\sigma}_3}\, v(x,\widehat{y})\, \widehat{\varpi}^{-1} \boldsymbol{\mathcal{K}}\, e^{i\widehat{\varpi} x \boldsymbol{\sigma}_3}$$

PWA : $v \longrightarrow \mathscr{V}_k \quad \Leftrightarrow \quad v(x,\widehat{y}) \longrightarrow \widehat{V}_k(x) := \widehat{\Pi}_k v(x,\widehat{y}) \widehat{\Pi}_k$

Transfer matrix for $\mathscr{V}_k$:

Recall: $\widehat{\mathbf{M}} : \mathscr{F}_k^2 \to \mathscr{F}_k^2$,

$$\widehat{\mathbf{M}} := \widehat{\boldsymbol{\Pi}}_k \mathscr{T} \exp\left\{-i\int_{-\infty}^{\infty} dx \widehat{\boldsymbol{\mathcal{H}}}(x)\right\} \widehat{\boldsymbol{\Pi}}_k$$

$$\widehat{\boldsymbol{\mathcal{H}}}(x) := \tfrac{1}{2}\, e^{-i\widehat{\varpi} x \boldsymbol{\sigma}_3}\, v(x,\widehat{y})\, \widehat{\varpi}^{-1} \boldsymbol{\mathcal{K}}\, e^{i\widehat{\varpi} x \boldsymbol{\sigma}_3}$$

PWA : $v \longrightarrow \mathscr{V}_k \quad \Leftrightarrow \quad v(x,\widehat{y}) \longrightarrow \widehat{V}_k(x) := \widehat{\Pi}_k v(x,\widehat{y}) \widehat{\Pi}_k$

$$\Rightarrow \quad \widehat{\boldsymbol{\Pi}}_k \widehat{V}_k(x) \widehat{\boldsymbol{\Pi}}_k = \widehat{V}_k(x)$$

Transfer matrix for $\mathscr{V}_k$:

Recall: $\widehat{\mathrm{M}}: \mathscr{F}_k^2 \to \mathscr{F}_k^2$,

$$\widehat{\mathrm{M}} := \widehat{\Pi}_k \mathscr{T} \exp\left\{-i\int_{-\infty}^{\infty} dx \widehat{\boldsymbol{\mathcal{H}}}(x)\right\}\widehat{\Pi}_k$$

$$\widehat{\boldsymbol{\mathcal{H}}}(x) := \tfrac{1}{2}\, e^{-i\widehat{\varpi} x \boldsymbol{\sigma}_3}\, v(x,\widehat{y})\, \widehat{\varpi}^{-1} \boldsymbol{\mathcal{K}}\, e^{i\widehat{\varpi} x \boldsymbol{\sigma}_3}$$

PWA : $v \longrightarrow \mathscr{V}_k \quad \Leftrightarrow \quad v(x,\widehat{y}) \longrightarrow \widehat{V}_k(x) := \widehat{\Pi}_k v(x,\widehat{y}) \widehat{\Pi}_k$

$$\Rightarrow \quad \widehat{\Pi}_k \widehat{V}_k(x) \widehat{\Pi}_k = \widehat{V}_k(x)$$

$$\widehat{\mathrm{M}} \longrightarrow \widehat{\mathrm{M}}_k := \widehat{\Pi}_k \mathscr{T} \exp\left\{-i\int_{-\infty}^{\infty} dx\, \widehat{\boldsymbol{\mathcal{H}}}_k(x)\right\}\widehat{\Pi}_k$$

$$\widehat{\boldsymbol{\mathcal{H}}}_k(x) := \tfrac{1}{2}\, e^{-i\widehat{\varpi} x \boldsymbol{\sigma}_3}\, \widehat{V}_k(x)\, \widehat{\varpi}^{-1} \boldsymbol{\mathcal{K}}\, e^{i\widehat{\varpi} x \boldsymbol{\sigma}_3} = \widehat{\Pi}_k \widehat{\boldsymbol{\mathcal{H}}}(x) \widehat{\Pi}_k$$

