Families of third-order shape-invariant Hamiltonians related to generalized Hermite and Okamoto polynomials

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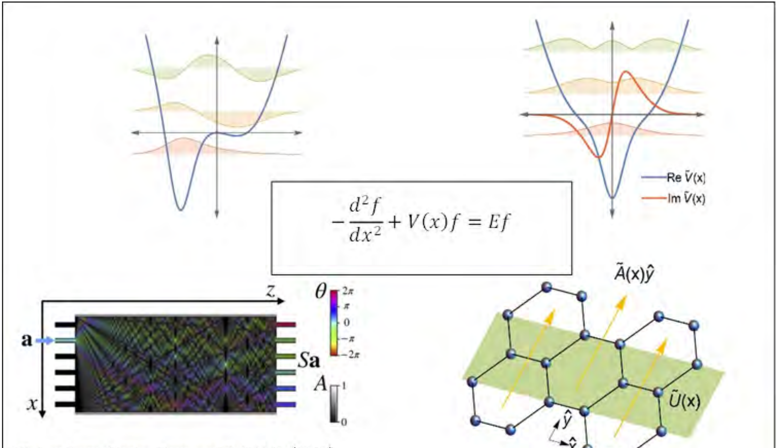
Outline

Why?



1Taken from *Optics Express* **29**, 38441 (2021)

Shape-invariant Hamiltonians in QM

The idea of shape-invariant Hamiltonians was introduced by Gendenshtein [1] in the context of SUSY QM to determine the spectral properties of some Hamiltonians.

Two Hamiltonians H_+ and H_- are said to be shape-invariant if they are related as $H_+(\{c_n\}) = H_-(f(\{c_n\})) + R(\{c_n\})\mathbb{I}$, with $\mathbb{I}$ the identity operator in the corresponding vector space, $\{c_n\}$ a set of parameters, and $f(\{c_n\})$ and $R(\{c_n\})$ functions of the set of parameters.

Oscillator:

$$H_-(\lambda) = P^2 + Q^2, \quad H_+(\lambda) = P^2 + Q^2 + 2\lambda \quad \lambda \in \mathbb{Z}^+,$$

Reflectionless hyperbolic Pöschl-Teller

$$H_-(\kappa) = P^2 + \kappa(\kappa+1)\,\mathrm{sech}(\kappa x), \quad \kappa \in \mathbb{Z}^+$$
$$H_+(\kappa) = P^2 + (\kappa - n)(\kappa - n + 1)\,\mathrm{sech}(\kappa x) \quad 0 < n < \kappa,$$

where clearly $f(\kappa) = k - n$.

[1] L.É. Gendenshteîn, *JETP Lett.* **38**, 356 (1983)

Third-order shape invariance

We are particularly interested in the class of shape-invariant Hamiltonians H fulfilling the relationship

$$HA = A(H - 2\lambda)\,, \quad H \equiv -\frac{d^2}{dx^2} + V(x), \quad \lambda > 0\,,$$

which in turn implies that A and $A^\dagger$ are annihilation and creation operators, respectively. In the latter, $V(x)$ has to be determined and the intertwining/ladder A operator has the form

$$A \equiv \frac{d^3}{dx^3} + A_2(x)\frac{d^2}{dx^2} + A_1(x)\frac{d}{dx} + A_0(x).$$

Although A has the most general form, we consider two convenient factorizations so that the Painlevé transcendent emerges naturally and the eigen-solutions can be determined.

Shape-invariant

$$\begin{tikzcd}
\boxed{H} \arrow[rr, "A^{\dagger} = Q^{\dagger} M = Q^{\dagger} M_2 M_1"] \arrow[d, "Q^{\dagger}"'] & & \boxed{H + 2\lambda} \\
\boxed{H_1 + 2\lambda} \arrow[rr, "M_2"] \arrow[rru, "\mathrm{M} = \mathrm{M}_2 \mathrm{M}_1"] & & \boxed{H_2 + 2\lambda} \arrow[u, "M_1"']
\end{tikzcd}$$

Figure:
Third-order shape-invariant Hamiltonian H. The arrows indicate the inter-twining relation among H and the *auxiliary* Hamiltonians H_1, and H_2.
For instance, the arrow on top indicates $HA^{\dagger} = A^{\dagger}(H + 2\lambda)$.
The direction of arrows is inverted by using the adjoint relations.

