On Dunkl-Quantum Mechanics

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E. P. Wigner,

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Do the Equations of Motion Determine the Quantum Mechanical Commutation Relations?

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The commutator of the Hamiltonian with the operator corresponding to any physical quantity gives the operator which corresponds to the time derivative of that quantity. One can ask, hence, whether the postulate, that the quantum mechanical operators obey the classical equations of motion, uniquely determines the commutation relations. The answer is found to depend on the form of the Hamiltonian and is in the negative for a free particle and for the harmonic oscillator.

1. SCHRÖDINGER[1] obtained his wave mechanical equation by postulating that the waves' motion correspond to the classical motion of a particle if the field of force in which it is moving does not change too rapidly with position. Later on, Ehrenfest[2] has shown that Schrödinger's work can be summarized most neatly by observing that the operators in the Heisenberg picture satisfy the classical differential equations:

$$\dot{q}=p/m; \quad \dot{p}=-\partial V/\partial x \qquad (1)$$

if one assumes that the Hamiltonian has the simple form

$$H=p^2/2m+V(x). \qquad (2)$$

As is well known, the time derivative of any operator in the Heisenberg picture is its commutator with the Hamiltonian so that (1) is equivalent with (2) and

$$(i/\hbar)[H,q]=p/m; \quad (i/\hbar)[H,p]=-\partial V/\partial x, \qquad (1a)$$

or

$$(i/\hbar)[\tfrac{1}{2}p^2,q]=p; \quad (i/\hbar)[V,p]=-\partial V/\partial x. \qquad (3)$$

These equations are usually derived from the Heisenberg-Born-Jordan relation

$$[p,q]=-i\hbar. \qquad (4)$$

Since, however, (1) and (1a) have a more immediate physical significance than (4) (see in particular Ehrenfest's discussion), it is natural to ask whether, conversely, (4) can be derived from (1a). The present writer has considered this question some time ago but its significance in consequence of Heisenberg's recent paper[3] and of their own work has been pointed out to him only recently by Pais and Uhlenbeck.

Some doubt on the fundamental nature of relations of the type (1a) must arise, of course, even apart from the results of the present analysis, by the observation that Dirac's equation of the electron does not lead to the classical equation of motion for the operators. Furthermore, because of the non-commuting character of the p and q, there are many forms in which the Hamiltonian can be written. In particular, in the example to be discussed below, $H=\frac{1}{2}(x+iv)(x-iv)$ could

[1] E. Schrödinger, *Abhandlungen zur Wellenmechanik* (J. A. Barth, Leipzig, 1927).
[2] P. Ehrenfest, Zeits. f. Physik 4, 455 (1927).
[3] W. Heisenberg, Zeits. f. Physik 123, 93 (1944), p. 108 ff.

have been written for the H of (5) and this would have altered the final result. In spite of these objections and ambiguities, it was felt that the above mentioned articles justify the publication of evidence that (4) is *not* a consequence of (1a).

2. The example which we shall consider is that of the harmonic oscillator. It was chosen because it seemed the simplest example except for the case of a free particle. This latter is, however, clearly anomalous because the second equation of (1a) is identically fulfilled. It so happens that the example of the harmonic oscillator is also the relevant one from the point of view of the considerations of Pais and Uhlenbeck.

Since the purpose of the above-mentioned considerations[4] is to avoid using Hamiltonian theory, we shall write the energy

$$H=\tfrac{1}{2}(x^2+v^2) \tag{5}$$

of an oscillator of mass one and classical frequency $1/2\pi$, in terms of coordinates and velocity rather than coordinates and momenta. If we choose units in which $\hbar=1$, the fundamental Eqs. (1a) become

$$v=\dot{x}=i[H,x] \tag{6a}$$

$$\dot{v}=-x=i[H,v]. \tag{6b}$$

3. The simplest method to solve Eqs. (5) and (6) seems to be essentially that of Born and Jordan.[5] One assumes that H is diagonal, its diagonal elements, which are because of (5) all positive, shall be denoted by $E_0, E_1, E_2, \cdots$. Then (6a) and (6b) read for the matrix elements x_{nm} and v_{nm} of x and v

$$v_{nm}=i(E_n-E_m)x_{nm} \tag{7a}$$

$$-x_{nm}=i(E_n-E_m)v_{nm}. \tag{7b}$$

Combining the two equations, we have

$$x_{nm}=(E_n-E_m)^2x_{nm}. \tag{8}$$

It follows that x_{nm} can be finite only if $E_n-E_m=\pm 1$ and it follows from (7a) that v_{nm} vanishes if x_{nm} does. As a result, the E_n which are connected by a finite matrix element of either x or v form an arithmetical series

$$E_n=E_0+n \tag{9}$$

if we restrict our attention to irreducible systems of operators satisfying (6).

---

[4] W. Heisenberg see reference 3; A. Pais and S. Uhlenbeck (to be published).

[5] M. Born and P. Jordan, Zeits. f. Physik 34, 858 (1927).

Off hand, every E_n of (9) could occur in the diagonal form of H several times. It appears, however,[6] that one can decompose any system of matrices in which E_n occurs more than once by means of a unitary transformation which leaves H unchanged. Hence we can assume that all characteristic values (9) are simple.

Among the matrix elements x_{nm} only those of the form x_{nn+1} and x_{n+1n} can be different from zero; the former can be made real and positive by a transformation with a unitary diagonal matrix. Because of the Hermitean nature of x, the $x_{n+1n}=x_{nn+1}$ will then be real also. The matrix elements of v will be purely imaginary:

$$\begin{aligned} v_{nn+1}&=-ix_{nn+1}=-ix_{n+1n} \\ v_{n+1n}&=ix_{n+1n}=-v_{nn+1} \end{aligned} \tag{10}$$

as follows from (7a) and (9).

So far, the $x_{01}, x_{12}, x_{23}, \cdots$ are entirely free but only (6) is satisfied. In order to fulfill (5), we have to calculate $\frac{1}{2}(x^2+v^2)$. One notes that this is, as a result of (10), automatically a diagonal matrix, the diagonal element corresponding to E_n being

$$E_n=E_0+n=x_{n-1n}{}^2+x_{nn+1}{}^2, \tag{11}$$

except that $x_{01}{}^2=E_0$. Hence the x_{nn+1} can be determined one after another

$$\begin{aligned} x_{nn+1}&=(E_0+\tfrac{1}{2}n)^{\frac{1}{2}} \quad \text{for even } n \\ x_{nn+1}&=(\tfrac{1}{2}n+\tfrac{1}{2})^{\frac{1}{2}} \quad \text{for odd } n. \end{aligned} \tag{12}$$

The commutator of v and x is also automatically diagonal as a result of (10), its diagonal elements are $-2ix_{01}{}^2, -2i(x_{12}{}^2-x_{01}{}^2), -2i(x_{23}{}^2-x_{12}{}^2), \cdots$. Because of (12), these are $-2iE_0, -2i(1-E_0), -2iE_0, -2i(1-E_0), \cdots$. The usual solution is, of course, $E_0=\frac{1}{2}$ and, hence, $[v, x]=-i$. Our somewhat more general solution can be written as

$$([v, x]+i)^2=-(2E_0-1)^2, \tag{13}$$

E_0 being a constant characterizing the solution.

It is worth noting that for large n, all solutions converge to the usual one. It may also be worth mentioning that the situation here described obtains for a large class of quantum mechanical problems. However, there are other cases in which the equations of motion entail the relation $[v, x]=-ih/m$. A trivial case of this nature is that of a potential which is a linear function of the coordinate, $V(x)=ax^3$ is a less obvious case therefore.

[6] This question will not be further pursued since not even those solutions of (5) and (6), in which every diagonal element of H occurs only once, are all equivalent to the usual solution.

L. M. Yang,
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A Note on the Quantum Rule of the Harmonic Oscillator

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The purpose of the present note is to show, with the aid of an elementary example, that the commutation rules, which are usually given the rule of postulates in quantum mechanics, are in fact not arbitrary, provided that a more stringent definition of the Hilbert space and a strict expansion theorem are adopted.

I. INTRODUCTION

IN this note the following problem, which has been recently treated by Wigner,[1] is considered again. In the formulation of quantum mechanics, one often wonders whether it is possible to reverse the ordinary procedure, in which the commutation relations are first postulated as a generalization of the classical poisson bracket and then the equations of motion are deduced. The problem is whether one can derive the commutation rules from the equation of motion taken over from the classical theory, together with the postulate that the energy is a time displacement operator, i.e.,

$$\dot{f}=[f, H], \qquad (1)$$

where f is any dynamical variable of a given system and H the total hamiltonian. By way of illustration, we shall consider the case of an harmonic oscillator where $H=\frac{1}{2}(x^2+\dot{x}^2)$ with $\hbar=\omega=1$. Here one notices that H is itself a function of x and $\dot{x}$, where $\dot{x}$ is defined by $\dot{x}=[x, H]$. Thus a relation of the type (1), with $f=f(x,\dot{x})$, is a complicated relation between the commutators. The conclusion which Wigner arrived at in this example is negative; i.e., the correct solution $[x,\dot{x}]=1$ does not follow uniquely. It will be shown in the present note that by properly formulating the conditions, including a more stringent definition of Hilbert space and a strict expansion theorem, the commutation rule will follow uniquely, though with less stringent definitions other solutions cannot be excluded.

It must be stressed that whether or not a state is physically permissible, cannot be seen clearly without referring to a special representation. Hence, it is the suitable boundary conditions in a special representation, laid down on physical grounds and in general being different for different systems, that serves to restrict wave functions to a certain special type, to include at the same time all permissible ones, and thereby to mark precisely the complete Hilbert space in which the state of the system is depicted and its operators apply.

