Summation of the An-harmonic part of the Propagator

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Motivation:

Functional integral for propagator of the an-harmonic oscillator is an example of φ^4 theory in $1+1$ dimension. We have studied this problem*) and found the solution in terms of the operator functions, in the spirit of the "umbral calculus."

In this talk we will express the operator functions in terms of classical functions and we will show the possibility to sum up the infinite series representing the operator functions.

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In the Euclidean theory propagator is defined as *conditional Wiener measure* path integral

$$\mathcal{W} = \int \; [D\varphi(\tau)] \; \exp\left(-E[\varphi]\right) \tag{1}$$

where the Euclidean continuous action is defined as:

$$E[\varphi] = \int_0^\beta d\tau \left[c(\tau)/2 \left(\frac{\partial\varphi(\tau)}{\partial\tau} \right)^2 + b(\tau)\varphi(\tau)^2 + a(\tau)\varphi(\tau)^4 \right]$$

where $c(\tau), b(\tau), a(\tau)$ are functions of the time.

The values $\varphi(0) = x_i$ and $\varphi(\beta) = x_f$ are fixed in the *conditional Wiener measure* path integral.

The path integral (1) is defined by the limit:

$$\mathcal{W} = \lim_{N \to \infty} \ \mathcal{W}_N,$$

where

$$\mathcal{W}_N = \left(\frac{1}{\sqrt{\frac{2\pi\triangle}{c_0}}} \right) \int_{-\infty}^{+\infty} \prod_{i=1}^{N-1} \frac{d\varphi_i}{\sqrt{\frac{2\pi\triangle}{c_i}}} \exp(-E_N),$$

with

$$E_N = \sum_{i=1}^{N} \triangle \left[c_i/2 \left(\frac{\varphi_i - \varphi_{i-1}}{\triangle} \right)^2 + b_i \varphi_i^2 + a_i \varphi_i^4 \right],$$

representing the standard time-slice discretization of $E[\varphi]$. $\triangle$ is the interval between ith and $i+1th$ time points: $\triangle = \beta/N$.

We found the result for the propagator $\mathcal{W}_\beta$:

$$\mathcal{W}_\beta = \frac{\mathcal{W}_\beta^{harm}}{\sqrt{f(\beta)}} \, \mathcal{W}_\beta^{an-harm},$$

where

$$\frac{\mathcal{W}_\beta^{harm}}{\sqrt{f(\beta)}},$$

is the harmonic part of the propagator, and

$$\mathcal{W}_\beta^{an-harm} = \frac{1}{1 - \frac{1}{1-h_\kappa \widehat{\mathcal{I}}_\kappa} \cdot \widehat{A}_{\kappa 0} \widehat{\mathcal{I}}_{\kappa 0}} \; (\exp{(h_\rho I_\rho)})$$

is the an-harmonic part, will be discussed bellow.

The an-harmonic part of propagator is expressed in the spirit of the "umbral calculus" as the operator defined function in the closed form.

The non-complicated operator function represent the higher transcendental function.

The sum of an expansion of the operator function is valid in the formal sense only.

The convergence must be checked for the final structures, obtained by the action of the operators $\widehat{A}_\kappa$ and $\widehat{I}_\kappa$ on the functions h_ν and I_ν.

Goals:

We show, that such final structures can be summed up, resulting to the exponential function, with sum of an infinite series in the exponent.

We show, that this series is absolutely convergent.

Definitions of functions h_κ and operator $\widehat{A}_\kappa$:

$$h_\kappa \equiv -4.4!\ \mathcal{H}_{4-\kappa,\kappa}(\phi_\beta, \phi_0|\gamma).$$

$$\mathcal{H}_{n-\kappa,\kappa}(\phi_\beta, \phi_0|\gamma) = \sum_{k=0}^{\min(n-\kappa,\kappa)} \frac{\phi_\beta^{n-\kappa-k}\phi_0^{\kappa-k}\gamma^k}{(n-\kappa-k)!k!(\kappa-k)!}$$

$$= (n-\kappa)!\kappa!\, H_{n-\kappa,\kappa}(\phi_\beta, \phi_0|\gamma),$$

where

$$H_{4-\kappa_\mu,\ \kappa_\mu}(\phi_\beta, \phi_0|\gamma)$$

are Dattoli's incomplete Hermite polynomials (Dattoli G., *Incomplete 2D Hermite polynomials: properties and applications*, J. Math. Anal. Appl. 284 (2003) 447-454.)

