# Non–self–adjoint relativistic point interaction in one dimension

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# General relativistic point interactions

$$\mathcal{D} := -ic\sigma_1 \frac{\mathrm{d}}{\mathrm{d}x} + mc^2\sigma_3,\ \mathcal{D}^{\mathbb{A}} := \mathcal{D} + c\mathbb{A}|\delta(x)\rangle\langle\delta(x)|,$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \mathbb{A} \in \mathbb{C}^{2,2}, m \geq 0, c > 0.$$

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$$\langle\delta(x)|\psi\rangle\delta(x)\varphi := \frac{\psi(0+) + \psi(0-)}{2}\varphi(0) \in \mathbb{C}^2.$$

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**Definition (Relativistic point interactions)**

$$\mathrm{Dom}(D^{\mathbb{A}}) = \{\psi \equiv \psi_- \oplus \psi_+ \in H^1(\mathbb{R}_-;\mathbb{C}^2) \oplus H^1(\mathbb{R}_+;\mathbb{C}^2) \mid (1) \text{ holds}\},$$

$$D^{\mathbb{A}}\psi = \mathcal{D}\psi_- \oplus \mathcal{D}\psi_+.$$

[Seba 89]

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$$D_{\text{min}}\psi = \mathcal{D}\psi,$$

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# Free Dirac operator

$$(D_0\psi)(x) = (\mathcal{D}\psi)(x), \forall x \in \mathbb{R},$$

$$\text{Dom } D_0 = H^1(\mathbb{R}) \otimes \mathbb{C}^2.$$

$$\sigma(D_0) = \{(-\infty, -mc^2] \cup [mc^2, +\infty)\} =: \mathbb{R}_{mc^2}.$$

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The resolvent

$$R_z(x,y) = \frac{i}{2c}(\mathbb{Z}(z) + \operatorname{sgn}(x-y)\sigma_1)\mathrm{e}^{ik(z)|x-y|}.$$

$$\mathbb{Z}(z) = \begin{pmatrix} \zeta(z) & 0 \\ 0 & \zeta(z)^{-1} \end{pmatrix},$$

$$\zeta(z) = \frac{z+mc^2}{ck(z)}, ck(z) = \sqrt{z^2-(mc^2)^2}, \mathrm{Im}\, k(z) \geq 0.$$

## The resolvent of $D^{\mathbb{A}}$

$$R_z^{\mathbb{A}}(x,y) = R_z(x,y) - cR_z(x,0)(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}R_z(0,y).$$

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$$R_{z,\varepsilon}^{\mathbb{A}}(x,y) = R_z(x,y) - c\int_{\mathbb{R}^2} R_z(x,\varepsilon s)v(s)M^\varepsilon(s,t)v(t)R_z(\varepsilon t,y)\,\mathrm{d}s\,\mathrm{d}t.$$

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### Theorem

*Let $\mathbb{A}$ and $z \in \mathbb{C}\setminus\mathbb{R}_{mc^2}$ such that*

$$(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))$$

*is invertible. Then*

$$(D_0 + cV_\varepsilon\mathbb{A} - z)^{-1} \xrightarrow[\varepsilon\to 0]{u} (D^{\mathbb{A}} - z)^{-1}.$$

# Spectrum

## Theorem

$$\sigma(D^{\mathbb{A}}) \setminus \mathbb{R}_{mc^2} = \sigma_p(D^{\mathbb{A}}).$$

*$z \in \mathbb{C} \setminus \mathbb{R}_{mc^2}$ is in the spectrum of $D^{\mathbb{A}}$ if and only if*

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*The eigenvalue $z$ has geometric multiplicity equal to $\dim(\mathrm{Ker}(2I + i\mathbb{A}\mathbb{Z}(z)))$ and the corresponding eigenfunction is*

$$\psi(x) = \begin{pmatrix} C\mathrm{e}^{ik(z)|x|} \\ \tilde{C}\zeta(z)^{-1}\mathrm{sgn}(x)\mathrm{e}^{ik(z)|x|} \end{pmatrix}, x \in \mathbb{R} \setminus \{0\},$$

$(-i(C + \tilde{C}), -i\zeta(z)(C - \tilde{C})) \in \mathrm{Ker}(2I + i\mathbb{A}\mathbb{Z}(z))$.