For the oscillator one requires that the eigenvalues of x and $\dot{x}$ form continuous spectra and extend from $-\infty$ to $+\infty$, and that H be positive definite.[2] These restrictions do not suffice to mark completely the appropriate Hilbert space. For this purpose one has to impose restrictions on the energy eigenfunctions $\psi_n(x)$. Here we have, as is obvious on physical grounds, a natural boundary condition, i.e., $\psi_n(x)\rightarrow 0$ as $x\rightarrow\pm\infty$ for all n.

[1] E. P. Wigner, Phys. Rev. 77, 711 (1950).

[2] It is then sufficient to deduce that H is discrete [see (10)].

We summarize the conditions for the deduction of the commutation rule in the case of the harmonic oscillator.

(a) Hamiltonian $H=\frac{1}{2}(x^2+\dot{x}^2)$

(b) Equation of motion $\ddot{x}+x=0$

(c) The complete Hilbert space for the system defined by the complete set of energy eigenfunctions $\psi_n(x)$ satisfying the boundary condition that $\psi_n(x)\to 0$ as $x\to\pm\infty$ $(-\infty<x<+\infty)$

(d) Superposition principle in the Hilbert space defined in (c). By (c) and (d) we require that any physically admissible state represented by a wave function satisfying the boundary condition in (c) shall be expansible in terms of the set of energy eigenfunctions. Here we need the stringent definition of the expansion theorem; for an arbitrary admissible wave function $f(x)$, we require that the expansion

$$\sum_{n=0}^{\infty} a_n\psi_n(x)$$

converges absolutely and uniformly to $f(x)$. If a less stringent definition is adopted, namely,

$$\sum_{n=0}^{\infty} a_n\psi_n(x)$$

converges to $f(x)$ only in the mean

$$\lim_{N\to\infty}\int_{-\infty}^{+\infty}|f(x)-\sum_{n=0}^{N}a_n\psi_n(x)|^2dx=0,$$

then we cannot rule out other possibilities than $[x,\dot{x}]=1$. These conclusions, however, cannot be reached without referring to the x-representation where a natural boundary condition can be laid down.

II. DEDUCTION OF THE COMMUTATION RULE

From (1) and (a) with $f=x$, it follows that

$$\dot{x}=[x,\tfrac{1}{2}\dot{x}^2]=\tfrac{1}{2}(\dot{x}[x,\dot{x}]+[x,\dot{x}]\dot{x}).$$

Introducing $S=[x,\dot{x}]-1$, one has

$$\{S,\dot{x}\}=0, \tag{2}$$

where the curly bracket is the anti-commutator. From (1) and (b), one has similarly

$$\{S,x\}=0. \tag{3}$$

From (2) and (3), it can easily be shown that

$$[S^2,x]=[S^2,\dot{x}]=0,\quad [S,H]=0$$

which shows that S is a constant of motion, and that S^2 is a real numerical constant.

In the x-representation, (2) becomes $(x'+x'')\times\langle x'|S|x''\rangle=0$. Hence it follows that

$$\langle x'|S|x''\rangle=c(x')\delta(x'+x'') \tag{4}$$

where $c(x')$ is an arbitrary function of x', and the hermitian property of S requires that $c(x')=c^*(-x')$. Hence in the x-representation one can write

$$S=c(x)R, \tag{5}$$

where R is the reflection operator defined by

$$R|x\rangle=|-x\rangle. \tag{6}$$

From this representation of S, one obtains the explicit operational form of $\dot{x}$;

$$\dot{x}=-i\frac{d}{dx}+g(x)+i\frac{c(x)}{2x}R, \tag{7}$$

where $g(x)$ is real. It can be shown that the term $g(x)$ can be removed by properly choosing the phase factor in the x-representation. Using a star to denote the operator in the new representation, one has

$$\left(\frac{d}{dx}\right)^*=e^{-iy}\frac{d}{dx}e^{iy}=\frac{d}{dx}+i\frac{dy}{dx}$$

$$R^*=e^{-iy}Re^{iy}=e^{-2iy_-}R$$

where y is a real function of x, and y_- is the odd part of y. If for y one chooses $y=\int^x g(x)dx$, (7) becomes

$$\dot{x}=-i\left(\frac{d}{dx}\right)^*+i\frac{c'(x)}{2x}R^*,\quad c'(x)=c(x)e^{2iy_-}.$$

Dropping the stars and the dash, and with the help of (2), one can show that $c(x)$ is a numerical constant, thus obtaining

$$\dot{x}=-i\frac{d}{dx}+\frac{ic}{2x}R. \tag{8}$$

Wigner algebra

$$
\begin{aligned}
\hat{p} &= -i\left(\frac{d}{dx} - \frac{c}{2x}\hat{R}\right), \\
\hat{x} &= x,
\end{aligned}
$$

$$[\hat{x}, \hat{p}] = i(1 + c\hat{R})$$

$$\hat{R}\hat{x} = -\hat{x}\hat{R} \quad , \quad \hat{R}\hat{p} = -\hat{p}\hat{R} \quad , \quad \hat{R}\hat{R} = 1$$

Wigner algebra is linked to the Calogero model.

$$H = -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} + \frac{\omega^2}{2}(x_1 - x_2)^2 + \frac{g}{(x_1 - x_2)^2}$$

$$H = \frac{1}{2}\left(-\frac{\partial^2}{\partial x^2} + \frac{c}{x^2}(c - \hat{R}) + x^2\right)$$

Yang-Hamiltonian is self-adjoint.

S. Watanabe,
J. Math. Phys. 30 (2), 376 (1989).

The problem of Yang differential

$$\hat{D}^{Yang} A \quad = \quad \left(\frac{d}{dx} - \frac{ic}{2x} \hat{R} \right) A = -\frac{ic}{2x} \hat{R} A \neq 0.$$

A is a constant.

C. F. Dunkl
Trans. Am. Math. Soc. 311 (1), 167 (1989).

$$\hat{D}^{\nu} = \left[\frac{d}{dx} + \frac{\nu}{x}(1-\hat{R})\right]$$

Superposition of Differential and Difference Operators

$$\hat{D}^{\nu} A = \left[\frac{d}{dx} + \frac{\nu}{x}(1-\hat{R})\right] A = 0$$

$$\begin{aligned} \hat{R} f_{even}(x) &= f_{even}(x), \\ \hat{R} f_{odd}(x) &= -f_{odd}(x) \end{aligned}$$

$$\hat{D}^{\nu} = \frac{d}{dx} \quad \text{for even functions}$$

$$\hat{D}^{\nu} = \frac{d}{dx} + \frac{2\nu}{x} \quad \text{for odd functions}$$

Some properties:

- Linear

$$D_x(af(x)+bg(x)) = aD_xf(x)+bD_xg(x).$$

- Dunkl Leibniz rule

$$D_x(f(x)g(x)) = (D_xf(x))g(x)+f(x)D_xg(x)-\frac{\nu}{x}[(1-P)f(x)][(1-P)g(x)].$$

If f(x) or g(x) is even, then we obtain the ordinary Leibniz rule

- Dunkl chain rule

$$D_xf(u(x)) = \frac{df}{du}\frac{du}{dx}+\frac{\nu}{x}[f(u(x))-f(u(-x))].$$

If u(x) is even, then we obtain the ordinary chain rule

Action on monomial

$$D_x x^n = [n]_\nu x^{n-1},$$

where ν-deformed number is defined by

$$[n]_\nu = n + \nu(1 - (-1)^n).$$

$$[0]_\nu = 0,$$

$$[1]_\nu = 1 + 2\nu$$

$$[2]_\nu = 2$$

$$[3]_\nu = 3 + 2\nu$$

$$[4]_\nu = 4.$$

$$[2k]_\nu = 2k, \quad [2k+1]_\nu = 2k + 1 + 2\nu \quad (k = 0, 1, 2, \ldots).$$

Dunkl-Wigner algebra

$$\begin{aligned}\hat{p} &= -iD^{\nu} = -i\left[\frac{d}{dx} + \frac{\nu}{x}(1-\hat{R})\right],\\ \hat{x} &= x,\end{aligned}$$

$$[\hat{x}, \hat{p}] = i(1 + 2\nu\hat{R})$$

Dunkl derivative is anti-Hermitian

$$\langle \psi_2 | D^\nu \psi_1 \rangle = -\langle D^\nu \psi_2 | \psi_1 \rangle$$

Dunkl-inner product

$$\langle f | g \rangle = \int_{-\infty}^{\infty} g^*(x) f(x) |x|^{2\nu} \, dx,$$

Dunkl-momentum operator is Hermitian

$$\boxed{\langle P \rangle - \langle P \rangle^* = \int_{-\infty}^{\infty} dx \psi^*(x) (\frac{\hbar}{i} D_x^\mu) \psi(x) - \int_{-\infty}^{\infty} dx \psi(x) (\frac{-\hbar}{i} D_x^\mu) \psi^*(x)}$$

Dunkl Laplacian

- In one dimension

$$\nabla_D^2 \quad = \quad (\hat{D}^\nu)^2 = \frac{d^2}{dx^2} + \frac{2\nu}{x}\frac{d}{dx} - \frac{\nu}{x}(1-\hat{R})$$

- In two dimensions (Two Wigner parameters and reflection operators)

$$\begin{aligned}\nabla_D^2 &= (\hat{D}_1{}^{\nu_1})^2 + (\hat{D}_2{}^{\nu_2})^2 \\ &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{2\nu_1}{x}\frac{\partial}{\partial x} + \frac{2\nu_2}{y}\frac{\partial}{\partial y} - \frac{\nu_1}{x}(1-\hat{R}_1) - \frac{\nu_2}{y}(1-\hat{R}_2).\end{aligned}$$

$$R_j D_j = -D_j R_j; \qquad [D_i, D_j] = 0; \qquad [x_i, D_j] = \delta_{ij}\left(1 + 2\mu_{\delta_{ij}} R_{\delta_{ij}}\right). \quad \text{(no summation)}$$