Transfer matrix for $\mathscr{V}_k$:

Recall: $\widehat{\mathbf{M}} : \mathscr{F}_k^2 \to \mathscr{F}_k^2$,

$$\widehat{\mathbf{M}} := \widehat{\Pi}_k \mathscr{T} \exp\left\{-i \int_{-\infty}^{\infty} dx \widehat{\mathcal{H}}(x)\right\} \widehat{\Pi}_k$$

$$\widehat{\mathcal{H}}(x) := \tfrac{1}{2}\, e^{-i\widehat{\varpi} x \boldsymbol{\sigma}_3}\, v(x,\widehat{y})\, \widehat{\varpi}^{-1} \boldsymbol{\mathcal{K}}\, e^{i\widehat{\varpi} x \boldsymbol{\sigma}_3}$$

PWA : $v \longrightarrow \mathscr{V}_k \quad \Leftrightarrow \quad v(x,\widehat{y}) \longrightarrow \widehat{V}_k(x) := \widehat{\Pi}_k v(x,\widehat{y}) \widehat{\Pi}_k$

$$\Rightarrow \quad \widehat{\Pi}_k \widehat{V}_k(x) \widehat{\Pi}_k = \widehat{V}_k(x)$$

$$\widehat{\mathbf{M}} \longrightarrow \widehat{\mathbf{M}}_k := \widehat{\Pi}_k \mathscr{T} \exp\left\{-i \int_{-\infty}^{\infty} dx\, \widehat{\mathcal{H}}_k(x)\right\} \widehat{\Pi}_k$$

$$\widehat{\mathcal{H}}_k(x) := \tfrac{1}{2}\, e^{-i\widehat{\varpi} x \boldsymbol{\sigma}_3}\, \widehat{V}_k(x)\, \widehat{\varpi}^{-1} \boldsymbol{\mathcal{K}}\, e^{i\widehat{\varpi} x \boldsymbol{\sigma}_3} = \widehat{\Pi}_k \widehat{\mathcal{H}}(x) \widehat{\Pi}_k$$

$$= \tfrac{1}{2}\, e^{-i\widehat{\varpi} x \boldsymbol{\sigma}_3}\, \widehat{\Pi}_k \widehat{V}_k(x) \widehat{\Pi}_k\, \widehat{\varpi}^{-1} \boldsymbol{\mathcal{K}}\, e^{i\widehat{\varpi} x \boldsymbol{\sigma}_3}$$

Transfer matrix for $\mathscr{V}_k$:

Recall: $\widehat{\mathrm{M}} : \mathscr{F}_k^2 \to \mathscr{F}_k^2$,

$$\widehat{\mathrm{M}} := \widehat{\Pi}_k \mathscr{T} \exp\left\{-i\int_{-\infty}^{\infty} dx \widehat{\mathcal{H}}(x)\right\} \widehat{\Pi}_k$$

$$\widehat{\mathcal{H}}(x) := \tfrac{1}{2} e^{-i\widehat{\varpi}x\boldsymbol{\sigma}_3} v(x,\widehat{y})\, \widehat{\varpi}^{-1}\boldsymbol{\mathcal{K}}\, e^{i\widehat{\varpi}x\boldsymbol{\sigma}_3}$$

PWA : $v \longrightarrow \mathscr{V}_k \quad \Leftrightarrow \quad v(x,\widehat{y}) \longrightarrow \widehat{V}_k(x) := \widehat{\Pi}_k v(x,\widehat{y}) \widehat{\Pi}_k$

$$\Rightarrow \quad \widehat{\Pi}_k \widehat{V}_k(x) \widehat{\Pi}_k = \widehat{V}_k(x)$$

$$\widehat{\mathrm{M}} \longrightarrow \widehat{\mathrm{M}}_k := \widehat{\Pi}_k \mathscr{T} \exp\left\{-i\int_{-\infty}^{\infty} dx\, \widehat{\mathcal{H}}_k(x)\right\} \widehat{\Pi}_k$$

