Quarks in a finite volume and deconfinement as percolation of center-electric fluxes in QCD

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PRD 106 (2022) 054513 [arXiv:2206.11697]

Bundesministerium für Bildung und Forschung

CRC-TR 211 Strong-interaction matter under extreme conditions

HFHF Helmholtz Forschungsakademie Hessen für FAIR

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Outline

- Quark in a finite volume — why bother?
- Center-vortex and electric-flux ensembles in pure gauge theory
- Stacks of closed center vortex sheets for full QCD
- Flux tube model for heavy-dense QCD
- Percolation of electric fluxes
- Summary and Outlook

- **Why bother?**
- **consider canonical partition functions:**

$$Z_c(T,V,N_q) = 0\,, \quad \text{for} \quad N_q \neq 0 \mod 3$$

→ Polyakov loop paradox

Kratochvila & de Forcrand, PRD 73 (2006) 114512

- **follows from Roberge-Weiss symmetry**

Roberge & Weiss, NPB 275 (1986) 734

- **imaginary chemical potential:** $\mu/T = i\theta$

fugacity expansion

→ Fourier series

period $2\pi/3$ →

$$Z^I(\theta) \equiv Z(T,V,i\theta T) = \sum_{N_q} e^{iN_q\theta}\, Z_c(T,V,N_q)$$

grand canonical at imaginary μ

canonical ensembles

→ only every 3rd Fourier coefficient ≠ 0

Single Quark in Finite Volume

…we've just changed the temporal b.c.'s without changing the spatial ones!

- **change spatial b.c.'s to account for Gauss' law**

back up — pure *SU*(*N*) gauge theory:

(d+1)-dim spacetime

$$Z_e(\vec{e}) = \frac{1}{N^d} \sum_{\vec{k} \in \mathbb{Z}_N^d} e^{2\pi i\, \vec{e}\cdot\vec{k}/N} \, Z_k(\vec{k})$$

't Hooft's electric flux ensembles

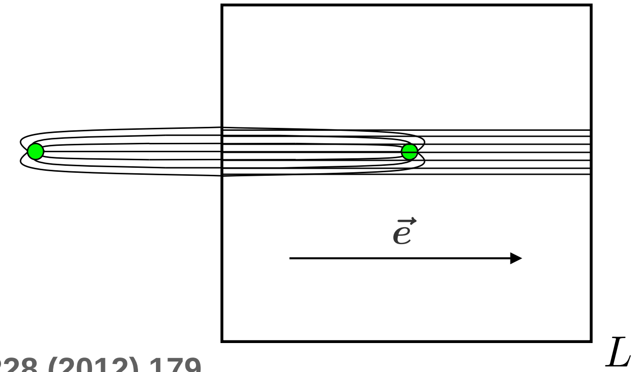
(mirror charges)

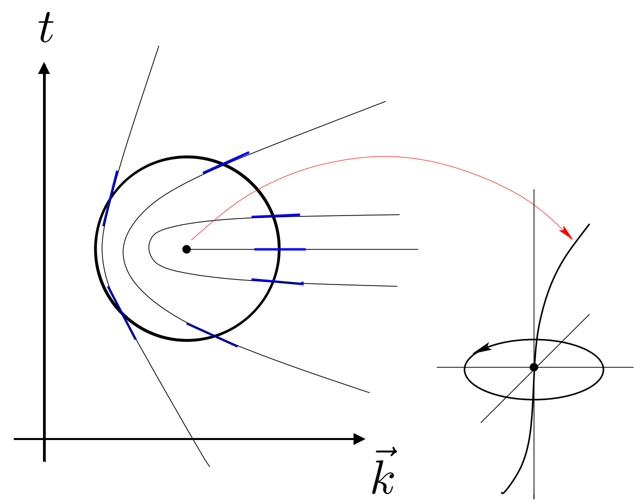
't Hooft's twisted b.c.'s

(no of center vortices mod N)