Factorization 1: Painlevé IV (PIV)

Be the factorization proposed by Andrianov *et al.*[2]

$$A = M^{\dagger} Q, \quad M^{\dagger} := \frac{d^2}{dx^2} - w(x)\frac{d}{dx} + B(x), \quad Q := -\frac{d}{dx} + W(x)\,.$$

After some calculations one gets

$$B(x) = \frac{w^2}{4} - \frac{w'}{2} - \frac{w''}{2w} + \frac{w'^2}{4w^2} + \frac{d}{w^2}\,, \qquad W(x) = -x - w,$$

where $w(x)$ fulfills the PIV equation

$$w'' = \frac{(w')^2}{2w} + \frac{3}{2}w^3 + 4xw^2 + 2(x^2-\alpha)w + \frac{\beta}{w}, \quad \alpha = \gamma + 1, \quad \beta = 2d\,,$$

with γ and d integration constants. The unknown potential becomes

$$V(x) = x^2 - (w' - 2xw - w^2) - 1.$$

[2] A. Andrianov, F. Cannata, M. Ioffe, and D. Nishnianidze, *Phys. Lett. A* **266**, 341 (2000).

Remarks on the solutions to PIV

So far, every unknown function has been expressed in terms of the solutions to PIV ($w(x)$). There is a plethora of families associated to PIV which may serve for our construction. Still, only those leading to regular potentials are of interest in this work.

Among the hierarchies of solutions one usually finds

- Solutions in terms of Riccati equation[3] ($\beta = -2(1 \pm \alpha)$)
- Complementary error functions hierarchy[3] ($\{\alpha, \beta\} = \{m_1 + 1, -2m_2^2\}$)
- Solutions in term of non-linear bound states[4] ($\{\alpha, \beta\} = \{2N + 1, 0\}$)
- Hierarchies of rational solutions[5] (α and β given in the next slides)

[3] V.I. Gromak, I. Laine, and S. Shimomura; Painlevé Differential Equations in the Complex Plane (2002)

[4] A.P. Bassom, P.A. Clarkson, Hicks A C and McLeod J B 1992; *Proc. R. Soc. Lond. A* **437**, 1 (1992)

[5] P.A. Clarkson, The fourth Painlevé equation and associated special polynomials, *J. Math. Phys.* **44**, 5350 (2003)

Factorization 2: Zero modes

The intertwining operators can be further decomposed through

$$A^\dagger = M^\dagger Q = M_1^\dagger M_2^\dagger Q, \quad A = Q^\dagger M = Q^\dagger M_2 M_1,$$
$$M_1^\dagger := \frac{d}{dx} + W_1(x)\,, \quad M_2^\dagger := \frac{d}{dx} + W_2(x)\,.$$

The factorization of second-order intertwining operators is refered to as *reducible factorization*, and it was introduced by Andrianov *et al.*[6], from which it is known that $W_{1,2}(x)$ are real-valued functions. One gets[7],

$$W_1(x) = \frac{(w' - 2xw - w^2) + 2\sqrt{-d}}{2w} + x,$$
$$W_2(x) = -\frac{(w' + 2xw + w^2) + 2\sqrt{-d}}{2w} + x\,.$$

[6] A. Andrianov, M. Ioffe, F. Cannata, and J-P. Dedonder, *Int. J. Mod. Phys. A* **10** (1995) 2683

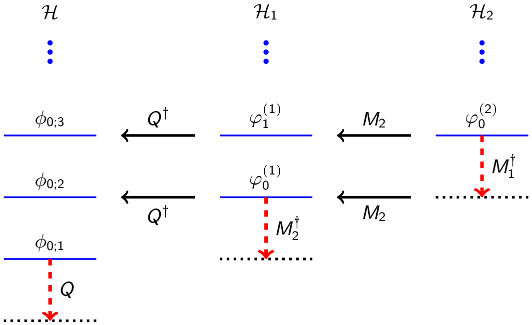
[7] V. Hussin, I. Marquette, and K. Zelaya, *J. Phys. A: Math. Theor.* **55** (2022) 045205