Finally, we define the operator acting only on the h_λ functions:

$$\widehat{A}_\kappa h_\lambda = \sum_{n=1}^{4} \frac{1}{2^n n!} \left(\partial^n_{\phi_0} h_\kappa\right) \left(\partial^n_{\phi_\beta}\ h_\lambda\right) \tag{2}$$

Definitions of the multi-index integrals and operator $\widehat{\mathcal{I}}_{\kappa_0}$:

$$I_{\kappa_1,\cdots,\kappa_\mu} = \int_0^\beta a(\tau_1) Q^4(\tau_1) J^{\kappa_1}(\tau_1) d\tau_1 \cdots \int_{\tau_{\mu-1}}^\beta a(\tau_\mu) Q^4(\tau_\mu) J^{\kappa_\mu}(\tau_\mu) d\tau_\mu$$

Operator $\widehat{\mathcal{I}}_{\kappa_0}$ acts on the multi-index integral $I_{\kappa_1,\cdots,\kappa_\mu}$ as:

$$\widehat{\mathcal{I}}_{\kappa_0}\ I_{\kappa_1,\cdots,\kappa_\mu} = I_{\kappa_0,\kappa_1,\cdots,\kappa_\mu}$$

$$\widehat{\mathcal{I}}_{\kappa_0} 1 = I_{\kappa_0}$$

Product of two multi-index integrals is the sum of $\begin{pmatrix} m+n \\ n \end{pmatrix}$ $m+n$ multi-index integrals:

$$I_{\rho_1,\cdots,\rho_m} I_{\kappa_1,\cdots,\kappa_n} = \sum_{\substack{i_1=1 \\ \sigma_{i_1}=\rho_1}}^{n+1} \sum_{\substack{i_2=i_1+1 \\ \sigma_{i_2}=\rho_2}}^{n+2} \cdots \sum_{\substack{i_m=i_{(m-1)}+1 \\ \sigma_{i_m}=\rho_m}}^{n+m} I_{\sigma_1,\cdots,\sigma_i,\cdots,\sigma_{m+n}}$$

The product of two multi-index integrals can be rewritten by help of the operator $\widehat{\mathcal{I}}$:

$$I_{\rho_1,\cdots,\rho_m} I_{\kappa_1,\cdots,\kappa_n} = \widehat{\mathcal{I}}_{\rho_1}\left[I_{\rho_2,\cdots,\rho_m} I_{\kappa_1,\cdots,\kappa_n}\right] + \widehat{\mathcal{I}}_{\kappa_1}\left[I_{\rho_1,\cdots,\rho_m} I_{\kappa_2,\cdots,\kappa_n}\right].$$

Above equation can be extended to the product of arbitrary numbers of integrals:

$$\begin{aligned}
& I_{\kappa_1,\kappa_2,\cdots,\kappa_{m_1}} \cdots I_{\nu_1,\nu_2,\cdots,\nu_{m_i}} \cdots I_{\rho_1,\rho_2,\cdots,\rho_{m_n}} = \\
= \; & \widehat{\mathcal{I}}_{\kappa_1}[I_{\kappa_2,\cdots,\kappa_{m_1}} \cdots I_{\nu_1,\nu_2,\cdots,\nu_{m_i}} \cdots I_{\rho_1,\rho_2,\cdots,\rho_{m_n}}] + \\
+ \; & \cdots + \\
+ \; & \widehat{\mathcal{I}}_{\nu_1}[I_{\kappa_1,\kappa_2,\cdots,\kappa_{m_1}} \cdots I_{\nu_2,\cdots,\nu_{m_i}} \cdots I_{\rho_1,\rho_2,\cdots,\rho_{m_n}}] + \\
+ \; & \cdots + \\
+ \; & \widehat{\mathcal{I}}_{\rho_1}[I_{\kappa_1,\kappa_2,\cdots,\kappa_{m_1}} \cdots I_{\nu_1,\nu_2,\cdots,\nu_{m_i}} \cdots I_{\rho_2,\cdots,\rho_{m_n}}].
\end{aligned}$$