dual ⟷

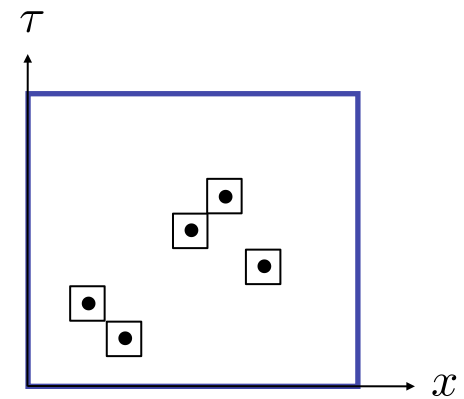
$$\frac{Z_e(\vec{e})}{Z_e(0)} = \frac{1}{N} \Big\langle \mathrm{tr}\big(P_\Omega(\vec{x}) P_\Omega^\dagger(\vec{x} + \vec{e}L)\big) \Big\rangle_{\text{no-flux}}$$





LvS, NPB (PS) 228 (2012) 179

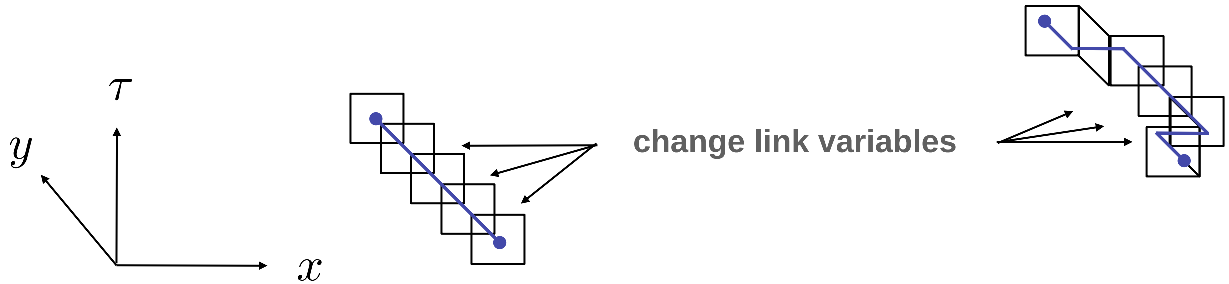
't Hooft's Twisted B.C.'s



- **fix total # mod. *N* of center vortices through planes**

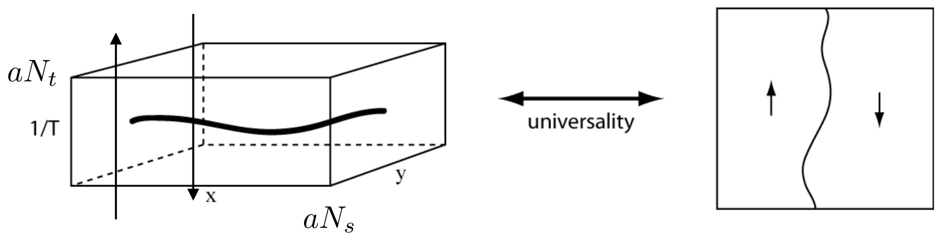
- **implement on lattice**

  multiply plaquette couplings by non-trivial center element



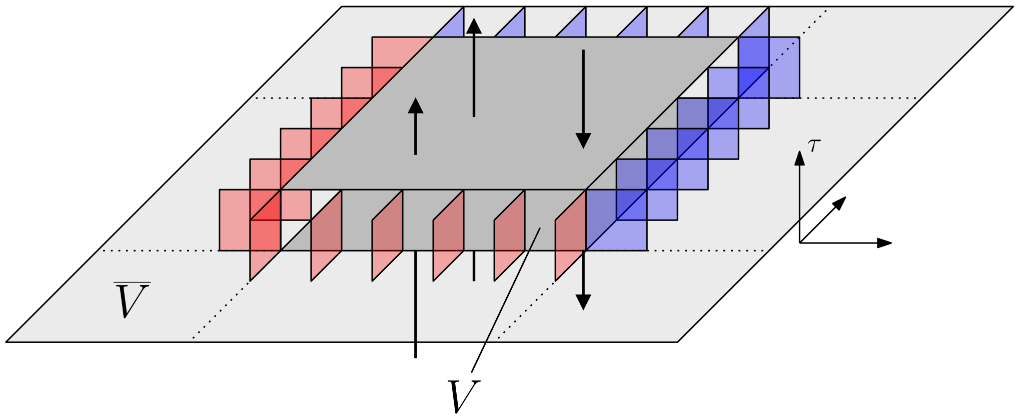
- **for Polyakov loops**

  act as interfaces in spin model dual to electric fluxes