Zero-modes

$$H_1 M_2 = M_2 H_2, \qquad H_2 = M_1 M_1^\dagger + \epsilon_1,$$

$$H_2 M_1 = M_1 H\,, \qquad H_1 = M_2 M_2^\dagger + \epsilon_2,$$

$$\epsilon_1 = \gamma - \sqrt{-d}, \qquad \epsilon_2 = \gamma + \sqrt{-d}\,.$$



The *zero-modes* are defined as the eigensolutions of H annihilated by A,

$$A\phi_{0;j}^{(k)} = M^{\dagger} Q\phi_{0;j}^{(k)} = M_1^{\dagger} M_2^{\dagger} Q\phi_{0;j}^{(k)} = 0\,, \quad j = 1, 2, 3.$$

Since A is of third-order, we have three zero-modes computed such that

- $\phi_{0;1}^{(k)}$: The zero-mode $\phi_{0;1}^{(k)} \in \mathcal{H}$ is annihilated by Q, that is, $Q\phi_{0;1} = 0$.
- $\phi_{0;2}^{(k)}$: Q does not annihilate the zero-mode, $Q\phi_{0;2}^{(k)} = f_2 \neq 0$, instead one has $M_2^{\dagger} f_2 = 0$.
- $\phi_{0;3}$: Neither Q nor $M_2^{\dagger}$ annihilate the zero-mode, $M_2^{\dagger} Q\phi_{0;3} = f_3 \neq 0$, instead one has $M_1^{\dagger} f_3 = 0$.

The procedure is analogous for the zero-modes $\Phi_{0;j}^{(k)}$ annihilated by $A^{\dagger} = Q^{\dagger} M_2 M_1$.

Hierarchies of rational solutions to PIV

$-1/x$ and $-2x$ hierarchies

Generalized Hermite polynomials

Rational solutions in the $-1/x$ and $-2x$ hierarchies are divided in three classes, the specific form of which is dictated by the values of α and β, and given by[8,9]

$$
\begin{aligned}
w_{p,q}^{[1]}(x) &= \frac{d}{dx}\ln\frac{H_{p+1,q}}{H_{p,q}}, && \alpha = 2p+q+1, \quad \beta = -2q^2, \\
w_{p,q}^{[2]}(x) &= \frac{d}{dx}\ln\frac{H_{p,q}}{H_{p,q+1}}, && \alpha = -(p+2q+1), \quad \beta = -2p^2, \\
w_{p,q}^{[3]}(x) &= -2x + \frac{d}{dx}\ln\frac{H_{p,q+1}}{H_{p+1,q}}, && \alpha = q-p, \quad \beta = -2\left(p+q+1\right)^2,
\end{aligned}
$$

with q and p integers, and $H_{m,n}(x)$ the *generalized Hermite polynomials* computed from the nonlinear recurrence relations

$$
\begin{aligned}
2pH_{p+1,q}H_{p-1,q} &= H_{p,q}H''_{p,q} - (H'_{p,q})^2 + 2pH_{p,q}^2\,, \\
2qH_{p,q+1}H_{p,q-1} &= -H_{p,q}H''_{p,q} + (H'_{p,q})^2 + 2qH_{p,q}^2\,,
\end{aligned}
$$

together with the initial conditions $H_{0,0} = H_{1,0} = H_{0,1} = 1$ and $H_{1,1} = 2x$.

[8] P.A. Clarkson, The fourth Painlevé transcendent, In *Differential Algebra and Related Topics II*, Li Guo and W.Y. Sit (eds.), World Scientific, Singapore, 2008.

[9] F. Marcellán, and W. Van Assche (Eds.), *Orthogonal Polynomials and Special Functions: Computation and Applications*, Springer-Verlag, Berlin, 2006.

Alternatively, $H_{p,q}$ has a representation in terms of Shcur-τ polynomials that reduces to the Wronskian representation [10]

$$H_{p,q}(x) \propto Wr(H_p(x),\ldots,H_{p+q-1}(x)),$$

with H_m the conventional Hermite polynomials. This is particularly useful as the total number of zeros can be determined with ease [11,12], leading to

$H_{p,q}$	$n_t = n_0 + 2n_+$
$H_{p,2q}$	0
$H_{p,2q+1}$	p

We thus fix $q \to 2q$ and arbitrary p to obtain the famly of regular potentials

$$V_{p,q}(x) \equiv V(x)\big|_{\substack{p\to p\\ q\to 2q}} = x^2 + 4q - 1 - \frac{d^2}{dx^2}\ln H_{p+1,2q}\,.$$

[10] P.A. Clarkson, The fourth Painlevé transcendent, In *Differential Algebra and Related Topics II*, Li Guo and W.Y. Sit (eds.), World Scientific, Singapore, 2008.