$$\begin{aligned}\widehat{\mathcal{H}}_k(x) &:= \tfrac{1}{2} e^{-i\widehat{\varpi}x\boldsymbol{\sigma}_3}\, \widehat{V}_k(x)\, \widehat{\varpi}^{-1}\boldsymbol{\mathcal{K}}\, e^{i\widehat{\varpi}x\boldsymbol{\sigma}_3} = \widehat{\Pi}_k \widehat{\mathcal{H}}(x) \widehat{\Pi}_k \\ &= \tfrac{1}{2} e^{-i\widehat{\varpi}x\boldsymbol{\sigma}_3}\, \widehat{\Pi}_k \widehat{V}_k(x) \widehat{\Pi}_k\, \widehat{\varpi}^{-1}\boldsymbol{\mathcal{K}}\, e^{i\widehat{\varpi}x\boldsymbol{\sigma}_3} \\ &= \widehat{\Pi}_k \widehat{\mathcal{H}}_k(x) \widehat{\Pi}_k\end{aligned}$$

Transfer matrix for $\mathscr{V}_k$:

Recall: $\widehat{\mathrm{M}} : \mathscr{F}_k^2 \to \mathscr{F}_k^2$,

$$\widehat{\mathrm{M}} := \widehat{\Pi}_k \mathscr{T} \exp\left\{-i\int_{-\infty}^{\infty} dx \widehat{\mathcal{H}}(x)\right\} \widehat{\Pi}_k$$

$$\widehat{\mathcal{H}}(x) := \tfrac{1}{2}\, e^{-i\widehat{\varpi}x\boldsymbol{\sigma}_3}\, v(x,\widehat{y})\, \widehat{\varpi}^{-1}\boldsymbol{\mathcal{K}}\, e^{i\widehat{\varpi}x\boldsymbol{\sigma}_3}$$

PWA : $v \longrightarrow \mathscr{V}_k \quad \Leftrightarrow \quad v(x,\widehat{y}) \longrightarrow \widehat{V}_k(x) := \widehat{\Pi}_k v(x,\widehat{y})\widehat{\Pi}_k$

$$\Rightarrow \quad \widehat{\Pi}_k \widehat{V}_k(x)\widehat{\Pi}_k = \widehat{V}_k(x)$$

$$\widehat{\mathrm{M}} \longrightarrow \widehat{\mathrm{M}}_k := \widehat{\Pi}_k \mathscr{T} \exp\left\{-i\int_{-\infty}^{\infty} dx\, \widehat{\mathcal{H}}_k(x)\right\} \widehat{\Pi}_k$$

$$\begin{aligned}
\widehat{\mathcal{H}}_k(x) &:= \tfrac{1}{2}\, e^{-i\widehat{\varpi}x\boldsymbol{\sigma}_3}\, \widehat{V}_k(x)\, \widehat{\varpi}^{-1}\boldsymbol{\mathcal{K}}\, e^{i\widehat{\varpi}x\boldsymbol{\sigma}_3} = \widehat{\Pi}_k \widehat{\mathcal{H}}(x)\widehat{\Pi}_k \\
&= \tfrac{1}{2}\, e^{-i\widehat{\varpi}x\boldsymbol{\sigma}_3}\, \widehat{\Pi}_k \widehat{V}_k(x)\widehat{\Pi}_k\, \widehat{\varpi}^{-1}\boldsymbol{\mathcal{K}}\, e^{i\widehat{\varpi}x\boldsymbol{\sigma}_3} \\
&= \widehat{\Pi}_k \widehat{\mathcal{H}}_k(x)\widehat{\Pi}_k
\end{aligned}$$

$$\Rightarrow \quad \widehat{\mathrm{M}}_k := \mathscr{T} \exp\left\{-i\int_{-\infty}^{\infty} dx\, \widehat{\mathcal{H}}_k(x)\right\}$$