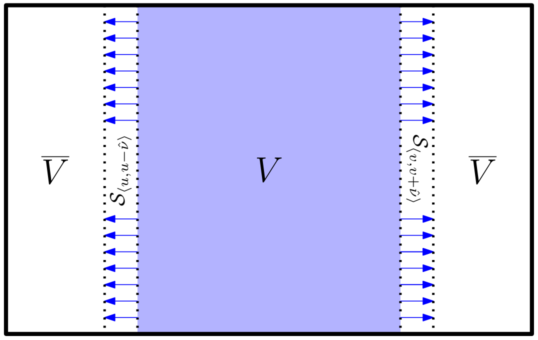
Closed Center Vortex Sheets

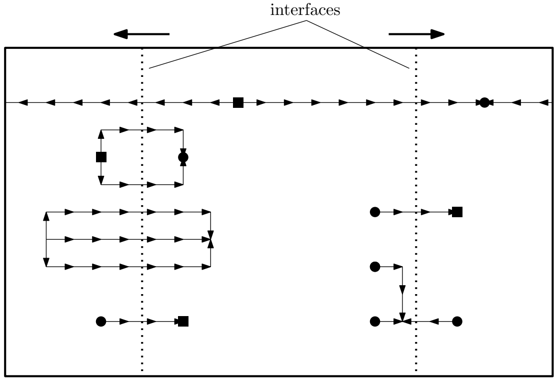
- **pure gauge theory**
  remove with variable transform
- **heavy-dense limit of QCD**
  static fermion determinant
- ***Z_3*-Fourier transform over closed center vortex sheets**



fix electric flux through $S = \partial V$

or net quark number mod. 3 inside

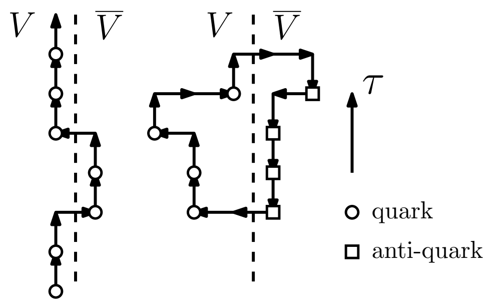




Full Lattice QCD

- **with arbitrary spatial hops**

  (anti-)quarks/diquarks can hop in and out of *V*



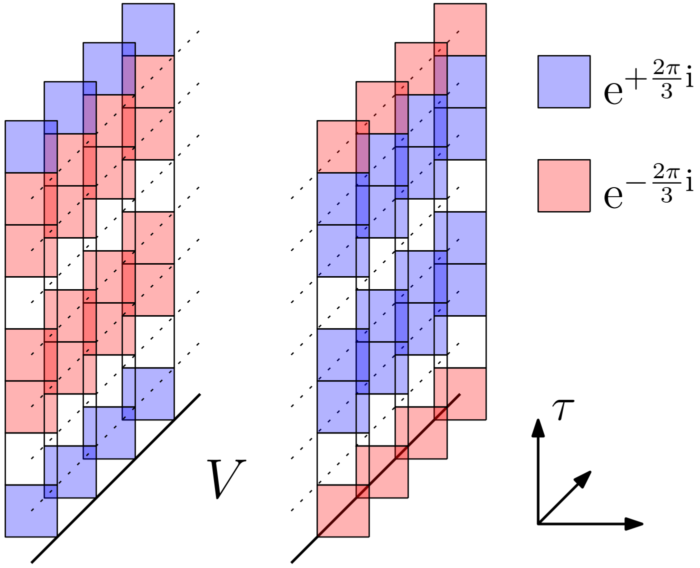
- **introduce between *all* time slices**