An-harmonic correction in the form of infinite operator series:

$$\frac{1}{1-\frac{1}{1-\sum_{\kappa=0}^{4}\ h_\kappa \widehat{\mathcal{I}}_\kappa}\cdot\sum_{\kappa_0=0}^{4}\widehat{A}_{\kappa_0}\widehat{\mathcal{I}}_{\kappa_0}}\ \exp\left(h_\rho I_\rho\right)= \quad (3)$$

$$=\ \sum_{\mu=0}^{\infty}\ \left(\frac{1}{1-\sum_{\kappa=0}^{4}\ h_\kappa \widehat{\mathcal{I}}_\kappa}\cdot\sum_{\kappa_0=0}^{4}\widehat{A}_{\kappa_0}\widehat{\mathcal{I}}_{\kappa_0}\right)^{\mu}\ \exp\left(h_\rho I_\rho\right)$$

Where the $\mu - th$ term is:

$$\left(\sum_{m_1=0}^{\infty}\left(h_\kappa \widehat{\mathcal{I}}_\kappa\right)^{m_1}\cdot\widehat{A}_{\kappa_0}\widehat{\mathcal{I}}_{\kappa_0}\right)\cdots\left(\sum_{m_\mu=0}^{\infty}\left(h_\kappa \widehat{\mathcal{I}}_\kappa\right)^{m_\mu}\cdot\widehat{A}_{\kappa_0}\widehat{\mathcal{I}}_{\kappa_0}\right)\exp\left(h_\rho I_\rho\right)$$

Operator $\widehat{\mathcal{I}}_{\kappa_0}$ acts on function $\exp(h_\rho I_\rho)$ via Taylor's expansion.

We can use the identity:

$$(h_\kappa I_\kappa)(h_\rho I_\rho) = h_\kappa h_\rho (I_{\kappa,\rho} + I_{\rho,\kappa}) = 2 h_\kappa h_\rho I_{\kappa,\rho},$$

and extend this characteristic for the products of the n terms:

$$(h_{\nu_1} I_{\nu_1}) \ \cdots (h_{\nu_n} I_{\nu_n}) = n! \ h_{\nu_1} \ . \cdots . \ h_{\nu_n} \ I_{\nu_1,\cdots,\nu_n},$$

Then we will use Taylor's expansion in the form:

$$\exp(h_\rho I_\rho) = \sum_{n=0}^{\infty} \frac{1}{n!} (h_\rho I_\rho)^n = \sum_{n=0}^{\infty} h_{\rho_1} \, h_{\rho_2} \ \cdots \ h_{\rho_n} \, I_{\rho_1,\cdots,\rho_n}.$$

Let us demonstrate the evaluation procedure on the simplest example of the operator (2) $\widehat{A}_{\kappa_0}$, acting on the functions h_λ:

$$\widehat{A}_{\kappa_0} = \frac{1}{2} \left(\partial_y h_{\kappa_0}\right) \left(\partial_x \ \cdot\right)$$

Then for $\mu = 1$ term of Eq. (3) we have:

$$R_1 = \left(\sum_{m=0}^{\infty} \left(h_\kappa \widehat{\mathcal{I}}_\kappa \right)^m \cdot \widehat{A}_{\kappa_0} \widehat{\mathcal{I}}_{\kappa_0} \right) \ \exp\left(h_\rho I_\rho\right) =$$

$$= \ \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} (h_{\kappa_1} \cdots \ h_{\kappa_m}) \widehat{A}_{\kappa_0} (h_{\rho_1} \cdots h_{\rho_n}) \ I_{\kappa_1,\cdots,\kappa_m,\kappa_0,\rho_1,\cdots,\rho_n}.$$

By the definition

$$A_1(h_{\kappa_0}, h_{\nu_i}) = \frac{1}{2}\,(\partial_y h_{\kappa_0})\,(\partial_x\ h_{\nu_i})$$

we red:

$$\widehat{A}_{\kappa_0}(h_{\nu_1}\cdots h_{\nu_n}) = \sum_{i=1}^{n}\ A_1(h_{\kappa_0}, h_{\nu_i})\frac{(h_{\nu_1}\cdots h_{\nu_n})}{h_{\nu_i}}$$