- Reflection operators with respect to x_i=0 plane

$$R_x f(x,y) = f(-x,y), \qquad R_y f(x,y) = f(x,-y).$$

- In three dimensions (Three Wigner parameters and reflection operators)

$$\boxed{\begin{aligned}&\nabla_D^2 = D_1^2 + D_2^2 + D_3^2\\ &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{2\nu_1}{x}\frac{\partial}{\partial x} + \frac{2\nu_2}{y}\frac{\partial}{\partial y} + \frac{2\nu_3}{z}\frac{\partial}{\partial z} - \frac{\nu_1}{x^2}(1-R_1) - \frac{\nu_2}{y^2}(1-R_2) - \frac{\nu_3}{z^2}(1-R_3)\end{aligned}}$$

$$\hat{R}_1 f(x,y,z) = f(-x,y,z), \qquad \hat{R}_2 f(x,y,z) = f(x,-y,z), \qquad \hat{R}_3 f(x,y,z) = f(x,y,-z).$$

- Dunkl angular momentum operator

$$\hat{L}_z^{\nu_1,\nu_2} = -i(x D_y{}^{\nu_2} - y D_x{}^{\nu_1})$$

- In polar coordinates (Two Wigner parameters and reflection operators)

$$\boxed{\begin{aligned}\nabla_D^2 &= \frac{\partial^2}{\partial\rho^2} + \frac{1+2\nu_1+2\nu_2}{\rho}\frac{\partial}{\partial\rho} - \frac{2}{\rho^2}B_\phi \\ B_\phi &= -\frac{1}{2}\frac{\partial^2}{\partial\varphi^2} + (\nu_1\tan\varphi - \nu_2\cot\varphi)\frac{\partial}{\partial\varphi} + \frac{\nu_1(1-R_1)}{2\cos^2\varphi} + \frac{\nu_2(1-R_2)}{2\sin^2\varphi}\end{aligned}}$$

Reflection operators with respect to the plane x_i=0

$$R_x f(\rho,\varphi) = f(\rho,\pi-\varphi), \quad R_y f(\rho,\varphi) = f(\rho,-\varphi).$$

- In spherical coordinates (Three Wigner parameters and reflection operators)

$$\boxed{\begin{aligned}\nabla_D^2 &= -\frac{1}{2}\frac{\partial^2}{\partial r^2} - \frac{1+\nu_1+\nu_2+\nu_3}{r}\frac{\partial}{\partial r} + \frac{1}{2}r^2 + \frac{2}{r^2\sin\theta}B_\phi + \frac{1}{r^2}N_\theta \\ N_\theta &= -\frac{1}{2}\frac{\partial^2}{\partial\theta^2} + (\nu_3\tan\theta - (\frac{1}{2}+\nu_1+\nu_2)\cot\theta)\frac{\partial}{\partial\theta} + \frac{\nu_3}{2\cos^2\theta}(1-R_3)\end{aligned}}$$

$$R_1\psi(r,\theta,\varphi) = \psi(r,\theta,\pi-\varphi); \quad R_2\psi(r,\theta,\varphi) = \psi(r,\theta,-\varphi); \quad R_3\psi(r,\theta,\varphi) = \psi(r,\pi-\theta,\varphi).$$

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The Dunkl oscillator in the plane: I. Superintegrability, separated wavefunctions and overlap coefficients

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Abstract
The isotropic Dunkl oscillator model in the plane is investigated. The model is

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The Dunkl–Coulomb problem in the plane

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ABSTRACT

The Dunkl–Coulomb system in the plane is considered. The model is defined in terms of the Dunkl Laplacian, which involves reflection operators, with an r^{-1} potential. The system is shown to be maximally superintegrable and exactly solvable. The spectrum of the Hamiltonian is derived algebraically using a realization of $\mathfrak{so}(2,1)$ in terms of Dunkl operators. The symmetry operators generalizing the Runge–Lenz vector are constructed. On eigenspaces of fixed energy, the invariance algebra they generate

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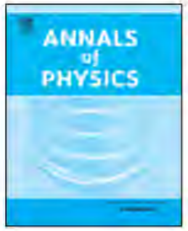

Exact solution of the relativistic Dunkl oscillator in (2 + 1) dimensions

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Landau levels for the (2 + 1) Dunkl–Klein–Gordon oscillator

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One-dimensional quantum mechanics with Dunkl derivative

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Dunkl–Maxwell equation and Dunkl-electrostatics in a spherical coordinate

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Thermal properties of relativistic Dunkl oscillators

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Abstract For a better understanding of the physical systems, even at the quantum level, the thermal quantities can be investigated. Recently, we realized that the parity of the system can also be examined simultaneously, by substituting the Dunkl operator to the ordinary differential operator in quantum mechanics. In this manuscript, we consider two relativistic Dunkl oscillators and investigate their thermal quantities with well-known statistical methods. Besides, we establish a relationship between the Dunkl–Dirac oscillator and the Dunkl–Anti–Jayne–Cummings model by defining the Dunk creation and annihilation operators. Therefore, we conclude that our model can be regarded as an appropriate scenario for the theory of an open quantum system coupled to a thermal bath.

Dunkl-Klein-Gordon oscillator

One dimensional stationary Klein-Gordon oscillator

$$\left[E^2 - \left(\frac{1}{i}\hat{D} + im\omega\hat{x}\right)\left(\frac{1}{i}\hat{D} - im\omega\hat{x}\right) - m^2\right]\psi(x) = 0,$$

Dunkl-Klein-Gordon oscillator

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{2\mu}{x}\frac{\mathrm{d}}{\mathrm{d}x} - \frac{\mu}{x^2}(1-s) - m^2\omega^2x^2 + m\omega(1+2\mu s) + E^2 - m^2\right]\psi^s(x) = 0.$$

where $\hat{R}\psi(x) = s\psi(x)$ with $s = \pm.$

Introduce $y = m\omega x^2$ Ansatz $\psi^s = y^{\frac{1-s}{4}} e^{-\frac{y}{2}} \Psi^s(y).$

$$\left[y\frac{\mathrm{d}^2}{\mathrm{d}y^2}+\left(1-\frac{s}{2}+\mu\right)\frac{\mathrm{d}}{\mathrm{d}y}-\frac{y}{4}+\frac{[1+2\mu s]}{4}+\frac{E^2-m^2}{4m\omega}\right]\Psi^s(y)=0.$$

Confluent hypergeometric equation

$$\Psi_n^s(y)=\mathcal{C}\mathbf{F}\left(\frac{(2\mu+1)(1-s)}{4}-\frac{E^2-m^2}{4m\omega},1-\frac{s}{2}+\mu;y\right).$$

Stationary wave function

$$\Psi_n^s(x)=\mathcal{C}(m\omega x)^{\frac{1-s}{2}}e^{-\frac{m\omega x^2}{2}}\mathbf{F}\left(\frac{(2\mu+1)(1-s)}{4}-\frac{E^2-m^2}{4m\omega},1-\frac{s}{2}+\mu;m\omega x^2\right).$$

$$\frac{E_n^s}{m}=\pm\sqrt{4nr+2r\left(\mu+\frac{1}{2}\right)\left(1-s\right)+1}\qquad\text{where } r\equiv\omega/m.$$