  N_τ **closed center-vortex sheets**

- ***Z*₃-Fourier transforms**

  over N_τ **closed center-vortex sheets**

  → selective static membrane at $S = \partial V$
  (only hadrons can pass)



Full Lattice QCD

- **final result, to fix charge in *V***

$$Z(q_V \mod 3 = e) = \frac{1}{3^{N_\tau}} \sum_{\{z_\tau \in Z_3\}} \left[\prod_{\tau=1}^{N_\tau} z_\tau^{-e} \right] Z(\{z_\tau\})$$

total charge (net quark number) modulo 3 in sub-volume *V*, write $q_V =_3 e$

with

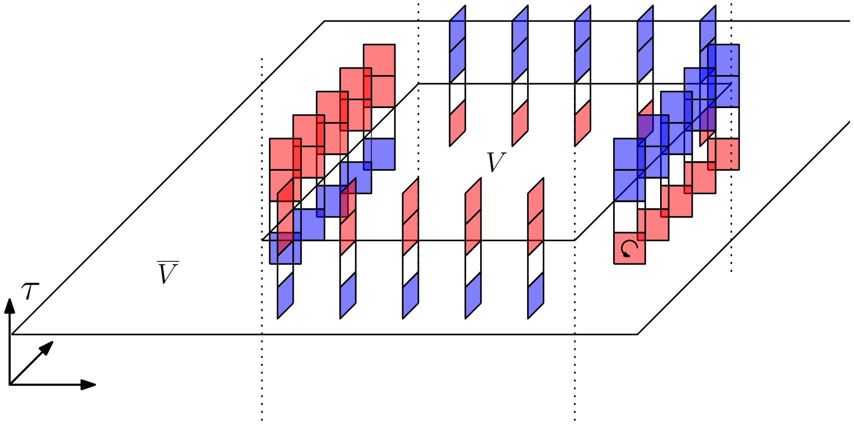
$$Z(\{z_\tau\}) = \int \mathcal{D}[\ldots]\, \mathrm{e}^{-S_G(\{z_\tau\},U) - S_F(U,\overline{\psi},\psi)}$$

only in gauge action

- **twisted plaquette action**

$$S_G(\{z_\tau\}, U) = -\frac{2}{g^2} \sum_p \mathrm{ReTr}\big(z(p)U_p\big)$$

$$z(p_{(i,\tau),\mu\nu}) = \begin{cases} z_\tau, & \nu = 4, \mu = k, \langle i, i+\hat{k}\rangle \in S^* \\ z_\tau^{-1}, & \nu = 4, \mu = k, \langle i+\hat{k}, i\rangle \in S^* \\ 1, & \text{otherwise} \end{cases}$$



- **effective Polyakov-loop theory**

(1 flavor Wilson)

$$Z_{\text{eff}} = \int \Big(\prod_i \mathrm{d}L_i \, J(L_i) \, Q(L_i) \Big) \prod_{\langle i,j \rangle} \left(1 + 2\lambda \, \mathrm{Re} L_i L_j^* \right)$$

