Aharonov–Casher theorem
on
domains with boundary

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talk based on my PhD thesis advised by J.P. Solovej

AAMP September, 2022

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QMATH

Problem: A relativistic massless charged particle in a plane region

Magnetic field $\vec{B} = (0, 0, B)$ (parts $B_0; B_1, B_2, \ldots$)
Vector potential $\vec{a} = (a_x, a_y, 0)$, $(\vec{\nabla} \times \vec{a} = \vec{B}, \ \mathrm{div}\vec{a} = 0)$
Flux $\Phi = \int \vec{B} \cdot \mathrm{d}\vec{S} = \oint \vec{a}\mathrm{d}\vec{s}$

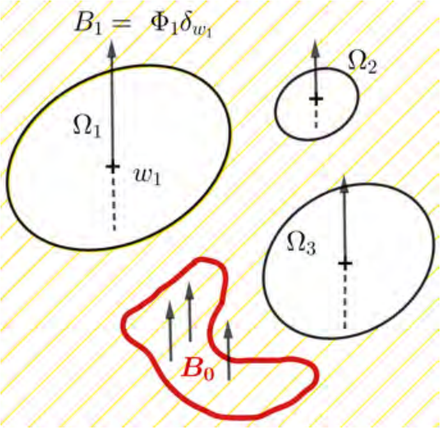


Figure: $\mathbb{R}^2$ with holes

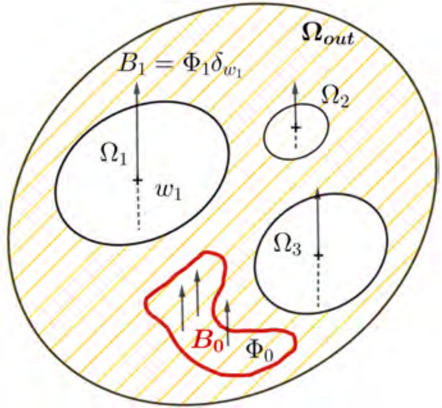


Figure: A disc with holes

(Formal) Hamiltonian for a particle in a plane region

Dirac operator

$$D_a = \sigma^1(-\mathrm{i}\partial_x - a_x) + \sigma^2(-\mathrm{i}\partial_y - a_y)$$

On $\mathbb{C}^2$ valued square integrable functions

Pauli operator

(non-relativistic limit, taking the interaction spin—mag. field into account)

$$D_a^2 = H_a = -\sum_{j\in\{x,y\}} (\partial_j - \mathrm{i}a_j)^2 I + B\sigma^3$$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Boundary conditions

Let M be $\mathbb{R}^2$ with holes or a disc with holes and ∂M its boundary
Let n be the coord. in the direction of *inward* normal ∂_n
Rewrite locally: $D_a = \sigma((\partial_n + \mathrm{i}a_n) + A)$, where
$\sigma : \partial M \to \mathrm{End}(\mathbb{C}^2)$,
A is a Dirac operator on ∂M

Denote $D_{max} \subset L^2(M, \mathbb{C}^2)$ the maximal domain of D_a

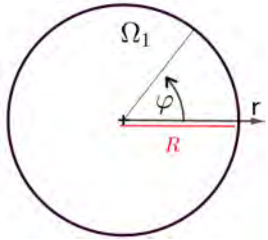
$dom(D_a) = \{u \in dom(D_{max}) \mid u|_{\partial M} \in$ a spectral subspace BC of a boundary operator$\}$

BC is called **boundary condition**

Atiyah–Patodi–Singer (APS) boundary condition

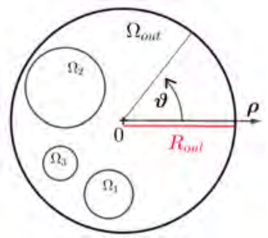
$=$ "the negative spectral subspace of a boundary operator"

APS boundary condition on circular boundaries



$$\begin{pmatrix}u^+\\u^-\end{pmatrix}\Bigg|_{\partial\Omega_1} = \left[\sum_{k>\frac{\Phi_1}{2\pi}-\frac{1}{2}} d'_k \begin{pmatrix}\mathrm{e}^{\mathrm{i}\varphi k}\\0\end{pmatrix} + \sum_{k\le\frac{\Phi_1}{2\pi}+\frac{1}{2}} b'_k \begin{pmatrix}0\\\mathrm{e}^{\mathrm{i}\varphi k}\end{pmatrix}\right]$$