$$\widehat{\mathbf{M}} := \widehat{\mathbf{\Pi}}_k \mathscr{T} \exp\left\{-i\int_{-\infty}^{\infty} dx \widehat{\boldsymbol{\mathcal{H}}}(x)\right\}\widehat{\mathbf{\Pi}}_k$$

$$= \widehat{\mathbf{I}} - i\int_{-\infty}^{\infty} dx_1 \widehat{\mathbf{\Pi}}_k \widehat{\boldsymbol{\mathcal{H}}}(x_1)\widehat{\mathbf{\Pi}}_k$$

$$- \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{x_2} dx_1 \widehat{\mathbf{\Pi}}_k \widehat{\boldsymbol{\mathcal{H}}}(x_2)\widehat{\boldsymbol{\mathcal{H}}}(x_1)\widehat{\mathbf{\Pi}}_k + \cdots$$

$$\widehat{\boldsymbol{\mathcal{H}}}(x) := \frac{1}{2} e^{-i\widehat{\varpi} x \boldsymbol{\sigma}_3}\, v(x, \widehat{y})\, \widehat{\varpi}^{-1} \boldsymbol{\mathcal{K}}\, e^{i\widehat{\varpi} x \boldsymbol{\sigma}_3}$$

$$\widehat{\mathbf{\Pi}}_k \widehat{\boldsymbol{\mathcal{H}}}(x_1)\widehat{\mathbf{\Pi}}_k = \widehat{\boldsymbol{\mathcal{H}}}_k(x_1)$$

[arXiv: 2204.05153]

$$\widehat{\mathbf{M}} := \widehat{\mathbf{\Pi}}_k \mathscr{T} \exp\left\{-i\int_{-\infty}^{\infty} dx \widehat{\mathcal{H}}(x)\right\} \widehat{\mathbf{\Pi}}_k$$
$$= \widehat{\mathbf{I}} - i\int_{-\infty}^{\infty} dx_1 \widehat{\mathbf{\Pi}}_k \widehat{\mathcal{H}}(x_1) \widehat{\mathbf{\Pi}}_k$$
$$- \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{x_2} dx_1 \widehat{\mathbf{\Pi}}_k \widehat{\mathcal{H}}(x_2) \widehat{\mathcal{H}}(x_1) \widehat{\mathbf{\Pi}}_k + \cdots$$

$$\widehat{\mathcal{H}}(x) := \tfrac{1}{2}\, e^{-i\widehat{\varpi} x \boldsymbol{\sigma}_3}\, v(x, \widehat{y})\, \widehat{\varpi}^{-1} \boldsymbol{\mathcal{K}}\, e^{i\widehat{\varpi} x \boldsymbol{\sigma}_3}$$

$$\widehat{\mathbf{\Pi}}_k \widehat{\mathcal{H}}(x_1) \widehat{\mathbf{\Pi}}_k = \widehat{\mathcal{H}}_k(x_1)$$

If $\widehat{\mathcal{H}}(x_2)\widehat{\mathcal{H}}(x_1) \approx \widehat{0}$ for all $x_1, x_2 \in \mathbb{R}$, $\widehat{\mathbf{M}} \approx \widehat{\mathbf{M}}_k$.

$\Rightarrow$ **PWA** is valid for weak potentials.