Fromm, Langelage, Lottini, Philipsen, JHEP 01 (2012) 042
Langelage, Neuman, Philipsen, JHEP 09 (2014) 131

leading order hopping expansion
static fermion determinat → site factors

$$Q(L) = \left(1 + hL + h^2 L^* + h^3\right)^2 \left(1 + \bar{h} L^* + \bar{h}^2 L + \bar{h}^3\right)^2$$

where

$$h(\mu) = e^{(\mu - m)/T}$$

$$\bar{h}(\mu) = h(-\mu)$$

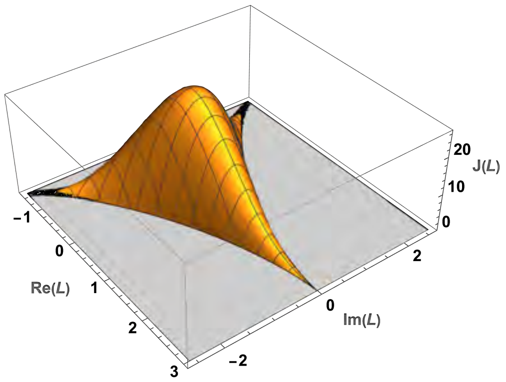
Pietri, Feo, Seiler, Stamatescu, PRD 76 (2007) 114501

Heavy-Dense QCD

- **reduce to Potts model**

maintain leading moments of reduced Haar measure

$$T_{m,n} = \int \mathrm{d}L\, J(L)\, L^m L^{*n}$$

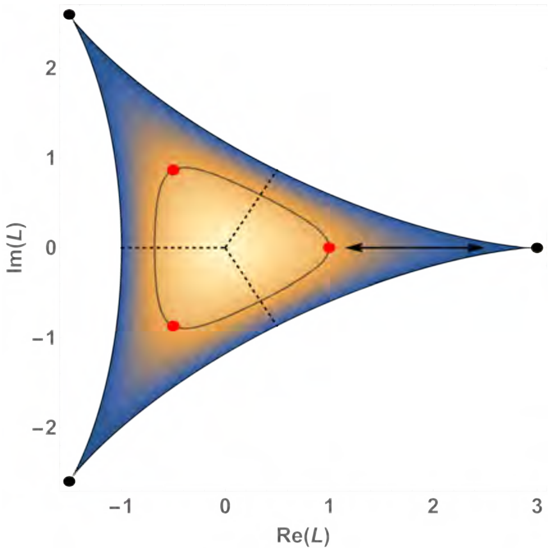


tabulated in

Uhlmann, Meinel, Wipf, J. Phys. A 40 (2007) 4367

replace group integrations

$$L \to z \in Z_3\,, \quad \int \mathrm{d}L\, J(L)\, f(L) \to \frac{1}{3}\sum_{z\in Z_3} f(z)$$



reproduces $T_{m,n}$ with $m+n<4$

- **o.k. in heavy-dense limit**

at strong coupling, or $T \lesssim T_c$

Smith, Dumitru, Pisarski, LvS, PRD 88 (2013) 054020
Endrodi, Gattringer, Schadler, PRD 89 (2014) 054509

Effective Potts Model

- **for QCD at strong coupling**

  with static fermion determinant

$$Z_{\rm eff} = \frac{1}{3^{N_s}} \sum_{\{z_i \in Z_3\}} \prod_{\langle i,j\rangle} \left(1 + 2\lambda \,\mathrm{Re}\, z_i z_j^*\right) \times$$

$$\left(\prod_i \left(1 + h z_i + h^2 z_i^* + h^3\right)^2 \left(1 + \bar{h} z_i^* + \bar{h}^2 z_i + \bar{h}^3\right)^2 \right)$$

$$= \mathcal{N} \sum_{\{z_i \in Z_3\}} \exp\left\{ \sum_{\langle i,j\rangle} 2\gamma \,\mathrm{Re}\, z_i z_j^* \right\} \times$$

$$\left(\prod_i \left(1 + h z_i + h^2 z_i^* + h^3\right)^2 \left(1 + \bar{h} z_i^* + \bar{h}^2 z_i + \bar{h}^3\right)^2 \right)$$

with $\gamma = \frac{1}{3} \ln\left(\frac{1+2\lambda}{1-\lambda}\right)$

- **Roberge-Weiss symmetric**

  from global Z_3 symmetry

$$Z_{\rm eff}(T, \mu = i\theta T) \equiv Z^I_{\rm eff}(\theta) = Z^I_{\rm eff}(\theta + 2\pi/3)$$