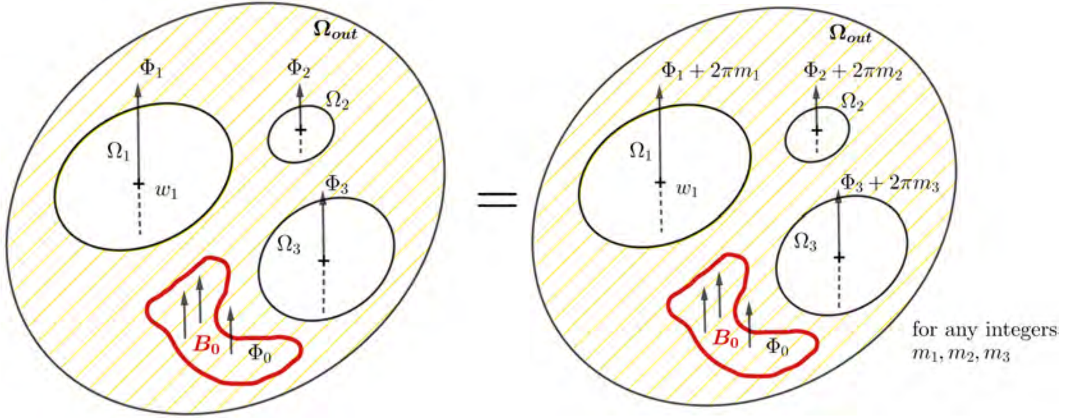
$$\times\ \mathrm{e}^{\mathrm{i}\int_0^{\varphi} \vec{a}(s)\,\mathrm{d}\vec{s} - \mathrm{i}\frac{\Phi_1}{2\pi}\varphi}$$



$$\begin{pmatrix}u^+\\u^-\end{pmatrix}\Bigg|_{\partial\Omega_{out}} = \left[\sum_{k<\frac{\Phi}{2\pi}-\frac{1}{2}} d''_k \begin{pmatrix}\mathrm{e}^{\mathrm{i}\vartheta k}\\0\end{pmatrix} + \sum_{k\ge\frac{\Phi}{2\pi}+\frac{1}{2}} b''_k \begin{pmatrix}0\\\mathrm{e}^{\mathrm{i}\vartheta k}\end{pmatrix}\right]$$

$$\times\ \mathrm{e}^{\mathrm{i}\int_0^{\vartheta} \vec{a}(s)\,\mathrm{d}\vec{s} - \mathrm{i}\frac{\Phi}{2\pi}\vartheta}$$

Gauge invariance



We can choose the fluxes inside the holes so that

$$\Phi_{1,2,\ldots,N} \in 2\pi \left[-\tfrac{1}{2}, \tfrac{1}{2}\right).$$

N is the number of the holes

$\lfloor y \rfloor \dots$ the biggest integer strictly smaller then y,
$\Phi \dots$ the total flux of the magnetic field a,
ZM$=$ zero modes.

Theorem

Let D_a be the Dirac operator on $\mathbb{R}^2 \setminus \cup_{k\leq N}\Omega_k$ with the magnetic field a in the AC gauge. Then if $|\frac{\Phi}{2\pi}| > 1$, the number of ZM of D_a with the APS boundary condition is $\left\lfloor \frac{|\Phi|}{2\pi} \right\rfloor$.

Theorem

Let D_a be the Dirac operator on $\Omega_{out} \setminus \cup_{k\leq N}\Omega_k$ with the magnetic field a in the AC gauge. Then the number of ZM of D_a with the APS boundary condition is $\left|\left\lfloor \frac{\Phi}{2\pi} + \frac{1}{2} \right\rfloor\right|$.

If $\Phi > 0 \Rightarrow$ ZM have spin up. If $\Phi < 0 \Rightarrow$ ZM have spin down.

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Proof idea [Aharonov–Casher 1979]

$D_a \begin{pmatrix} u^+ \\ u^- \end{pmatrix} = 0$, Ansatz: $u^\pm = \mathrm{e}^{\pm h} g^\pm$. Then

$$0 = \left[\partial_{\bar z} - \frac{\mathrm{i}a}{2}\right] u^+ = \mathrm{e}^h \left[\partial_{\bar z} + \underbrace{\partial_{\bar z} h(z) - \frac{\mathrm{i}a}{2}}_{\text{make this } 0}\right] g^+ ,$$

$$0 = \left[\partial_z - \frac{\mathrm{i}\bar a}{2}\right] u^- = \mathrm{e}^{-h} \left[\partial_z \overbrace{-\partial_z h(z) - \frac{\mathrm{i}\bar a}{2}}\right] g^-$$

$\Rightarrow \partial_{\bar z} g^+ = 0 \Rightarrow g^+(z) = \sum_{k\geq 0} d_k z^k$

Aharonov–Casher gauge: $\partial_{\bar z} h(z) = \frac{\mathrm{i}a}{2} \overset{\operatorname{div} a = 0}{\overbrace{\Rightarrow}} -\Delta h = B \in C_0^\infty(\mathbb{R}^2)$,

Choose $h(z) = \frac{-1}{2\pi} \int \log|z - z'| B(z')\,\mathrm{d}z'\,\mathrm{d}\bar z'$, $z \notin \operatorname{supp} B$.