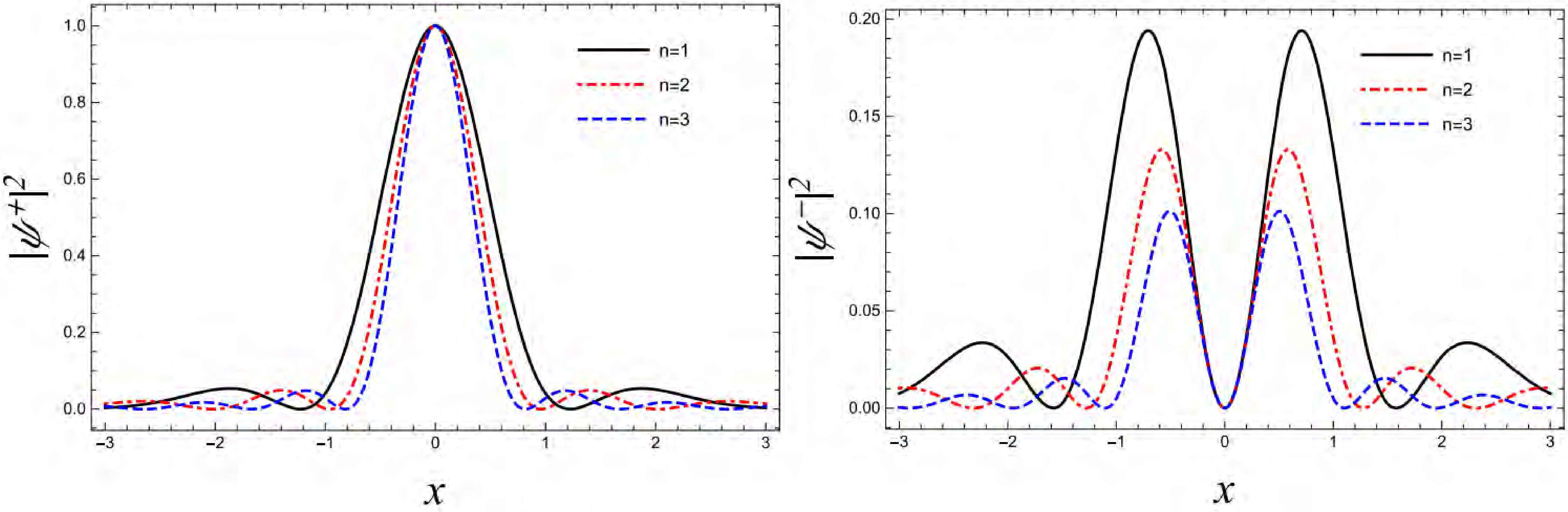
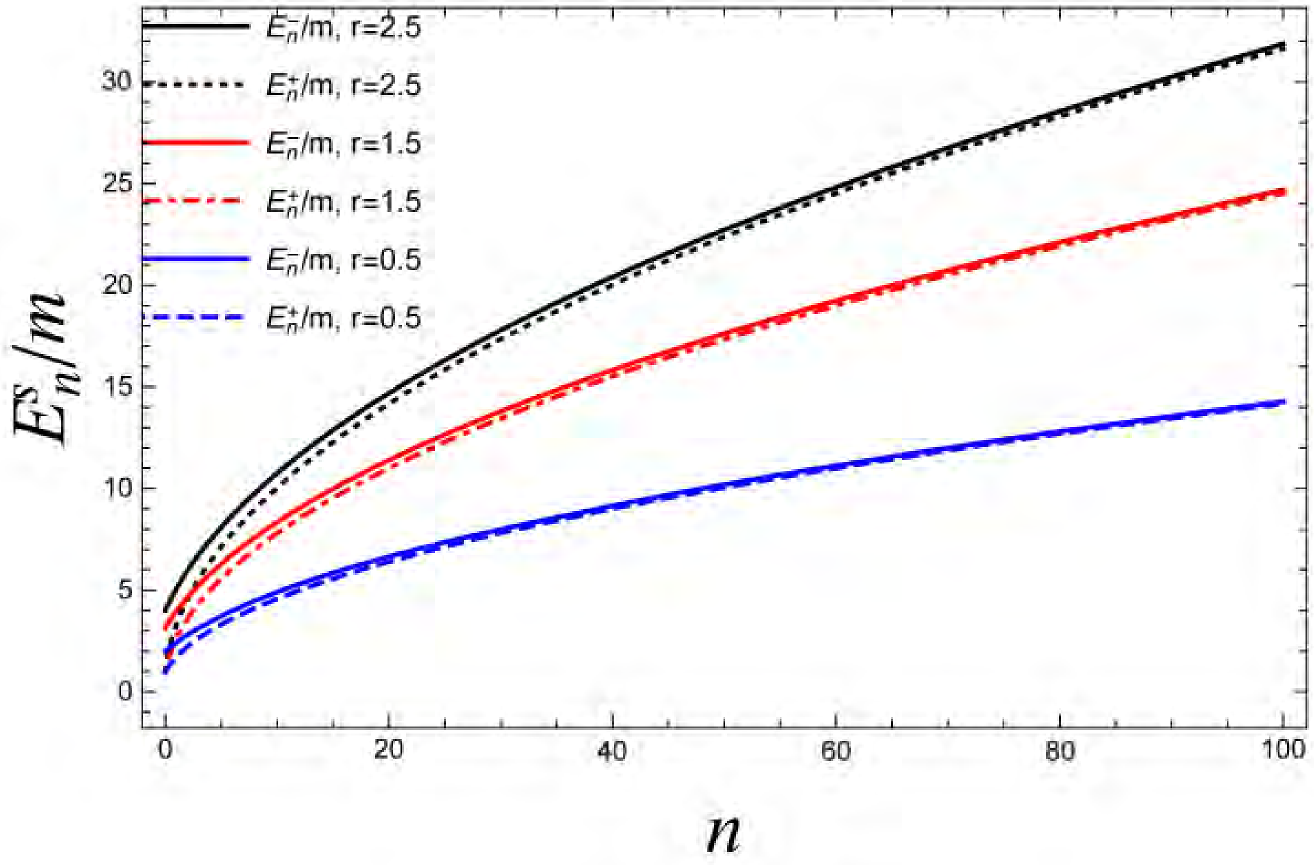


Fig. 1 Reduced probability densities versus coordinate

Fig. 2 Dunkl relativistic energy spectra versus the node numbers, where $r = \omega/m$



Dunkl-Dirac oscillator

$$\hat{\mathcal{H}}_D \psi = E^s \psi^s,$$

Dunkl-Dirac Hamilton operator

$$\hat{\mathcal{H}}_D = \left[\alpha_x \left(\frac{1}{i}\hat{D} - i\beta m\omega\hat{x}\right) + \beta m\right],$$

Considering the following spinor and Dirac matrices

$$\psi = \begin{pmatrix} \Phi \\ F \end{pmatrix} \qquad \alpha_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\psi_n^s = \mathcal{N}_s (m\omega x)^{\frac{1-s}{2}} e^{-\frac{m\omega x^2}{2}} \left(\begin{array}{c} 1 \\ \frac{-i}{(E+m)} \left[\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\left(\mu+\frac{1}{2}\right)(1-s)}{x} \right] \end{array} \right) \mathbf{F}\left(-n, 1 - \frac{s}{2} + \mu; m\omega x^2\right).$$

Establish a connection between Dunkl-Dirac oscillator and quantum optics, we introduce Dunkl-creation and annihilation operators

$$\hat{a}_D^\dagger = \frac{1}{\sqrt{2m\hbar\omega}}\left(m\omega\hat{x} - \hbar\hat{D}\right),$$
$$\hat{a}_D = \frac{1}{\sqrt{2m\hbar\omega}}\left(m\omega\hat{x} + \hbar\hat{D}\right),$$

satify

$$\left[\hat{a}_D^\dagger, \hat{a}_D\right] = 1 + 2\mu\hat{R}.$$

Dunkl – Dirac Hamiltonian

$$\hat{\mathcal{H}}_D = g\left(\sigma^-\hat{a}_D + \sigma^+\hat{a}_D^\dagger\right) + m\sigma_z,$$

with

$$\sigma^\mp = \frac{1}{2}(\sigma_x \mp \sigma_y).$$

corresponds to Dunkl-Anti-Jaynes-Cummings model.

It reduces to AJC model, if one takes Wigner parameter as zero.

Thermodynamics of relativistic Dunkl-oscillators

- The system is in thermal equilibrium with a termal bath at the temperature, T.

$$Z^s = \sum_{n=0} e^{-\frac{E_n^s - E_0^s}{K_B T}},$$

where K_B is the Boltzmann constant, and E_0^s is the ground-state energy corresponding to $n = 0$.

After testing the convergency of the series

$$\int_0^{+\infty} e^{-\frac{1}{\tau}\sqrt{ax+b}}\mathrm{d}x = \frac{2\tau^2}{a}\left(1 + \tau\sqrt{b}\right)e^{-\frac{1}{\tau}\sqrt{b}}, \qquad \text{for } \tau > 0.$$

We use Euler-MacLaurin formula

$$\sum_{n=0} f(n) = f(0) + \int_0^{+\infty} f(x)\mathrm{d}x - \sum_{p=1} \frac{B_{2p}}{(2p)!} f^{(2p-1)}(0),$$

where B_{2p} are the Bernoulli numbers, i.e., $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, and $B_6 = \frac{1}{42}$.

$$\sum_{n=0} f(n) = f(0) + \int_0^{+\infty} f(x)dx - \frac{1}{12} f^{(1)}(0) + \frac{1}{720} f^{(3)}(0) - \frac{1}{30240} f^{(5)}(0) + \cdots$$

Partition function

$$Z^s = \frac{1}{2} + \frac{\sqrt{\alpha_s}\tau}{2r} + \frac{\tau^2}{2r} + \frac{r}{6\sqrt{\alpha_s}\tau} + \mathcal{O}\left(\frac{1}{\tau^3}\right).$$

$$\alpha_s \equiv 2r\left(\frac{1}{2} + \mu\right)(1-s) + 1, \qquad \tau \equiv \frac{K_B T}{m} = \frac{T}{T_0}, \qquad r \equiv \frac{\omega}{m},$$

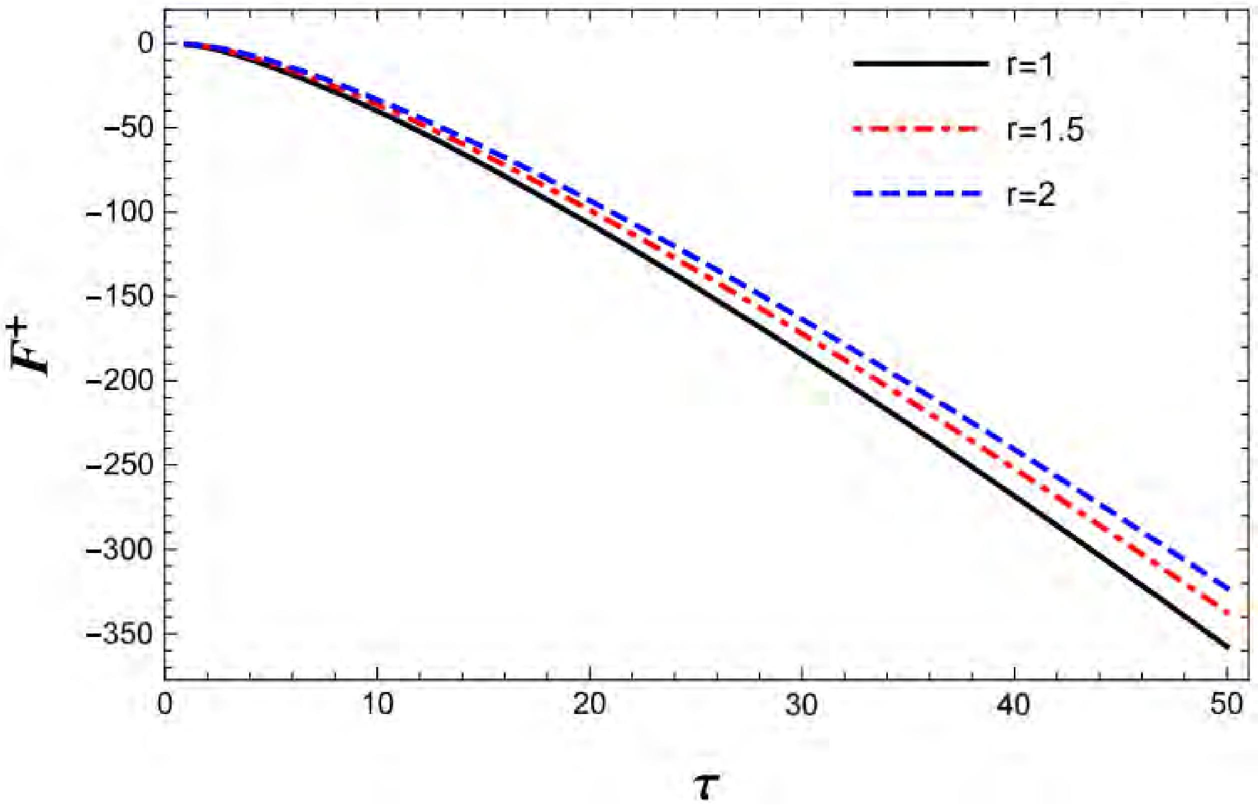
where $T_0 = \frac{m}{K_B}$ is the characteristic temperature that splits the range of temperature to very low temperature, $T << T_0$, and very high temperature, $T >> T_0$, regions