[arXiv: 2204.05153]

$$
\begin{aligned}
\widehat{\mathbf{M}} &:= \widehat{\mathbf{\Pi}}_k \mathscr{T} \exp\left\{-i\int_{-\infty}^{\infty} dx\, \widehat{\boldsymbol{\mathcal{H}}}(x)\right\}\widehat{\mathbf{\Pi}}_k \\
&= \widehat{\mathbf{I}} - i\int_{-\infty}^{\infty} dx_1 \widehat{\mathbf{\Pi}}_k \widehat{\boldsymbol{\mathcal{H}}}(x_1)\widehat{\mathbf{\Pi}}_k \\
&\quad - \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{x_2} dx_1 \widehat{\mathbf{\Pi}}_k \widehat{\boldsymbol{\mathcal{H}}}(x_2)\widehat{\boldsymbol{\mathcal{H}}}(x_1)\widehat{\mathbf{\Pi}}_k + \cdots
\end{aligned}
$$

$$
\widehat{\boldsymbol{\mathcal{H}}}(x) := \tfrac{1}{2}\, e^{-i\widehat{\varpi} x \boldsymbol{\sigma}_3}\, v(x,\widehat{y})\, \widehat{\varpi}^{-1} \boldsymbol{\mathcal{K}}\, e^{i\widehat{\varpi} x \boldsymbol{\sigma}_3}
$$

$$
\widehat{\mathbf{\Pi}}_k \widehat{\boldsymbol{\mathcal{H}}}(x_1)\widehat{\mathbf{\Pi}}_k = \widehat{\boldsymbol{\mathcal{H}}}_k(x_1)
$$

If $\widehat{\boldsymbol{\mathcal{H}}}(x_2)\widehat{\boldsymbol{\mathcal{H}}}(x_1) \approx \widehat{0}$ for all $x_1, x_2 \in \mathbb{R}$, $\widehat{\mathbf{M}} \approx \widehat{\mathbf{M}}_k$.

$\Rightarrow$ **PWA** is valid for weak potentials.

For $v(x,y) = \delta(x)g(y)$, $\widehat{\boldsymbol{\mathcal{H}}}(x_2)\widehat{\boldsymbol{\mathcal{H}}}(x_1) = \widehat{0}$ & $\widehat{\mathbf{M}} = \widehat{\mathbf{M}}_k$

$\Rightarrow$ **PWA** is exact. [arXiv: 2204.05153]

A class of potentials for which PWA is exact:

Theorem: Let $v:\mathbb{R}^2\to\mathbb{C}$ be a potential such that

$\tilde{v}(x,p)=0$ for $p\leq 0$, or $\tilde{v}(x,p)=0$ for $p\geq 0$.

Then **PWA** is exact for v.

[arXiv: 2204.05153]

A class of potentials for which PWA is exact:

Theorem: Let $v : \mathbb{R}^2 \to \mathbb{C}$ be a potential such that

$\tilde{v}(x,p) = 0$ for $p \leq 0$, or $\tilde{v}(x,p) = 0$ for $p \geq 0$.

Then **PWA** is exact for v.

Example: Let $g : \mathbb{R} \to \mathbb{C}$ be a function such that

$$x\, g(x) \to 0 \quad \text{for} \quad x \to \pm\infty,$$

and

$$v(x,y) = \frac{g(x)}{(\alpha - iy)^{\ell+1}}, \qquad \alpha \in \mathbb{R}^+,\ \ell \in \mathbb{Z}^+.$$

Then $\tilde{v}(x,p) = 0$ for $p \leq 0$, and **PWA** is exact for v.

[arXiv: 2204.05153]

Existence of the transfer matrix

$$\widehat{\mathbf{M}} := \widehat{\mathcal{U}}(\infty, -\infty) := \lim_{x_\pm \to \pm\infty} \widehat{\mathcal{U}}(x_+, x_-)$$

$$\widehat{\mathcal{U}}(x, x_0) := \widehat{\mathbf{I}} - i \int_{x_0}^{x} dx_1 \widehat{\mathbf{\Pi}}_k \widehat{\mathcal{H}}(x_1) \widehat{\mathbf{\Pi}}_k - \int_{x_0}^{x} dx_2 \int_{x_0}^{x_2} dx_1 \widehat{\mathbf{\Pi}}_k \widehat{\mathcal{H}}(x_2) \widehat{\mathcal{H}}(x_1) \widehat{\mathbf{\Pi}}_k + \cdots$$

$$\widehat{\mathcal{H}}(x) := \frac{1}{2} e^{-i\widehat{\varpi} x \boldsymbol{\sigma}_3} v(x, \widehat{y})\, \widehat{\varpi}^{-1} \mathcal{K}\, e^{i\widehat{\varpi} x \boldsymbol{\sigma}_3}$$