By the summation transformation:

$$\sum_{m=0}^{\infty}\ \sum_{n=1}^{\infty}\ \to\ \sum_{\mu=2}^{\infty}\ \sum_{n=1}^{\mu-1}\ ,\quad \mu = m+n+1$$

We find for R_1:

$$R_1 = \sum_{\mu=2}^{\infty}\ \left(\sum_{i=1}^{\mu-1}\ \sum_{j=i+1}^{\mu}\ A_1(h_{\nu_i}, h_{\nu_j})\ \frac{h_{\nu_1}\cdots\ h_{\nu_\mu}}{h_{\nu_i}, h_{\nu_j}}\right) I_{\nu_1,\cdots,\nu_\mu}$$

In equation

$$\sum_{i=1}^{\mu-1} \sum_{j=i+1}^{\mu} A_1(h_{\nu_i}, h_{\nu_j}) \frac{h_{\nu_1} \cdots h_{\nu_\mu}}{h_{\nu_i}, h_{\nu_j}} I_{\nu_1,\cdots,\nu_\mu}$$

the order of indexes in the integral $I_{\nu_1,\cdots,\nu_\mu}$ is "frozen", The content of functions h_{ν_i} in A_1 varies in the first part of the equation. By summation indexes transformations, we "freeze" the order of h_{ν_i} in the first part, and we find the variations in the indexes of the integral:

$$A_1(h_{\nu_1}, h_{\nu_2})\ h_{\nu_3} h_{\nu_4} \cdots h_{\nu_\mu} \sum_{\substack{i=1 \\ \sigma_i=\nu_1}}^{\mu-1} \sum_{\substack{j=i+1 \\ \sigma_j=\nu_2}}^{\mu} I_{\sigma_1,\cdots,\sigma_i,\cdots,\sigma_j,\cdots,\sigma_\mu}$$

This sum is product of two integrals

$$\sum_{\substack{i=1\\ \sigma_i=\nu_1}}^{\mu-1} \sum_{\substack{j=i+1\\ \sigma_j=\nu_2}}^{\mu} I_{\sigma_1,\cdots,\sigma_i,\cdots,\sigma_j,\cdots,\sigma_\mu} = I_{\nu_1,\nu_2} I_{\nu_3,\cdots,\nu_\mu}$$

Result for $\mu = 1$ can be red:

$$R_1 = A_1(h_{\nu_1}, h_{\nu_2}) I_{\nu_1,\nu_2} \exp\left(h_\kappa I_\kappa\right)$$

For the following evaluations we define the "single term"

$$\mathcal{A}_i = (\hat{A}_{\kappa_i}(\hat{A}_{\kappa_{i-1}} \cdots (\hat{A}_{\kappa_1} h_\rho) \cdots) I_{\kappa_i,\kappa_{i-1},\cdots,\kappa_1,\rho}$$

and also the "non-complete single term"

$$\overline{\mathcal{A}}_{i,\kappa_i} = (\hat{A}_{\kappa_i}(\hat{A}_{\kappa_{i-1}} \cdots (\hat{A}_{\kappa_1} h_\rho) \cdots) I_{\kappa_{i-1},\cdots,\kappa_1,\rho}$$

Connection between them is: $\mathcal{A}_i = \hat{\mathcal{I}}_{\kappa_i}[\overline{\mathcal{A}}_{i,\kappa_i}]$

Let us show the process of evaluation for $\mu = 2$:

$$\begin{aligned} R_2 &= \left(\sum_{m=0}^{\infty} \left(h_\sigma \widehat{\mathcal{I}}_\sigma \right)^m \cdot \widehat{A}_{\kappa_0} \widehat{\mathcal{I}}_{\kappa_0} \right) A_1(\kappa_1, \kappa_2) I_{\kappa_1,\kappa_2} \ \exp\left(h_\rho I_\rho\right) = \\ &= A_2(\kappa_1, \kappa_2, \kappa_3) I_{\kappa_1,\kappa_2,\kappa_3} \exp\left(h_\rho I_\rho\right) + \\ &+ \widehat{\mathcal{I}}_{\kappa_0} \left[\mathcal{A}_1(\kappa_1, \kappa_2) \overline{\mathcal{A}}_{1,\kappa_0}(\kappa_0, \nu) \right] \exp\left(h_\rho I_\rho\right) \end{aligned}$$