$$u^+ = \mathrm{e}^h g^+(z) = |z|^{-\Phi/2\pi}(1 + \mathcal{O}(|z|^{-1})) \sum_{k_0 \geq k \geq 0} d_k z^k \text{, as } |z| \to \infty$$

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Extending the Aharonov–Casher idea to the case with boundary

Using the APS boundary condition

- **problem**: function g^+ is analytic only outside of the holes, we have Laurent series $g^+(z) = \sum_{k\in\mathbb{Z}} d_k z^k$
- **solution**: use the boundary condition to find out if d_k vanish for some k
- **means**: multiply u^+ by a convenient function e^G whose restriction to the boundary cancels the exponential term in the boundary condition. The exponential e^{G+h} turns out to be analytic inside Ω_1 and so is, consequently, g^+

$$\mathrm{e}^G u^+ = \mathrm{e}^{G+h} g^+ \qquad \text{by Aharonov–Casher ansatz}$$

$$u^+\big|_{\partial\Omega_1} = \sum_{k>\frac{\Phi_1}{2\pi}-\frac{1}{2}} d'_k \begin{pmatrix} \mathrm{e}^{\mathrm{i}\varphi k} \\ 0 \end{pmatrix} \quad \times \mathrm{e}^{\mathrm{i}\int_0^\varphi \vec{a}(s)\,\mathrm{d}\vec{s} - \mathrm{i}\frac{\Phi_1}{2\pi}\varphi}, \qquad d'_k \in \mathbb{C}$$

Sphere with N holes

Stereographic projection

Conformal transformation of the Dirac operator with the APS boundary condition

All the fluxes sum to zero

Theorem

Let D_a be the Dirac operator on $\mathbb{S}^2 \setminus \cup_{k \le N} \Omega_k$ with magnetic field a such that $\int_{\mathbb{S}^2} B = 0$. Denote $\widehat{\Phi} = \Phi'_2 + \cdots \Phi'_N + \Phi_0$ with $[-\pi, \pi) \ni \Phi'_k = \Phi_k + 2\pi m_k$, $k = 2, \ldots, N$ and Φ_0 the flux in the bulk. Then there are

$$\left| \left\lfloor \frac{\widehat{\Phi}}{2\pi} + \frac{1}{2} \right\rfloor \right|$$

ZM of the operator D_a with the APS boundary conditions.

If $\widehat{\Phi} > 0 \Rightarrow$ ZM have spin up. If $\widehat{\Phi} < 0 \Rightarrow$ ZM have spin down.

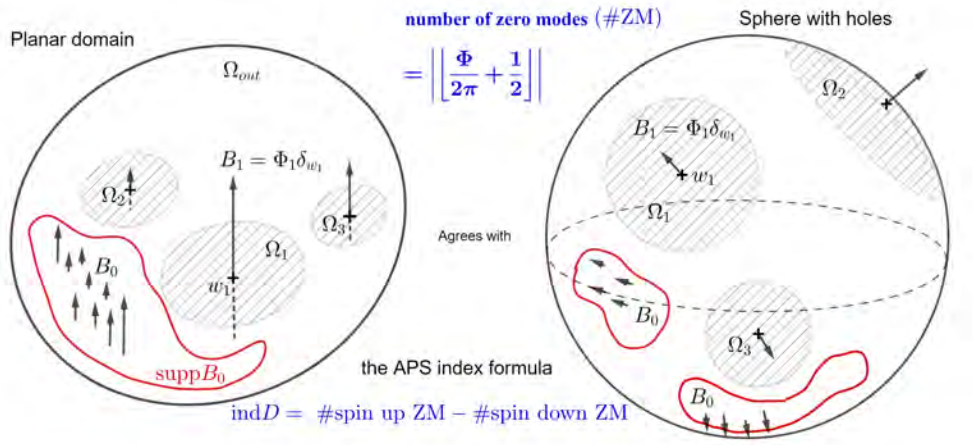

Figure: Setting of the bounded region with magnetic field.

Concluding remarks

- Dirac operator
- Atiyah–Patodi–Singer boundary condition
- Zero modes (Aharonov–Casher theorem, Index theorem)

- Anomalies in QFT, Domain-Wall-Fermions

Boundary conditions

Let M be a manifold with compact boundary ∂M
Let V be a "suitable" vector space
Let D be a Dirac operator, $D_{max} \subset L^2(M, V)$ its maximal domain
Denote by ∂_n the inward normal vector field on ∂M
Rewrite locally: $D = \sigma(\partial_n + A)$, where

$\sigma : \partial M \to \mathrm{End}(V)$,
A is a Dirac operator on ∂M

$dom(D) = \{u \in dom(D_{max}) \mid u|_{\partial M} \in$ a spectral subspace BC of a boundary operator $\}$

BC is called **boundary condition**

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