# Spectral transitions

**Theorem**

*We have*

$$\sigma(D^{\mathbb{A}}) = \sigma_p(D^{\mathbb{A}}) \cup \mathbb{R}_{mc^2}$$

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- $m \neq 0 \wedge \det \mathbb{A} = 4 \wedge \alpha = \delta = 0 \Rightarrow \sigma_p(D^{\mathbb{A}}) = \mathbb{C} \setminus \mathbb{R}_{mc^2}$.

*In all other cases we have at most two eigenvalues of* $D^{\mathbb{A}}$.

# Schrödinger operator

$$H_0 = -\frac{1}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2},$$
$$\mathrm{Dom}\, H_0 = W^{2,2}(\mathbb{R}).$$

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The resolvent

$$K_z(x,y) = \frac{im}{\mu(z)}\mathrm{e}^{i\mu(z)|x-y|}.$$

[Grod, Kuzhel 14]

$$H^{\mathbb{A}} = -\frac{1}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\alpha}{2m}|\delta(x)\rangle\langle\delta(x)| + \frac{i\beta}{2m}|\delta(x)\rangle\langle\delta'(x)| - \frac{i\gamma}{2m}|\delta'(x)\rangle\langle\delta(x)| + \frac{\delta}{2m}|\delta'(x)\rangle\langle\delta'(x)|.$$

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$H^{\mathbb{A}}$ will be define as

$$H^{\mathbb{A}}\varphi(x) = -\frac{1}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\varphi(x),\ \forall x \in \mathbb{R}\setminus\{0\},$$

$$\mathrm{Dom}\, H^{\mathbb{A}} = \{\varphi \in W^{2,2}(\mathbb{R}\setminus\{0\}) \mid \Gamma_1\varphi = \mathbb{V}\mathbb{A}\mathbb{V}^*\Gamma_0\varphi\},$$

where

$$\Gamma_0\varphi = \frac{1}{2}\begin{pmatrix}\varphi(0+)+\varphi(0-)\\ -\varphi'(0+)-\varphi'(0-)\end{pmatrix},\ \Gamma_1\varphi = \begin{pmatrix}\varphi'(0+)-\varphi'(0-)\\ \varphi(0+)-\varphi(0-)\end{pmatrix},$$

and

$$\mathbb{V} = \begin{pmatrix} i & 0\\ 0 & 1\end{pmatrix}.$$

# Non–relativistic limit of point interactions

[Benvegnu, Dabrowski 94] Subtract the rest energy $mc^2$ and

$$\mathbb{A} \mapsto \mathbb{A}_c = \begin{pmatrix} \frac{1}{2mc}\alpha & \beta \\ \gamma & 2mc\delta \end{pmatrix}.$$

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**Theorem**

*$z \in \mathbb{C} \setminus [0, +\infty)$ such that*

$$4 - \det \mathbb{A} + 2i\frac{1}{\mu(z)}\alpha + 2i\mu(z)\delta \neq 0.$$

*Then*

$$(D^{\mathbb{A}_c} - mc^2 - z)^{-1} \xrightarrow[c \to +\infty]{u} (H^{\mathbb{A}} - z)^{-1} P_+.$$

$$P_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

# Non–relativistic limit of non–local potentials

## Theorem

$1 + \frac{\tilde{E}_\varepsilon}{\mu(z)}(\alpha + \delta\mu(z)^2) + \tilde{E}_\varepsilon^2 \det \mathbb{A} \neq 0,\ \tilde{E}_\varepsilon = \frac{i}{2}\int_{\mathbb{R}^2} v_\varepsilon(x)\mathrm{e}^{i\mu(z)|x-y|} v_\varepsilon(y)\,\mathrm{d}x\,\mathrm{d}y.$

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*where*

$$W_\varepsilon^{\mathbb{A}} = \frac{1}{2m(1-\delta\|v_\varepsilon\|_{L^2}^2)} \langle \mathbb{W}_\varepsilon | \hat{\mathbb{A}} \rangle_2 + \frac{1}{2m} \frac{\beta\gamma}{1-\delta\|v_\varepsilon\|_{L^2}^2} |v_\varepsilon\rangle\langle v_\varepsilon|,$$