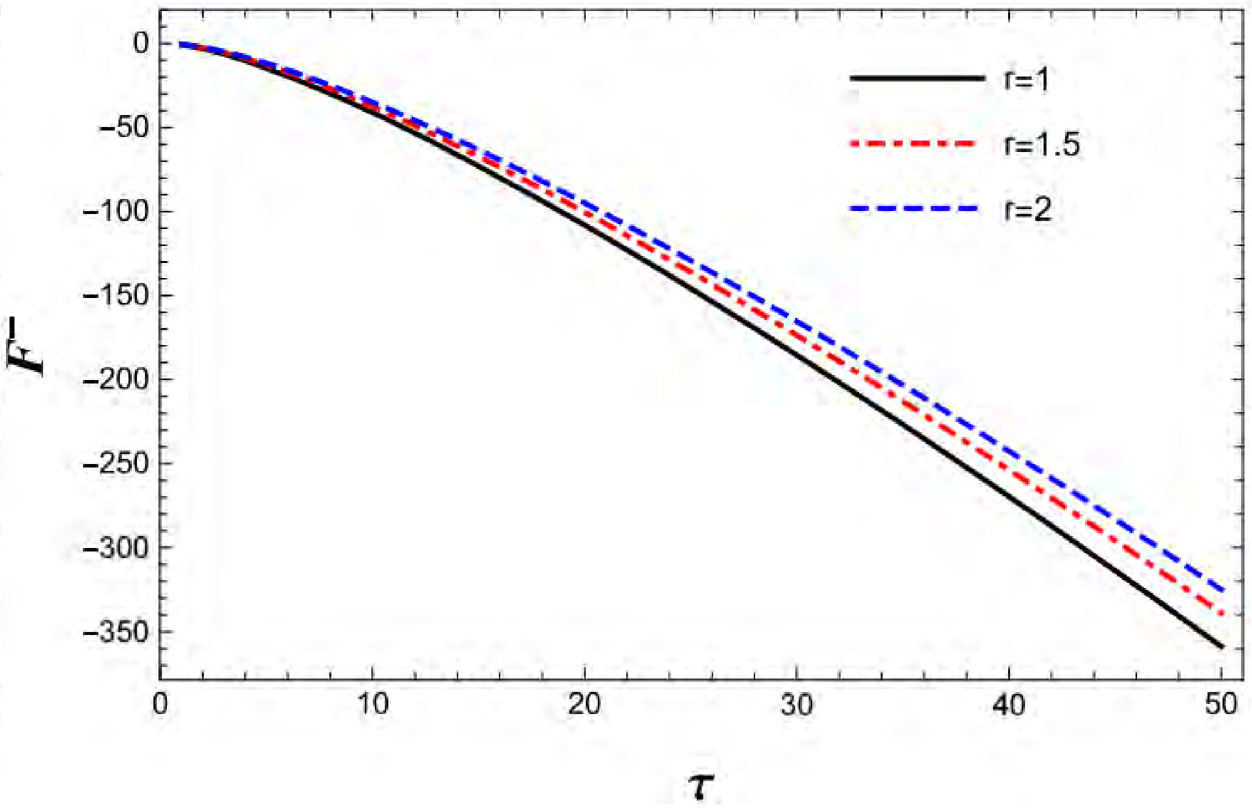
We observe that only in the odd case the Wigner parameter value takes a role.

in the high-temperature regime, $T \to \infty$, the Dunkl parameter's contribution remains negligible, $Z^+ = Z^- = \frac{\tau^2}{2r}$.

$$F^s = -\tau \ln Z^s,$$

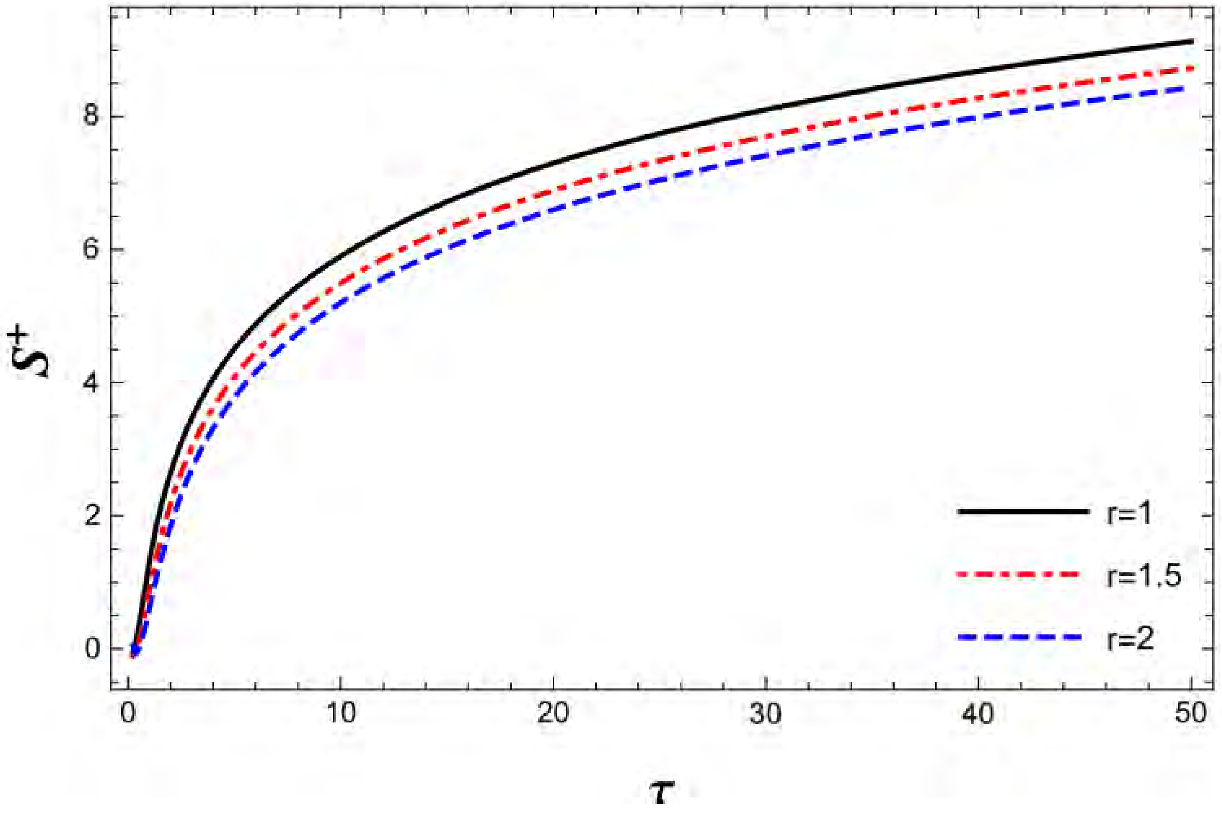
$$F^s = -\tau \ln\left[\frac{1}{2} + \frac{1}{2r}\left(\tau^2 + \tau\sqrt{\alpha_s}\right) + \frac{r}{6\tau\sqrt{\alpha_s}}\right].$$