Equivalent Flux-Tube Model

- **flux-tube model representation (dual)**

$$Z_{\text{eff}}(T,\mu) = \sum_{\{n,l\}_{\text{phys}}} \exp\Big\{-\beta\Big(H(n,l) - \mu \sum_i q_i\Big)\Big\}$$

analogous to:

Patel, NPB 243 (1984) 411

Bernard, DeGrand, DeTar, Gottlieb, Krasnitz, Sugar, Toussain, PRD 49 (1994) 6051

Condella & DeTar, PRD 61(2000) 074023

here with:

string tension

$$H(n,l) = \sum_{\langle i,j\rangle} \sigma |l_{\langle i,j\rangle}| + \sum_{i,s} m(n_{i,s} + \bar{n}_{i,s})$$

fluxes represented by link variables: $l_{\langle i,j\rangle} \in \{-1,0,1\}$

(anti-)quark occupation numbers: $n_{i,s} \in \{0,\ldots,3\}$ and $\bar{n}_{i,s} \in \{0,\ldots,3\}$ spin *s* = {↑,↓}

- ***Z*₃-Gauss' law:**

(Poisson equation)

$$\sum_{j\sim i} l_{\langle i,j\rangle} - \sum_s (n_{i,s} - \bar{n}_{i,s}) = 0 \mod 3$$

flux from volume around site *i*

$$\phi_i = q_i \mod 3$$

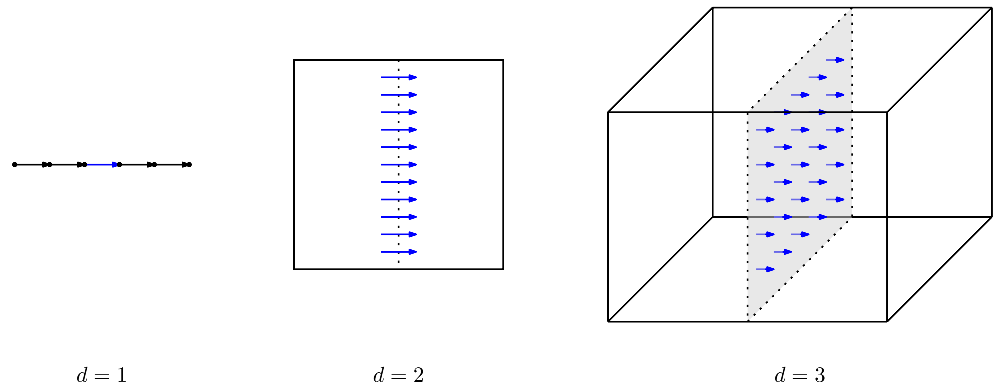
net-quark number modulo 3

Electric fluxes

- **interfaces in flux-tube model**

  dual stacks of links S^*

$$H(n,l) = \sum_{\langle i,j\rangle} \sigma |l_{\langle i,j\rangle}| + \sum_{i,s} m(n_{i,s} + \bar{n}_{i,s})$$