*where*

$$\hat{\mathbb{A}} = \begin{pmatrix} \alpha(1-\delta\|v_\varepsilon\|_{L^2}^2) & i\beta \\ -i\gamma & \delta \end{pmatrix} \text{ and}$$

$$\mathbb{W}_\varepsilon = \begin{pmatrix} |v_\varepsilon\rangle\langle v_\varepsilon| & |v_\varepsilon\rangle\langle (v_\varepsilon)'| \\ |(v_\varepsilon)'\rangle\langle v_\varepsilon| & |(v_\varepsilon)'\rangle\langle (v_\varepsilon)'| \end{pmatrix}.$$

$$(H_0 + W_\varepsilon^{\mathbb{A}} - z)^{-1} \xrightarrow[\varepsilon \to 0]{u} (H^{\mathbb{A}} - z)^{-1}.$$

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Then

$$\begin{tikzcd}
(D_0 + cV_\varepsilon \mathbb{A}_c - mc^2) \arrow[r, "\varepsilon\to 0", "u"'] \arrow[d, "c\to+\infty"', "u"] & (D^{\mathbb{A}_c} - mc^2) \arrow[d, "u"', "c\to+\infty"] \\
(H_0 + W_\varepsilon^{\mathbb{A}}) \arrow[r, "u", "\varepsilon\to 0"'] & H^{\mathbb{A}}
\end{tikzcd}$$

- S. Benvegnu, L. Dabrowski, *Relativistic point interaction in one dimension.* Letters in Mathematical Physics 30, 1994, 159-167.
- A. Grod, S. Kuzhel, *Schrödinger operators with non–symmetric zero–range potentials.* MFAT, 2014.
- R. J. Hughes, *Relativistic point interactions: approximation by smooth potentials.* Reports on Mathematical Physics 39, 1997.
- R. J. Hughes, *Finite-rank perturbations of the Dirac operator.* Journal of Mathematical Analysis and Applications 238, 1999.
- A. Hussein, D. Krejčiřík, P. Siegl, *Non–self–adjoint graphs.* Transactions of the American Mathematical Society 367, 2015, 2921-2957.
- P. Šeba, *Klein's Paradox and the Relativistic Point Interaction.* Letters in Mathematical Physics 18, 1989, 77-86.
- M. Tušek, *Approximation of one-dimensional relativistic point interactions by regular potentials revised.* Letters in Mathematical Physics 110, 2020, 2585-2601.

$$z \in \sigma_p(D_\varepsilon^{\mathbb{A}}) \text{ iff } \det(I + E_\varepsilon \mathbb{A}\mathbb{Z}(z)), E_\varepsilon = \frac{i}{2}\int_{\mathbb{R}^2} v(x)\mathrm{e}^{ik(z)\varepsilon|x-y|}v(y)\,\mathrm{d}y\,\mathrm{d}x.$$

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Asymptotic behaviour of eigenvalues of approximations nearby the eigenvalues of the limiting operator

$$\det(I + E_\varepsilon \mathbb{A}\mathbb{Z}(z)) \xrightarrow{\varepsilon \to 0} \det(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z)).$$

$$\det(I + E_\varepsilon \mathbb{A}\mathbb{Z}(z)) = \det(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z)) + O(\varepsilon).$$

$$D_0 = -ic(\sigma_1 \partial_x + \sigma_2 \partial_y) + m\sigma_3,$$

$$\mathrm{Dom}\, D_0 := H^1(\mathbb{R}^2; \mathbb{C}^2).$$

The smooth boundary $\Sigma := \partial\Omega$. $\mathbb{R}^2 = \Omega_+ \cup \Sigma \cup \Omega_-$. Distribution $\delta_\Sigma$.

$$D_0 + (\alpha I + \beta\sigma_3 + \gamma(\sigma \cdot \vec{t}) + \delta(\sigma \cdot \vec{n}))\delta_\Sigma.$$

Closed densely defined symmetric operator $A$ on $\mathcal{H}$. $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ boundary triplet for $A^*$ if and only if

- $\forall f, g \in \mathrm{Dom}\, A^*$

$$\langle A^* f | g \rangle_{\mathcal{H}} - \langle f | A^* g \rangle_{\mathcal{H}} = \langle \Gamma_1 f | \Gamma_0 g \rangle_{\mathcal{G}} - \langle \Gamma_0 f | \Gamma_1 g \rangle_{\mathcal{G}}.$$

- The map $f \in \mathrm{Dom}\, A^* \mapsto (\Gamma_0 f, \Gamma_1 f) \in \mathcal{G} \times \mathcal{G}$ is surjective.