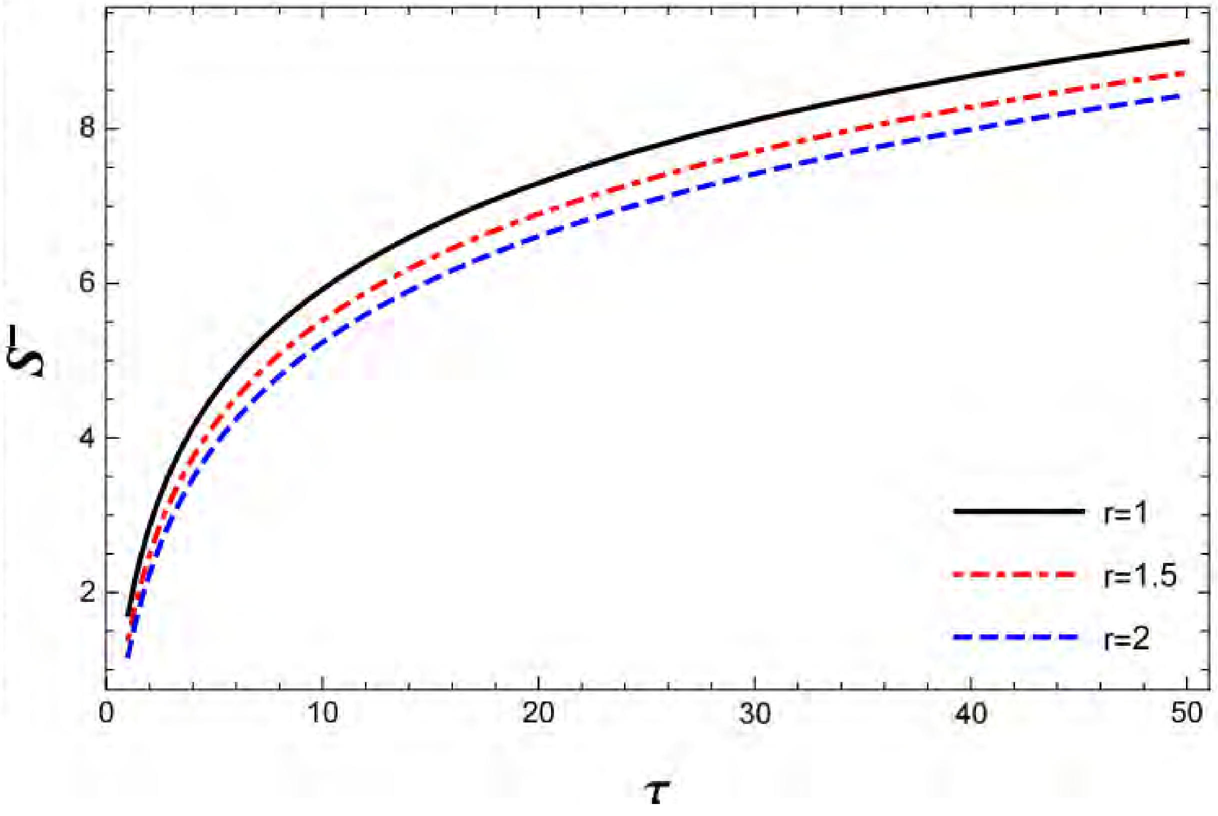




$$S^s = \ln Z^s + \tau \frac{\partial}{\partial \tau} \ln Z^s .$$

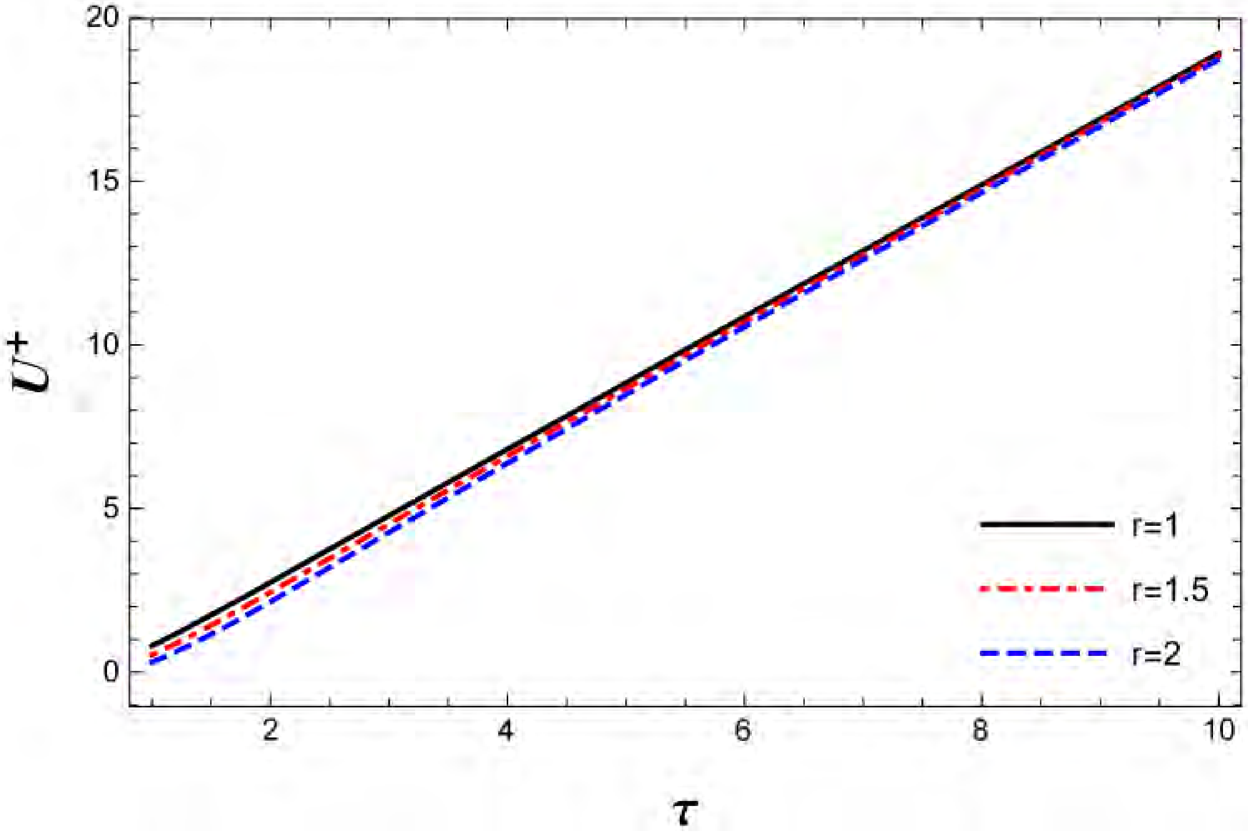
$$S^s = \ln\left[\frac{1}{2} + \frac{1}{2r}\left(\tau^2 + \tau\sqrt{\alpha_s}\right) + \frac{r}{6\tau\sqrt{\alpha_s}}\right] + \tau\left[\frac{\sqrt{\alpha_s}+2\tau}{2r} - \frac{r}{6\tau^2\sqrt{\alpha_s}}\right]\left[\frac{1}{2} + \frac{r}{6\tau\sqrt{\alpha_s}} + \frac{\tau\sqrt{\alpha_s}+\tau^2}{2r}\right]^{-1} .$$