Identity:

$$\begin{aligned} \widehat{\mathcal{I}}_{\kappa_0} \left[\mathcal{A}_1(\kappa_1, \kappa_2) \overline{\mathcal{A}}_{1,\kappa_0}(\kappa_0, \nu) \right] &= \frac{1}{2} \widehat{\mathcal{I}}_{\kappa_0} \left[\mathcal{A}_1(\kappa_1, \kappa_2) \overline{\mathcal{A}}_{1,\kappa_0}(\kappa_0, \nu) \right] + \\ + \ \frac{1}{2} \widehat{\mathcal{I}}_{\kappa_1} \left[\mathcal{A}_1(\kappa_0, \nu) \overline{\mathcal{A}}_{1,\kappa_1}(\kappa_1, \kappa_2) \right] &= \frac{1}{2} \mathcal{A}_1(\kappa_1, \kappa_2) \mathcal{A}_1(\kappa_0, \nu) \end{aligned}$$

Because

$$\widehat{\mathcal{I}}_{\kappa_0}[I_{\kappa_1,\kappa_2} I_\nu] + \widehat{\mathcal{I}}_{\kappa_1}[I_{\kappa_0,\nu} I_{\kappa_2}] = I_{\kappa_1,\kappa_2} I_{\kappa_0,\nu}.$$

The process of evaluation of the expansion terms in the form of final structures can be characterized by:
<u>Lemma 1</u>: *Let* $\mathcal{A}_{i_1} \cdots \mathcal{A}_{i_n}$ *is the product of* n *single terms. Then*

$$\left(\sum_{m=0}^{\infty} \left(h_\sigma \widehat{\mathcal{I}}_\sigma\right)^m \cdot \widehat{A}_{\kappa_0} \widehat{\mathcal{I}}_{\kappa_0}\right) [\mathcal{A}_{i_1} \cdots \mathcal{A}_{i_n}] \exp\left(h_\rho I_\rho\right) =$$

$$= \Big(\widehat{\mathcal{I}}_{\kappa_0} [\overline{\mathcal{A}}_{i_1+1,\kappa_0} \cdots \mathcal{A}_{i_n}] + \cdots + \widehat{\mathcal{I}}_{\kappa_0} [\mathcal{A}_{i_1} \cdots \overline{\mathcal{A}}_{i_n+1,\kappa_0}] +$$

$$+ \widehat{\mathcal{I}}_{\kappa_0} [\mathcal{A}_{i_1} \cdots \mathcal{A}_{i_n} \overline{\mathcal{A}}_{1,\kappa_0}]\Big) \exp\left(h_\rho I_\rho\right).$$

<u>Lemma 2</u>: *Let* $\mathcal{A}_{i_1} \cdots \mathcal{A}_{i_n}$ *is the product of* n *single terms. Then*

$$\mathcal{A}_{i_1} \cdots \mathcal{A}_{i_n} = \widehat{\mathcal{I}}_{\kappa_0} [\overline{\mathcal{A}}_{i_1,\kappa_0} \cdots \mathcal{A}_{i_n}] + \cdots +$$

$$+ \widehat{\mathcal{I}}_{\kappa_0} [\mathcal{A}_{i_1} \cdots \overline{\mathcal{A}}_{i_n,\kappa_0}]$$

Following the evaluations in the above spirit, we find for $\mu = 2, 3$ the results:

$$R_2 = \left(\mathcal{A}_2 + \frac{1}{2}(\mathcal{A}_1)^2\right)\exp\left(h_\rho I_\rho\right),$$

$$R_3 = \left(\mathcal{A}_3 + \hat{\mathcal{I}}_{\kappa_0}[\overline{\mathcal{A}}_{1,\kappa_0}\mathcal{A}_2] + \hat{\mathcal{I}}_{\kappa_0}[\overline{\mathcal{A}}_{2,\kappa_0}\mathcal{A}_1] + \right.$$

$$\left. + \frac{1}{2}\hat{\mathcal{I}}_{\kappa_0}[(\mathcal{A}_1)^2\overline{\mathcal{A}}_{1,\kappa_0}]\right)\exp\left(h_\rho I_\rho\right),$$

$$R_3 = \left(\mathcal{A}_3 + \mathcal{A}_1\mathcal{A}_2 + \frac{1}{3!}(\mathcal{A}_1)^3\right)\exp\left(h_\rho I_\rho\right),$$

etc., see Table 1.