Existence of the transfer matrix

$$\widehat{\mathrm{M}} := \widehat{\mathcal{U}}(\infty, -\infty) := \lim_{x_\pm \to \pm\infty} \widehat{\mathcal{U}}(x_+, x_-)$$

$$\widehat{\mathcal{U}}(x, x_0) := \widehat{\mathbf{I}} - i \int_{x_0}^{x} dx_1 \widehat{\mathbf{\Pi}}_k \widehat{\mathcal{H}}(x_1) \widehat{\mathbf{\Pi}}_k - \int_{x_0}^{x} dx_2 \int_{x_0}^{x_2} dx_1 \widehat{\mathbf{\Pi}}_k \widehat{\mathcal{H}}(x_2) \widehat{\mathcal{H}}(x_1) \widehat{\mathbf{\Pi}}_k + \cdots$$

$$\widehat{\mathcal{H}}(x) := \tfrac{1}{2}\, e^{-i\widehat{\varpi} x \boldsymbol{\sigma}_3}\, v(x, \widehat{y})\, \widehat{\varpi}^{-1} \mathcal{K}\, e^{i\widehat{\varpi} x \boldsymbol{\sigma}_3}$$

Qxn: What is an appropriate function space in which $\widehat{\mathcal{H}}(x)$ acts as a densely-defined operator?

Existence of the transfer matrix

$$\widehat{\mathbf{M}} := \widehat{\mathcal{U}}(\infty, -\infty) := \lim_{x_\pm \to \pm\infty} \widehat{\mathcal{U}}(x_+, x_-)$$

$$\widehat{\mathcal{U}}(x, x_0) := \widehat{\mathbf{I}} - i \int_{x_0}^{x} dx_1 \widehat{\mathbf{\Pi}}_k \widehat{\mathcal{H}}(x_1) \widehat{\mathbf{\Pi}}_k - \int_{x_0}^{x} dx_2 \int_{x_0}^{x_2} dx_1 \widehat{\mathbf{\Pi}}_k \widehat{\mathcal{H}}(x_2) \widehat{\mathcal{H}}(x_1) \widehat{\mathbf{\Pi}}_k + \cdots$$

$$\widehat{\mathcal{H}}(x) := \tfrac{1}{2}\, e^{-i\widehat{\varpi} x \boldsymbol{\sigma}_3}\, v(x, \widehat{y})\, \widehat{\varpi}^{-1} \boldsymbol{\mathcal{K}}\, e^{i\widehat{\varpi} x \boldsymbol{\sigma}_3}$$

Qxn: What is an appropriate function space in which $\widehat{\mathcal{H}}(x)$ acts as a densely-defined operator?

Qxn: Does $\widehat{\mathcal{H}}(x)$ generate a dynamics, i.e., does the Dyson series for its evolution operator $\widehat{\mathcal{U}}(x, x_0)$ converge over a dense domain?

Existence of the transfer matrix

$$\widehat{\mathbf{M}} := \widehat{\mathcal{U}}(\infty, -\infty) := \lim_{x_\pm \to \pm\infty} \widehat{\mathcal{U}}(x_+, x_-)$$

$$\widehat{\mathcal{U}}(x, x_0) := \widehat{\mathbf{I}} - i \int_{x_0}^{x} dx_1 \widehat{\mathbf{\Pi}}_k \widehat{\mathcal{H}}(x_1) \widehat{\mathbf{\Pi}}_k - \int_{x_0}^{x} dx_2 \int_{x_0}^{x_2} dx_1 \widehat{\mathbf{\Pi}}_k \widehat{\mathcal{H}}(x_2) \widehat{\mathcal{H}}(x_1) \widehat{\mathbf{\Pi}}_k + \cdots$$

$$\widehat{\mathcal{H}}(x) := \frac{1}{2} e^{-i\widehat{\varpi} x \boldsymbol{\sigma}_3} \, v(x, \widehat{y}) \, \widehat{\varpi}^{-1} \mathcal{K} \, e^{i\widehat{\varpi} x \boldsymbol{\sigma}_3}$$

Qxn: What is an appropriate function space in which $\widehat{\mathcal{H}}(x)$ acts as a densely-defined operator?