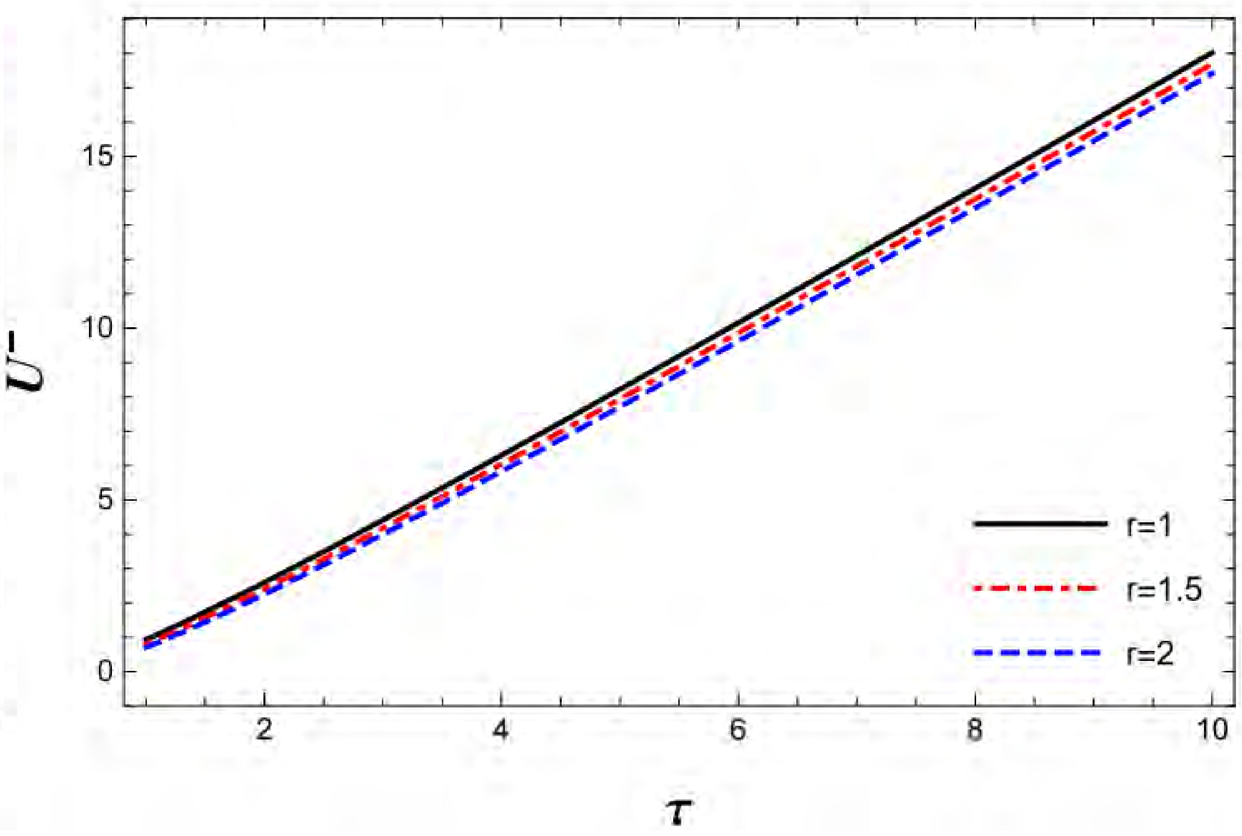




$$U^s = \tau^2 \frac{\partial}{\partial \tau} \ln Z^s.$$

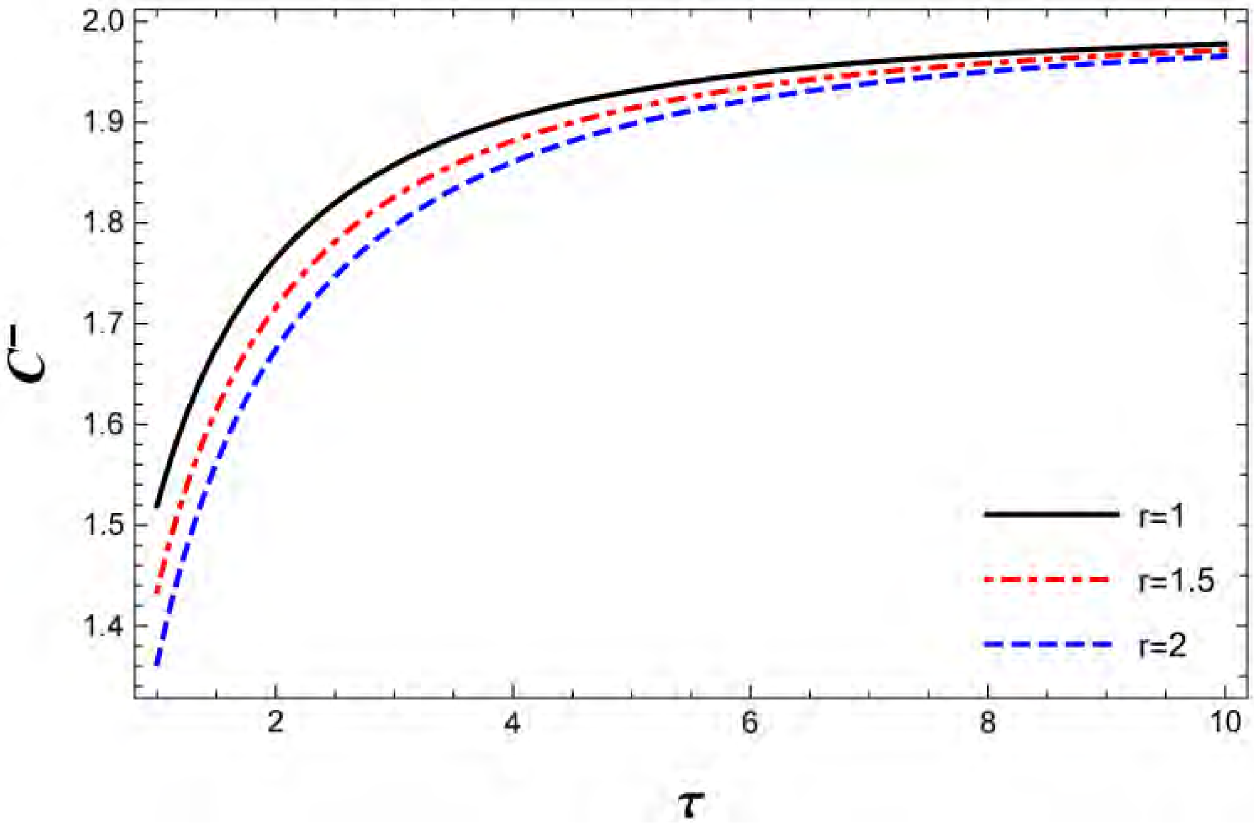
$$U^s = \tau^2 \left[\frac{\sqrt{\alpha_s} + 2\tau}{2r} - \frac{r}{6\sqrt{\alpha_s}\tau^2} \right] \left[\frac{1}{2} + \frac{r}{6\sqrt{\alpha_s}\tau} + \frac{\sqrt{\alpha_s}\tau + \tau^2}{2r} \right]^{-1}.$$

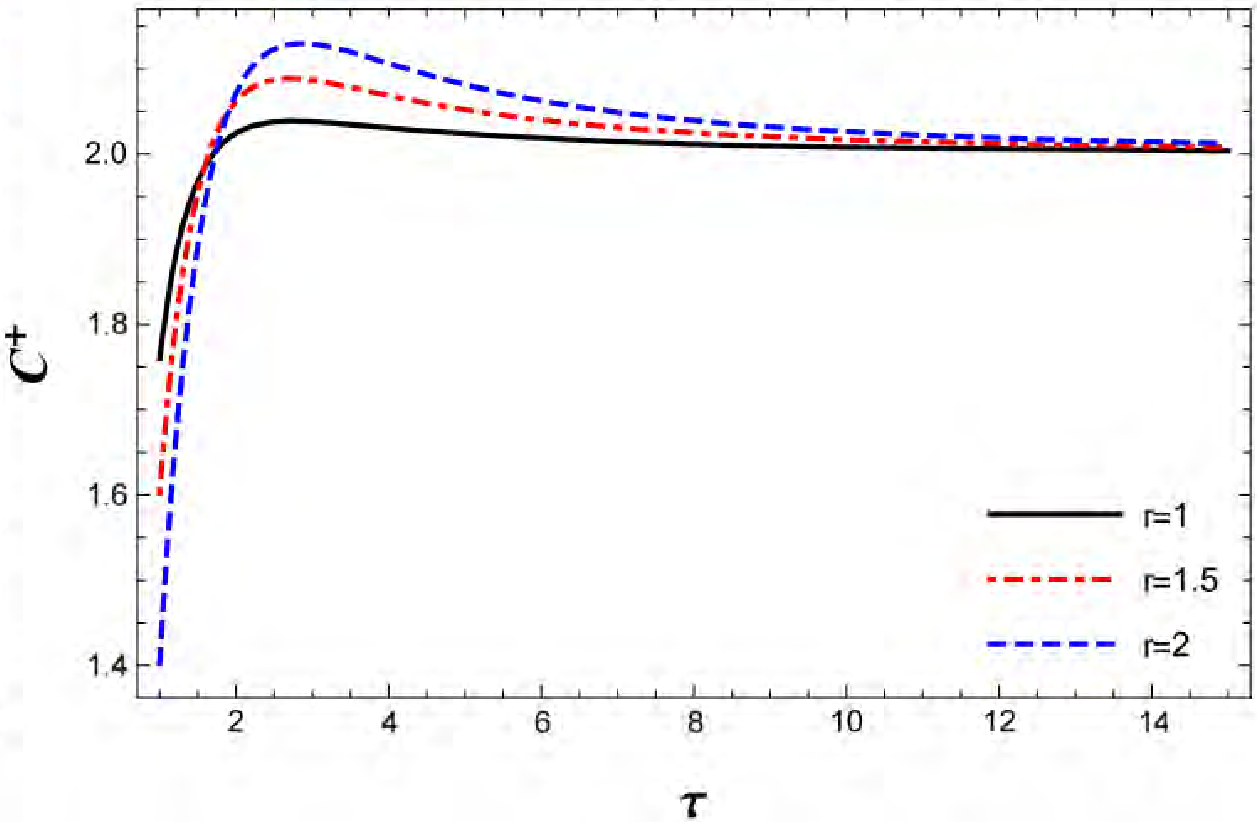




$$C^s = 2\tau \frac{\partial}{\partial \tau} \ln Z^s + \tau^2 \frac{\partial^2}{\partial \tau^2} \ln Z^s.$$

$$C^s = 2\tau \frac{\frac{\sqrt{\alpha_s}+2\tau}{2r} - \frac{r}{6\sqrt{\alpha_s}\tau^2}}{\frac{\sqrt{\alpha_s}\tau+\tau^2}{2r} + \frac{r}{6\sqrt{\alpha_s}\tau} + \frac{1}{2}} + \tau^2 \frac{\frac{r}{3\sqrt{\alpha_s}\tau^3} + \frac{1}{r}}{\frac{\sqrt{\alpha_s}\tau+\tau^2}{2r} + \frac{r}{6\sqrt{\alpha_s}\tau} + \frac{1}{2}} - \tau^2 \frac{\left(\frac{\sqrt{\alpha_s}+2\tau}{2r} - \frac{r}{6\sqrt{\alpha_s}\tau^2}\right)^2}{\left(\frac{\sqrt{\alpha_s}\tau+\tau^2}{2r} + \frac{r}{6\sqrt{\alpha_s}\tau} + \frac{1}{2}\right)^2}.$$





arXiv:2112.09948v1 [quant-ph]

Dunkl-Klein-Gordon equation in three-dimensions: The Klein-Gordon oscillator and Coulomb Potential

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Abstract

Recent studies show that deformations in quantum mechanics are inevitable. In this contribution, we consider a relativistic quantum mechanical differential equation in the presence of Dunkl operator-based deformation and we investigate solutions for two important problems in three-dimensional spatial space. To this end, after introducing the Dunkl quantum mechanics, we examine the Dunkl-Klein-Gordon oscillator solutions with the Cartesian and spherical coordinates. In both coordinate systems, we find that the differential equations are separable and their eigenfunctions can be given in terms of the associate Laguerre and Jacobi polynomials. We observe how the Dunkl formalism is affecting the eigenvalues as well as the eigenfunctions. As a second problem, we examine the Dunkl-Klein-Gordon equation with the Coulomb potential. We obtain the eigenvalue, their corresponding eigenfunctions, and the Dunkl-fine structure terms.

Dunkl-Klein-Gordon oscillator in 3-dimensions

$$\left\{E^2-\left(\frac{1}{i}D_j+im\omega x_j\right)\left(\frac{1}{i}D_j-im\omega x_j\right)-m^2\right\}\psi=0;\quad \text{with } j=\overline{1;3},$$

$$\left\{-D_1^2-D_2^2-D_3^2-2m\omega\left(\mu_1R_1+\mu_2R_2+\mu_3R_3+\frac{3}{2}\right)+m^2\omega^2\left(x_1^2+x_2^2+x_3^2\right)\right\}\psi=\left(E^2-m^2\right)\psi.$$