An-harmonic part of the propagator is the sum of all terms of Tab.1:

$$\left(1+\sum_{i=1}^{\infty}\frac{1}{i!}\left(\sum_{k=1}^{\infty}\mathcal{A}_k\right)^i\right)\exp\left(h_\rho I_\rho\right)$$

We identify that this is Taylor's expansion of the exponential function

$$\exp\left(\sum_{k=1}^{\infty}\mathcal{A}_k\right)$$

We find for an-harmonic part of the propagator the function:

$$\exp\left(h_\kappa I_\kappa+\sum_{k=1}^{\infty}\mathcal{A}_k\right)=\exp\left(\frac{1}{1-\widehat{A}_{\kappa_0}\widehat{I}_{\kappa_0}}h_\nu I_\nu\right)$$

$\mathcal{A}_n$ is the product of $n+1$ functions h_κ, the signs in series $\mathcal{A}_n$ alternate. The simple inequality:

$$|\mathcal{A}_1| = |A_1(\kappa_1,\kappa_2) I_{\kappa_1,\kappa_2}| =$$

$$= \frac{1}{2} |A_1(\kappa_1,\kappa_2) I_{\kappa_1,\kappa_2} + A_1(\kappa_2,\kappa_1) I_{\kappa_2,\kappa_1}| \leq \frac{1}{2} \mathcal{M}(\kappa_1,\kappa_2) I_{\kappa_1} I_{\kappa_2},$$

where

$$\mathcal{M}(\kappa_1,\kappa_2) = \max\left(|A_1(\kappa_1,\kappa_2)|, |A_1(\kappa_2,\kappa_1)|\right),$$

can be extended by mathematical induction:

$$|\mathcal{A}_{n-1}| \leq \frac{1}{n!} \mathcal{M}(\kappa_1,\cdots,\kappa_n) I_{\kappa_1} \cdots I_{\kappa_n}$$

For series with alternating signs the condition $\lim_{n\to\infty} |\mathcal{A}_n| \to 0$ is sufficient condition of the absolute convergence of the sum of the series $\mathcal{A}_n$.

Beyond the simplest example. Let $\widehat{A}_{\kappa_0}$ is:

$$
\begin{aligned}
\widehat{A}_{\kappa_0}\widehat{\mathcal{I}}_{\kappa_0} &= \widehat{\mathcal{I}}_{\kappa_0}\sum_{n=1}^{2}\frac{1}{2^n n!}\left(\partial_y^n h_{\kappa_0}\right)(\partial_x^n\cdot) \qquad (4)\\
&= \widehat{\mathcal{I}}_{\kappa_0}\widehat{A}_1(\kappa_0,\cdot)+\widehat{\mathcal{I}}_{\kappa_0}\widehat{B}_1(\kappa_0,\cdot,\cdot)
\end{aligned}
$$

where we defined

$$
\widehat{A}_1(\kappa_0,\cdot)=\sum_{n=1}^{2}\frac{1}{2^n n!}\left(\partial_y^n h_{\kappa_0}\right)(\widetilde{\partial_x^n}\ \cdot),
$$

$$
\widehat{B}_1(\kappa_0,\cdot,\cdot)=\frac{1}{4}\left(\partial_y^2 h_{\kappa_0}\right)(\partial_x\ \cdot)(\partial_x\ \cdot)
$$

$\widetilde{\partial_x^n}$ acts only on the one member of the product of the single terms and h_λ functions.