Qxn: Does $\widehat{\mathcal{H}}(x)$ generate a dynamics, i.e., does the Dyson series for its evolution operator $\widehat{\mathcal{U}}(x, x_0)$ converge over a dense domain?

Qxn: Does $\lim_{x_\pm \to \pm\infty} \widehat{\mathcal{U}}(x_+, x_-)$ exist?

Existence of the transfer matrix

$$\widehat{\mathbf{M}} := \widehat{\mathcal{U}}(\infty, -\infty) := \lim_{x_\pm \to \pm\infty} \widehat{\mathcal{U}}(x_+, x_-)$$

$$\widehat{\mathcal{U}}(x, x_0) := \widehat{\mathbf{I}} - i\int_{x_0}^{x} dx_1 \widehat{\mathbf{\Pi}}_k \widehat{\mathcal{H}}(x_1)\widehat{\mathbf{\Pi}}_k - \int_{x_0}^{x} dx_2 \int_{x_0}^{x_2} dx_1 \widehat{\mathbf{\Pi}}_k \widehat{\mathcal{H}}(x_2)\widehat{\mathcal{H}}(x_1)\widehat{\mathbf{\Pi}}_k + \cdots$$

$$\widehat{\mathcal{H}}(x) := \tfrac{1}{2}\, e^{-i\widehat{\varpi} x \boldsymbol{\sigma}_3}\, v(x, \widehat{y})\, \widehat{\varpi}^{-1} \boldsymbol{\mathcal{K}}\, e^{i\widehat{\varpi} x \boldsymbol{\sigma}_3}$$

Qxn: What is an appropriate function space in which $\widehat{\mathcal{H}}(x)$ acts as a densely-defined operator?

Qxn: Does $\widehat{\mathcal{H}}(x)$ generate a dynamics, i.e., does the Dyson series for its evolution operator $\widehat{\mathcal{U}}(x, x_0)$ converge over a dense domain?

Qxn: Does $\lim_{x_\pm \to \pm\infty} \widehat{\mathcal{U}}(x_+, x_-)$ exist?

We could answer these questions for $\widehat{\mathcal{H}}_k(x)$ and prove the existence of $\widehat{\mathrm{M}}_k$.

[arXiv: 2207.10054]

Definition: Let $v : \mathbb{R}^2 \to \mathbb{C}$ be a bounded integrable function and $s, k \in \mathbb{R}^+$. We say that $v \in \mathscr{C}_{s,k}$ if

1) $\exists \alpha, \beta, \sigma \in \mathbb{R}^+$ such that $\sigma > s$ and for all $(x, y) \in \mathbb{R}^2$,

$$|v(x,y)| \le \frac{\beta}{(1+|x|)^\sigma} \quad \text{for} \quad |x| \ge \alpha.$$

2) The function $\check{v}_k : \mathbb{R} \to L^\infty(-2k, 2k)$ defined by

$$\big(\check{v}_k(x)\big)(p) := \tilde{v}(x,p), \qquad x \in \mathbb{R},\ p \in (-2k, 2k),$$

is piecewise continuous.

[arXiv: 2207.10054]

Theorem: Let $v \in \mathscr{C}_{3,k}$. Then the following hold.