[11] S. Karlin and G. Szegö, *J. Anal. Math.* **8**, 1 (1960)

[12] G. Felder, A.D. Hemery, and A.P. Veselov, *Physica D* **241**, 2131 (2012)

$$\phi_{n=0;1}^{(p,q)}(x) \equiv \phi_{0;1}(x) = \frac{1}{\mathcal{N}_{0;1}^{(p,q)}} \left(\frac{e^{-\frac{x^2}{2}}}{H_{p+1,2q}} \right) H_{p,2q}, \qquad E_{0;1} = 0,$$

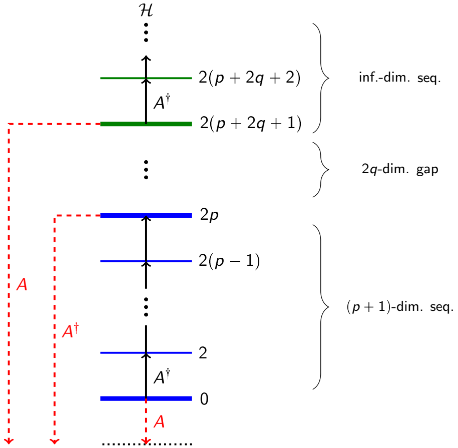
$$\phi_{n=p;1}^{(p,q)}(x) \equiv \Phi_{0;2}(x) = \frac{1}{\mathcal{N}_{p;1}^{(p,q)}} \left(\frac{e^{-\frac{x^2}{2}}}{H_{p+1,2q}} \right) H_{p,2q+1}, \qquad E_{p;1} = 2p,$$

$$\phi_{n=0;2}^{(p,q)}(x) \equiv \phi_{0;3}(x) = \frac{1}{\mathcal{N}_{0;2}^{(p,q)}} \left(\frac{e^{-\frac{x^2}{2}}}{H_{p+1,2q}} \right) H_{p+1,2q+1}, \qquad E_{0;2} = 2p + 4q + 2,$$

The rest of the eigenvalues are determined through the iterated action of $A^{\dagger}$, which increases the eigenvalue by 2λ units. This hierarchy leads to

$$E_{n;1}^{(p,q)} = 2n, \qquad n = 0, \ldots, p,$$

$$E_{n;2}^{(p,q)} = 2n + 2p + 4q + 2, \qquad n = 0, 1, \ldots,$$


$\mathcal{H}$
$2(p+2q+2)$
$A^\dagger$
$2(p+2q+1)$
inf.-dim. seq.
2q-dim. gap
$2p$
$2(p-1)$
$(p+1)$-dim. seq.
A
$A^\dagger$
2
$A^\dagger$
0
A

$-2x/3$ hierarchy

Generalized Okamoto polynomials

The third hierarchy of rational solutions is defined as[13]

$$w_{m,n}^{[1]}(x) = -\frac{2x}{3} + \frac{d}{dx}\ln\frac{Q_{m+1,n}}{Q_{m,n}}, \qquad \alpha = 2m+n, \quad \beta = -2\left(n-\frac{1}{3}\right)^2,$$

$$w_{m,n}^{[2]}(x) = -\frac{2x}{3} + \frac{d}{dx}\ln\frac{Q_{m,n}}{Q_{m,n+1}}, \qquad \alpha = -m-2n, \quad \beta = -2\left(m-\frac{1}{3}\right)^2,$$

$$w_{m,n}^{[3]}(x) = -\frac{2x}{3} + \frac{d}{dx}\ln\frac{Q_{m,n+1}}{Q_{m+1,n}}, \qquad \alpha = n-m, \quad \beta = -2\left(m+n+\frac{1}{3}\right)^2,$$

where $Q_{m,n} \equiv Q_{m,n}(x)$ stands for the *generalized Okamoto polynomials*[13], computed from the nonlinear recurrence relations