For $\mu = 1$, we evaluate the basic single term $\mathcal{Z}_1$:

$$R_1 = \left(\sum_{m=0}^{\infty} (h_\kappa \widehat{\mathcal{I}}_\kappa)^m \cdot \widehat{A}_{\kappa_0} \widehat{\mathcal{I}}_{\kappa_0} \right) \, \exp\left(h_\kappa I_\kappa\right) = \mathcal{Z}_1 \exp\left(h_\kappa I_\kappa\right),$$

$$\mathcal{Z}_1 = A_1(h_{\nu_1}, h_{\nu_2}) I_{\nu_1,\nu_2} + B_1(h_{\nu_1}, h_{\nu_2}, h_{\nu_3}) \; I_{\nu_1,\nu_2,\nu_3}$$

$$A_1(h_{\nu_1}, h_{\nu_2}) = \sum_{n=1}^{2} \frac{1}{2^n n!} \left(\partial_y^n h_{\nu_1}\right) (\partial_x^n h_{\nu_2}),$$

$$B_1(h_{\nu_1}, h_{\nu_2}, h_{\nu_3}) = \frac{1}{4} \left(\partial_y^2 h_{\nu_1}\right) (\partial_x h_{\nu_2})(\partial_x h_{\nu_3})$$

We found the operator:

$$\hat{\mathcal{O}}_{\kappa_0}(\triangle) = \sum_{n=1}^{2} \frac{1}{2^n n!} \left(\partial_y^n h_{\kappa_0}\right) (\widetilde{\partial_x^n} \triangle) + \frac{1}{4} \left(\partial_y^2 h_{\kappa_0}\right) (\partial_x\, h_\mu I_\mu)(\partial_x \triangle)$$

With help of this operator we define the single terms corresponding to the operator (4):

$$\mathcal{Z}_i = \left(\hat{\mathcal{I}}_{\kappa_0} \hat{\mathcal{O}}_{\kappa_0}\right)^{i-1} \ \mathcal{Z}_1.$$

and sum of all this terms:

$$S_1 = \sum_{i=1}^{\infty} \mathcal{Z}_i$$

Also, we evaluated

$$S_2 = \sum_{i=0}^{\infty} \left(\widehat{I}_{\nu_0} \widehat{\mathcal{O}}_{\nu_0}\right)^i \left(\frac{1}{2!}\mathcal{B}_1(S_1, S_1)\right),$$

where

$$\mathcal{B}_1(S_1, S_1) = \frac{1}{4}\widehat{\mathcal{I}}_{\nu_0} \left(\partial_y^2 h_{\nu_0}\right)(\partial_x S_1)(\partial_x S_1)$$

By mathematical induction we find the structures:

$$S_j = \sum_{k=1}^{j-1} \sum_{i=0}^{\infty} \left(\widehat{I}_{\nu_0} \widehat{\mathcal{O}}_{\nu_0}\right)^i \left(\frac{1}{2!}\mathcal{B}_1(S_{j-k}, S_k)\right), \ j \geq 2.$$

The an-harmonic part of the propagator in this case is:

$$\exp\left(h_\kappa I_\kappa + \sum_{j=1}^{\infty} S_j\right)$$

Q1: The convergence of the sum of the series S_j is dependent on the convergence of the sum of series of the single terms $\mathcal{Z}_i$. Proof of the convergence is not so simple as in the case of "simplest example."

$$S_1 = \sum_{i=1}^{\infty} \mathcal{Z}_i = \frac{1}{1 - \widehat{I}_{\nu_0}\widehat{\mathcal{O}}_{\nu_0}}\mathcal{Z}_1.$$

Q2:

If $\mathcal{S} = \sum_{j=1}^{\infty} S_j$, we can read:

$$\mathcal{S} = S_1 + \sum_{i=0}^{\infty} \left(\hat{I}_{\nu_0}\hat{\mathcal{O}}_{\nu_0}\right)^i \left(\frac{1}{2!}\mathcal{B}_1(\mathcal{S},\mathcal{S})\right)$$

Because

$$S_1 = \frac{1}{1-\hat{I}_{\nu_0}\hat{\mathcal{O}}_{\nu_0}}\mathcal{Z}_1 \ = \ \sum_{i=0}^{\infty} \left(\hat{I}_{\nu_0}\hat{\mathcal{O}}_{\nu_0}\right)^i \mathcal{Z}_1 \ ,$$

one can find the equation:

$$(1-\hat{I}_{\nu_0}\hat{\mathcal{O}}_{\nu_0})\mathcal{S} = \mathcal{Z}_1 + \frac{1}{2!}\mathcal{B}_1(\mathcal{S},\mathcal{S})$$

Is this equation a method to evaluate $\mathcal{S}$ without evaluation and summation of all S_j?