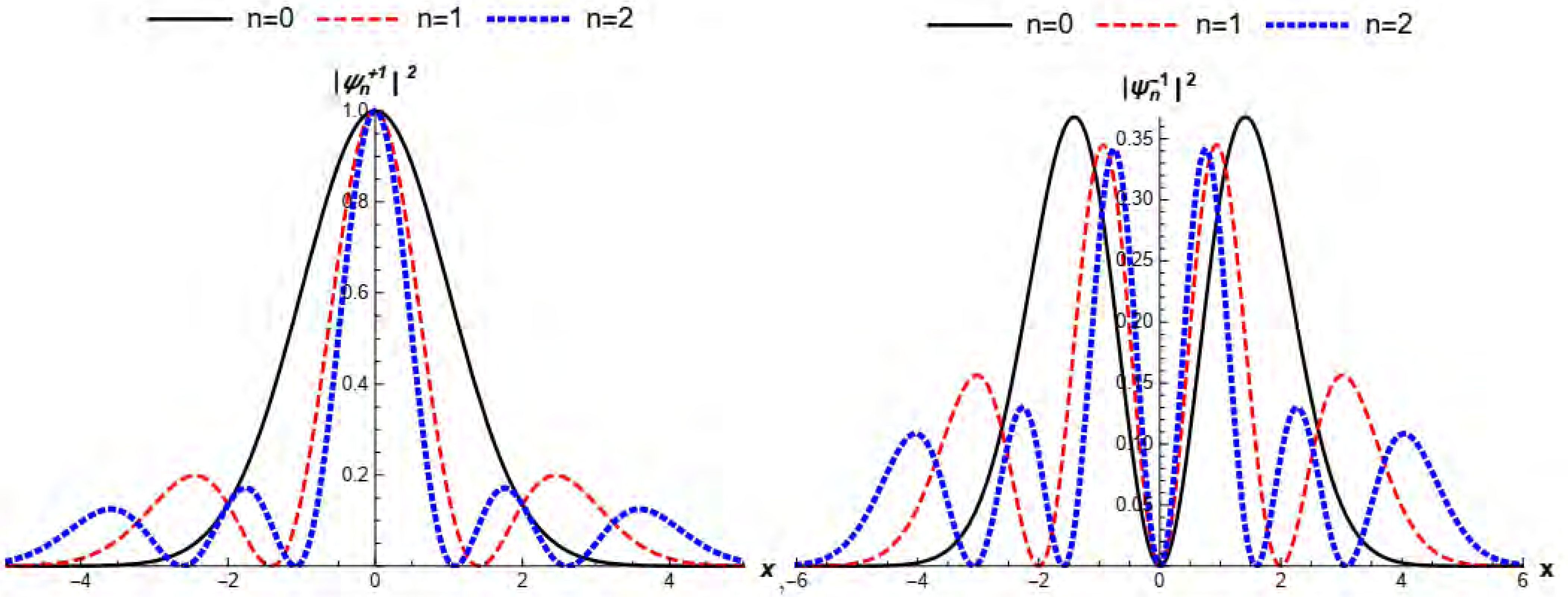
Solution in Cartesian coordinates

$$\begin{aligned} E^2-m^2 &= \mathcal{E}_1+\mathcal{E}_2+\mathcal{E}_3, \\ \psi &= \psi(x_1)\,\psi(x_2)\,\psi(x_3), \\ \mathcal{H} &= \mathcal{H}_1+\mathcal{H}_2+\mathcal{H}_3, \end{aligned}$$

where

$$\mathcal{H}_j=-D_j^2-m\omega\left(1+2\mu_jR_j\right)+m^2\omega^2x_j^2,\quad j=1,2,3.$$

$$E_n^{(s_1,s_2,s_3)} = \pm\sqrt{2m\omega\left[2n + \left(\mu_1 + \frac{1}{2}\right)(1-s_1) + \left(\mu_2 + \frac{1}{2}\right)(1-s_2) + \left(\mu_3 + \frac{1}{2}\right)(1-s_3)\right] + m^2},$$



Reduced probability densities versus coordinate.

Solution in Spherical coordinates

$$\begin{aligned}
\left\{\frac{\partial^2}{\partial\varphi^2}+2\left(\mu_2\cot\varphi-\mu_1\tan\varphi\right)\frac{\partial}{\partial\varphi}-\frac{\mu_1}{\cos^2\varphi}\left(1-R_1\right)-\frac{\mu_2}{\sin^2\varphi}\left(1-R_2\right)+\Omega^2\right\}\Phi\left(\varphi\right) &= 0,\\
\left\{\frac{\partial^2}{\partial\theta^2}+\cot\theta\frac{\partial}{\partial\theta}+2\left(\left(\mu_1+\mu_2\right)\cot\theta-\mu_3\tan\theta\right)\frac{\partial}{\partial\theta}-\frac{\mu_3}{\cos^2\theta}\left(1-R_3\right)-\frac{\Omega^2}{\sin^2\theta}+\varpi^2\right\}\Theta\left(\theta\right) &= 0,\\
\left\{\frac{\partial^2}{\partial r^2}+\frac{2\left(1+\mu_1+\mu_2+\mu_3\right)}{r}\frac{\partial}{\partial r}-m^2\omega^2r^2+2m\omega\left(\frac{3}{2}+\mu_1R_1+\mu_2R_2+\mu_3R_3\right)-\frac{\varpi^2}{r^2}+E^2-m^2\right\}F\left(r\right) &= 0,
\end{aligned}$$

$$E_{N,\nu,\ell}^{s_1,s_2,s_3}=\pm\sqrt{2m\omega\left[2\left(N+\nu+\ell\right)+\mu_1\left(1-s_1\right)+\mu_2\left(1-s_2\right)+\mu_3\left(1-s_3\right)\right]+m^2}.$$

We observe that the energy spectra depends on parity and Wigner parameters in addition to the usual quantum numbers.

Coulomb potential

$$V(r) = -\frac{Ze^2}{r}$$

Stationary Dunkl-Klein-Gordon equation with Coulomb potential

$$\left\{\left(E + \frac{Ze^2}{r}\right)^2 + \frac{\partial^2}{\partial r^2} + \frac{2(1+\mu_1+\mu_2+\mu_3)}{r}\frac{\partial}{\partial r} + \frac{\mathcal{J}_\varphi}{r^2\sin^2\theta} + \frac{\mathcal{J}_\theta}{r^2} - m^2\right\}\psi = 0$$

$$E_{n,\ell,\nu}(Z) = m\left\{1 + \frac{Z^2e^4}{\left(n + \frac{1}{2} + \sqrt{\left(\mu_1+\mu_2+\mu_3+2\nu+2\ell+\frac{1}{2}\right)^2 - Z^2e^4}\right)^2}\right\}^{-1/2}.$$

arXiv:2208.11729v1 [cond-matt.mes-hall]

Dunkl-Graphene in constant magnetic field

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Abstract

Graphene-based materials are thought to revolutionize entire industries. Therefore, many research are being carried on graphene theoretically and experimentally. On the other hand, recent studies show that the use of Dunkl derivative, instead of ordinary derivative, allows the concept of parity to be interpreted together with other physical quantities. In this manuscript, we investigate the thermal quantities of graphene under the constant magnetic field with the Dunkl-formalism. We observe that only at low temperatures Dunkl-parameters, thus parity, modify the conventional results.

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Investigation of the Dunkl-Schrödinger equation for Position Dependent Mass in the presence of a Lie algebraic approach

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Abstract

Recent studies have shown that the use of Dunkl derivatives instead of ordinary derivatives leads to deriving parity-dependent dynamic solutions. According to this motivation in this manuscript, we formulate the Dunkl-Schrödinger equation within the position-dependent mass formalism and derive an algebraic solution out of it. Our systematic approach lets us observe some new findings in addition to the earlier ones. For example, we find that the solution of the Dunkl-Schrödinger equation with position-dependent mass cannot be considered independent from the choice of parameters. Similarly, through the sl(2) algebra, the energy spectrum and the corresponding wave functions are derived in terms of possible Dunkl, (μ), and mass, (α), parameters.

Keywords: Dunkl derivative; Position-dependent mass; Quasi-Exactly Solvable (QES); sl(2) Lie algebra.

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Energy corrections of the two dimensional Dunkl harmonic oscillator in the Non-Commutative phase-space

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Abstract

In this paper, we examine the harmonic oscillator problem in non-commutative phase space (NCPS) by using the Dunkl derivative instead of the habitual one. After defining the Hamilton operator, we use the perturbation method to derive the binding energy eigenvalues. We find eigenfunctions that correspond to these eigenvalues in terms of the Laguerre functions. We observe that the Dunkl-Harmonic Oscillator (DHO) in the NCPS differs from the ordinary one in the context of providing additional information on the even and odd parities. Therefore, we conclude that working with the Dunkl operator could be more appropriate because of its rich content.

Generalized-Dunkl operator

$$D_x^{\alpha,\beta,\gamma} = \frac{\partial}{\partial x} + \frac{\alpha}{x} + \frac{\beta}{x}R + \gamma\frac{\partial}{\partial x}R,$$

W. S. Chung, H. Hassanabadi, Eur. Phys. J. Plus. **136**, 239 (2021).

$$\gamma_1 = \gamma,\ \beta_1 = -\alpha_1$$

$$\tilde{D}_1 \equiv \frac{\partial}{\partial x} + \frac{\mu_1}{x}(1 - R_1) + \gamma\frac{\partial}{\partial x}R_1 = D_1^{\mu_1} + \gamma\frac{\partial}{\partial x}R_1,$$

R. D. Mota, D. Ojeda-Guillén, arXiv:2207.10048v2 [quant-ph].

$\gamma_i = 0$ and $\beta_i = -\alpha_i$

Ordinary-Dunkl operator

arXiv:2208.10471v1 [quant-ph]

Relativistic Solutions of Generalized-Dunkl Harmonic and Anharmonic Oscillators

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Abstract

Dunkl derivative enriches solutions by discussing parity due to its reflection operator. Very recently, one of the authors of this manuscript presented one of the most general forms of Dunkl derivative that depends on three Wigner parameters to have a better tuning. In this manuscript, we employ the latter generalized Dunkl derivative in a relativistic equation to examine two dimensional harmonic and anharmonic oscillators solutions. We obtain the solutions by Nikiforov-Uvarov and QES methods, respectively. We show that degenerate states can occur according to the Wigner parameter values.

Non-relativistic particles in the polar coordinates in the presence of Generalized Dunkl derivatives

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Abstract

In this paper, we considering the generalized Dunkl derivatives for non-relativistic particles. First, we written them in Cartesian coordinates and then converted them to polar coordinates. Then we examined the eigenfunctions and eigenvalues once in the presence of the harmonic oscillator potential and once in the presence of the Coulomb potential.

THANK YOU