$$Q_{m+1,n}Q_{m-1,n} = \frac{9}{2}\left[Q_{m,n}Q''_{m,n} - (Q'_{m,n})^2\right] + \left[2x^2 + 3(2m+n-1)\right]Q_{m,n}^2,$$

$$Q_{m,n+1}Q_{m,n-1} = \frac{9}{2}\left[Q_{m,n}Q''_{m,n} - (Q'_{m,n})^2\right] + \left[2x^2 + 3(1-m-2n)\right]Q_{m,n}^2,$$

for $m,n = 0,1,\cdots$, together with the initial conditions $Q_{0,0} = Q_{1,0} = Q_{0,1} = 1$ and $Q_{1,1} = \sqrt{2}\,x$.

[13] P.A. Clarkson, The fourth Painlevé transcendent, In *Differential Algebra and Related Topics II*, Li Guo and W.Y. Sit (eds.), World Scientific, Singapore, 2008.

We thus require nodeless polynomials $Q_{m,n}$ to generate regular solutions $w_{m,n}^{[j]}$, and consequently regular potentials and eigensolutions. This can be analyzed by simply looking at the zero distributions of $Q_{m,n}$, which can be done using the *oscillation theorem for Wronskians composed of orthogonal eigensolutions* introduced in [M.Á. García-Ferrero and D. Gómez-Ullate, *Lett. Math. Phys.* **105**, 551 (2015).]. By using the latter one obtains

$Q_{m,n}$	n_0	n_+	$n_t = n_0 + 2n_+$
$Q_{2m,2n}$	0	n	$2n$
$Q_{2m,2n+1}$	0	$n+m$	$2(n+m)$
$Q_{2m+1,2n}$	0	n	$2n$
$Q_{2m+1,2n+1}$	1	n	$2(n+m)+1$

We thus impose $n=0$ and $m=k\in\mathbb{Z}^+$, leading to the regular potential

$$V^{(k)}(x) \equiv \frac{x^2}{9} - 2\frac{d^2}{dx^2}\ln Q_{k+1} + \frac{4k}{3} - \frac{1}{3},$$
$$Q_k(x) \equiv Q_{k,0}(x).$$

From the latter, the zero-modes of the system in question become

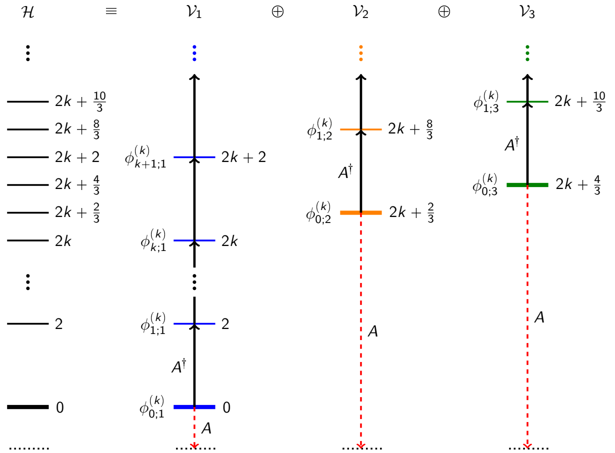
$$\phi_{0;1}^{(k)}(x) = \mathcal{N}_{0;1}\left(\frac{e^{-\frac{x^2}{6}}}{Q_{k+1}}\right)Q_k, \qquad E_{0;1} = 0,$$

$$\phi_{0;2}^{(k)}(x) = \mathcal{N}_{0;2}\left(\frac{e^{-\frac{x^2}{6}}}{Q_{k+1}}\right)Q_{k+1,1}, \qquad E_{0;2} = 2k+1-\frac{1}{3},$$

$$\phi_{0;3}^{(k)}(x) = \mathcal{N}_{0;3}\left(\frac{e^{-\frac{x^2}{6}}}{Q_{k+1}}\right)Q_{k+2,-1}, \qquad E_{0;3} = 2k+1+\frac{1}{3}.$$

The complete set of eigenvalues take the form

$$E_{n;1} = 2n, \quad E_{n;2} = 2k+2n+\frac{2}{3}, \quad E_{n;3} = 2k+2n+\frac{4}{3}.$$