1) For all $x \in \mathbb{R}$, $\widehat{\mathcal{H}}_k(x)$ defines an unbounded non-self-adjoint linear operator acting in

$$\mathscr{H} := L^2(-k,k)^2 = \left\{ \begin{bmatrix} \phi_+ \\ \phi_- \end{bmatrix} \;\middle|\; \phi_\pm \in L^2(-k,k) \right\}$$

[arXiv: 2207.10054]

Theorem: Let $v \in \mathscr{C}_{3,k}$. Then the following hold.

1) For all $x \in \mathbb{R}$, $\widehat{\boldsymbol{\mathcal{H}}}_k(x)$ defines an unbounded non-self-adjoint linear operator acting in

$$\mathscr{H} := L^2(-k,k)^2 = \left\{ \left[\begin{array}{c} \phi_+ \\ \phi_- \end{array} \right] \middle| \ \phi_\pm \in L^2(-k,k) \right\}$$

and having the domain:

$$\mathscr{D} \; = \; \left\{ \left[\begin{array}{c} \phi_+ \\ \phi_- \end{array} \right] \in \mathscr{H} \middle| \ \phi_+ + \phi_- \in \mathscr{R} \right\},$$

where $\mathscr{R} := \left\{ \sqrt{k^2 - \widehat{p}^2}\ \phi \mid \phi \in L^2(-k,k) \right\},$

$$\left(\sqrt{k^2 - \widehat{p}^2}\ \phi\right)(p) := \sqrt{k^2 - p^2}\ \phi(p) \quad \text{for} \quad p \in (-k,k)$$

[arXiv: 2207.10054]

Theorem: Let $v \in \mathscr{C}_{3,k}$. Then the following hold.

1) For all $x \in \mathbb{R}$, $\widehat{\mathcal{H}}_k(x)$ defines an unbounded non-self-adjoint linear operator acting in

$$\mathscr{H} := L^2(-k,k)^2 = \left\{ \left[\begin{array}{c} \phi_+ \\ \phi_- \end{array} \right] \,\middle|\, \phi_\pm \in L^2(-k,k) \right\}$$

and having the domain:

$$\mathscr{D} = \left\{ \left[\begin{array}{c} \phi_+ \\ \phi_- \end{array} \right] \in \mathscr{H} \,\middle|\, \phi_+ + \phi_- \in \mathscr{R} \right\},$$

where $\mathscr{R} := \left\{ \sqrt{k^2 - \widehat{p}^2}\, \phi \mid \phi \in L^2(-k,k) \right\}$,

$$\left(\sqrt{k^2 - \widehat{p}^2}\, \phi\right)(p) := \sqrt{k^2 - p^2}\, \phi(p) \quad \text{for} \quad p \in (-k,k)$$

2) The Dyson series for $\widehat{\mathcal{U}}_k(x, x_0)$ converges strongly on $\mathscr{D}$.

3) $\lim_{x_\pm \to \pm\infty} \widehat{\mathcal{U}}(x_+, x_-)$ exist as strong limits on $\mathscr{D}$.

[arXiv: 2207.10054]

Conclusions

- Propagating-wave approximation (PWA)

- Scattering by nonlocal potential achieving PWA

- Existence of the transfer matrix for these potentials

- Numerical implementations: $L^2(-k,k)$ & Hilbert-Schmidt (compact) operators.

Conclusions

- Propagating-wave approximation (PWA)

- Scattering by nonlocal potential achieving PWA

- Existence of the transfer matrix for these potentials

- Numerical implementations: $L^2(-k,k)$ & Hilbert-Schmidt (compact) operators.

References:

- arXiv: 2204.05153 (to appear in Phys Rev A)

- arXiv: 2207.10054.

Conclusions

- Propagating-wave approximation (PWA)

- Scattering by nonlocal potential achieving PWA

- Existence of the transfer matrix for these potentials

- Numerical implementations: $L^2(-k,k)$ & Hilbert-Schmidt (compact) operators.

References:

- arXiv: 2204.05153 (to appear in Phys Rev A)

- arXiv: 2207.10054.

Thank you for your attention.