$\mathcal{H}$
$\equiv$
$\mathcal{V}_1$
$\oplus$
$\mathcal{V}_2$
$\oplus$
$\mathcal{V}_3$
$2k+\frac{10}{3}$
$2k+\frac{8}{3}$
$2k+2$
$2k+\frac{4}{3}$
$2k+\frac{2}{3}$
$2k$
2
0
$\phi^{(k)}_{k+1;1}$
$\phi^{(k)}_{k;1}$
$\phi^{(k)}_{1;1}$
$\phi^{(k)}_{0;1}$
$A^\dagger$
A
$\phi^{(k)}_{1;2}$
$\phi^{(k)}_{0;2}$
$\phi^{(k)}_{1;3}$
$\phi^{(k)}_{0;3}$

Excited states, orthogonal polynomials, and three-term recurrence relations

The higher-modes (excited states) are determined from the iterated action of $A^{\dagger}$; however, direct calculations become a challenging task. We thus introduce the following form

- $-\frac{1}{x}, -2x \quad : \quad \phi_{n;j}^{(p,q)}(x) \propto \mu_{n;j}^{(p,q)}(x) \mathcal{P}_{n;j}^{(p,q)}(x) \quad \begin{cases} j=1, & n=0,\ldots,p \\ j=2, & n=0,1,\ldots \end{cases}$
- $-\frac{2x}{3} \quad : \quad \phi_{n;j}^{(k)}(x) \propto \mu_{n;j}^{(k)}(x) \mathcal{P}_{n;j}^{(k)}(x) \quad \begin{cases} j=1,2,3, \\ n=0,1,\ldots \end{cases}$

for the general set of solutions such that

- $\mu^{(p,q)} = \frac{e^{-\frac{x^2}{2}}}{H_{p,2q}}, \quad \mathcal{P}_{0;1}^{(p,q)} = H_{p,2q}, \quad \mathcal{P}_{p;1}^{(p,q)} = H_{p,2q+1}, \quad \mathcal{P}_{0;2}^{(p,q)} = H_{p+1,2q+1}$
- $\mu^{(k)} = \frac{e^{-\frac{x^2}{6}}}{Q_k}, \quad \mathcal{P}_{0;1}^{(k)} = Q_k, \quad \mathcal{P}_{0;2}^{(k)} = Q_{k+1,1}, \quad \mathcal{P}_{0;3}^{(k)} = Q_{k+2,-1}.$

With the latter, the zero-modes are recovered for all the hierarchies. We now have to determine how to determine the remaining functions $\mathcal{P}_{n;j}^{(p,q)}$ and $\mathcal{P}_{n;j}^{(k)}$.

From H we get $T^{(p,q)}\mathcal{P}_{n;j}^{(p,q)}=0$ and $T^{(k)}\mathcal{P}_{n;j}^{(k)}=0$, with the second-order differential operators

$$T^{(p,q)}:=\frac{d^2}{dx^2}-2\left(x+\frac{H'_{p+1,2q}}{H_{p+1,2q}}\right)\frac{d}{dx}+\left(\frac{H''_{p+1,2q}}{H_{p+1,2q}}+2x\frac{H'_{p+1,2q}}{H_{p+1,2q}}+E_{n;j}^{(p,q)}-4q\right),$$

$$T^{(k)}:=\frac{d^2}{dx^2}-\left(\frac{2x}{3}+2\frac{Q'_{k+1}}{Q_{k+1}}\right)\frac{d}{dx}+\left(\frac{Q''_{k+1}}{Q_{k+1}}+\frac{2x}{3}\frac{Q'_{k+1}}{Q_{k+1}}-\frac{4k}{3}+E_{n;j}^{(k)}\right).$$

Likewise, $\mathcal{P}_{n-1;j}^{(p,q)}$ and $\mathcal{P}_{n-1;j}^{(k)}$ solve the second-order difference equations

$$\mathcal{P}_{n+1;j}^{(p,q)}(x)+F_n^{(p,q)}(x)\mathcal{P}_{n-1;j}^{(p,q)}(x)+G_n^{(p,q)}(x)\mathcal{P}_{n;j}^{(p,q)}(x)=0,$$
$$\mathcal{P}_{n+1;j}^{(k)}(x)+F_n^{(k)}(x)\mathcal{P}_{n-1;j}^{(k)}(x)+G_n^{(k)}(x)\mathcal{P}_{n;j}^{(k)}(x)=0,$$

respectively, with $F_0^{(p,q)}=F_0^{(k)}=0$, and $F_n^{(p,q)}$, $G_n^{(p,q)}$, $F_n^{(k)}$, and $G_n^{(k)}$ rational functions, whose explicit forms are given in [14,15]. Furthermore, one needs the boundary conditions $\mathcal{P}_{-1;j}^{(p,q)}=\mathcal{P}_{-1;j}^{(k)}=0$.

[14] I. Marquette and K. Zelaya, arXiv:2203.05631 `[math-ph]`

[15] V. Hussin, I. Marquette, and K. Zelaya, *J. Phys. A: Math. Theor.* **55**, 045205 (2022)

From the latter, it is then proved that $P_{n;j}^{(p,q)}(x)$ and $P_{n;j}^{(k)}(x)$ are polynomial solutions, and also satisfy the orthonormality relation

$$\int_{\mathbb{R}} dx[\mu^{(p,q)}(x)]^2 P_{n;j}^{(p,q)}(x) P_{m;j}^{(p,q)}(x) = \delta_{n,m}\delta_{r,s}[\mathcal{N}_{n;j}^{(p,q)}]^2, \qquad \begin{cases} j=1 & n=0,\ldots,p \\ j=2 & n=0,\ldots \end{cases}$$

$$\int_{\mathbb{R}} dx[\mu^{(k)}(x)]^2 P_{n;j}^{(k)}(x) P_{m;j}^{(k)}(x) = \delta_{n,m}\delta_{j,k}[\mathcal{N}_{n;j}^{(p,q)}]^2, \qquad \begin{cases} j=1 & n=0,\ldots,p \\ j=2 & n=0,\ldots \end{cases}$$

That is, $\{P_{n;1}^{(p,q)}\}_{n=0}^{p}$, $\{P_{n;2}^{(p,q)}\}_{n=1}^{\infty}$, and $\{P_{n;j}^{(k)}\}_{n=0}^{\infty}$ are sets of orthogonal polynomials that simultaneously solve a second-order differential and a second-order difference equation.

Final remarks

Time-dependent case

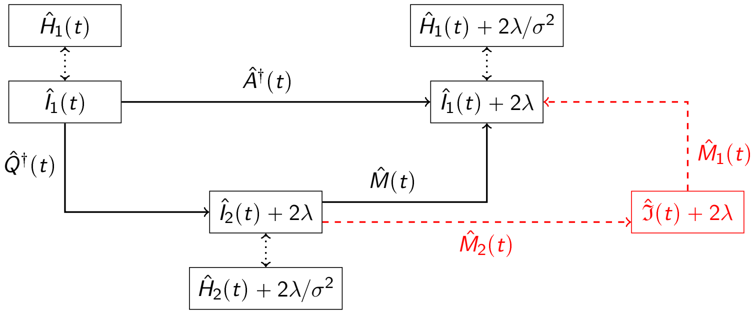


Figure:
K. Zelaya, I. Marquette, and V. Hussin, *J. Phys. A: Math. Theor.* **54**, 015206 (2021).

Concluding Remarks

- To test the validity of our results, one of the particular cases discussed in [V.G. Bagrov and B.F. Samsonov, *Pramana* **49**, 563 (1996)] has been recovered. Our approach requires only three-term recurrence relations to compute the excited states, contrary to the fifth-order one used by the authors mentioned above.
- The authors of [A. Andrianov, F. Cannata, M. Ioffe, and D. Nishnianidze, *Phys. Lett. A* **266**, 341 (2000).] claim that zero-modes of third-order intertwining operators have at most two nodes. However, that is not entirely correct, and here we presented several counterexamples of such a claim.
- Ladder operators A and $A^{\dagger}$ for the $-1/x$ and $-2x$ hierarchies can be modified utilizing the f-oscillators approach so that the new resulting operators fulfill the $su(1,1)$ and $su(2)$ algebraic relationships when acted on the finite- and infinite-dimensional space of solutions, respectively. In this form, we have an alternative representation for such algebras in terms of the newly introduced polynomials $P_{m;j}^{(p,q)}(x)$.

References

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Thanks for your attention