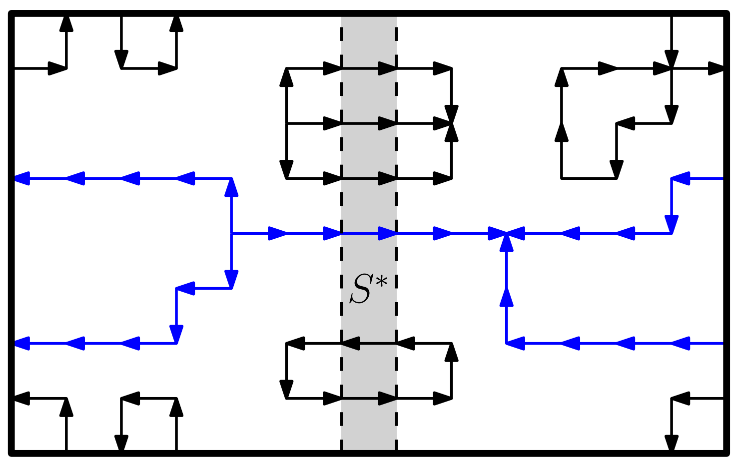
- **flux through interface**

$$\phi_S = \sum_{\langle i,j\rangle \in S^*} l_{\langle i,j\rangle}$$

- **add constraint to fix**

$$\phi_S = e \mod 3$$

example flux config without (anti-)quarks



$\phi_S \mod 3 = 1$

Ensembles with Fixed Flux

- **flux-tube model with interface**

$$Z(q_V =_3 e)$$

$$= \sum_{\{l,n\}} \exp\Big\{ -\beta\Big(H(\{l,n\}) - \mu \sum_i q_i\Big)\Big\} \underbrace{\Big(\frac{1}{3}\sum_{z\in Z_3} z^{\phi_S - e}\Big)}_{\text{flux constraint}} \underbrace{\prod_j \Big(\frac{1}{3}\sum_{z\in Z_3} z^{\phi_j - q_j}\Big)}_{\text{Gauss' law}}$$

$$= \frac{1}{3}\sum_{z\in Z_3} z^{-e} \sum_{\{l,n\}} \exp\Big\{ -\beta\Big(H(\{l,n\}) - \mu \sum_i q_i\Big)\Big\} z^{\phi_S} \prod_j \Big(\frac{1}{3}\sum_{z\in Z_3} z^{\phi_j - q_j}\Big)$$

- **back to dual Potts model**

$$\sum_{\{l\}} \Big(\prod_{\langle i,j\rangle} e^{-\beta\sigma |l_{\langle i,j\rangle}|}\Big) z^{\phi_S} \Big(\prod_k z_k^{\phi_k}\Big)$$

$$= \prod_{\langle i,j\rangle} \big(1 + 2\lambda\, \mathrm{Re}\,\big(z^{-s_{\langle i,j\rangle}} z_i z_j^*\big)\big)$$

with interface:

$$s_{\langle i,j\rangle} = \begin{cases} 1, & \langle i,j\rangle \in \mathcal{S}^* \\ -1, & \langle j,i\rangle \in \mathcal{S}^* \\ 0, & \text{otherwise} \end{cases}$$

Ensembles with Fixed Flux

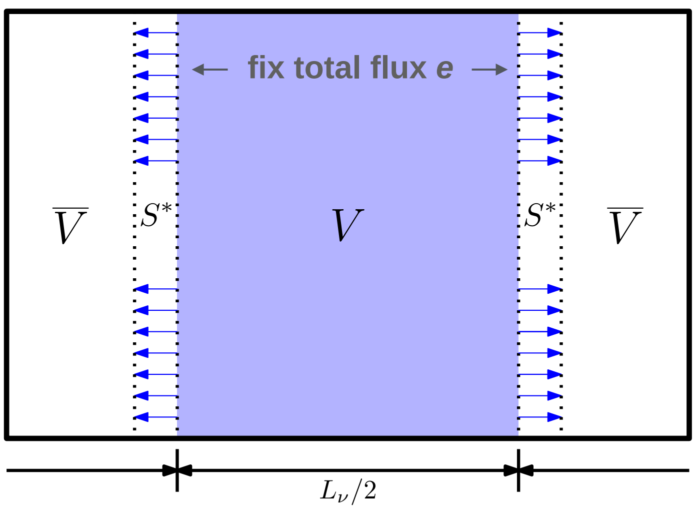
- **Z_3-Fourier transform**

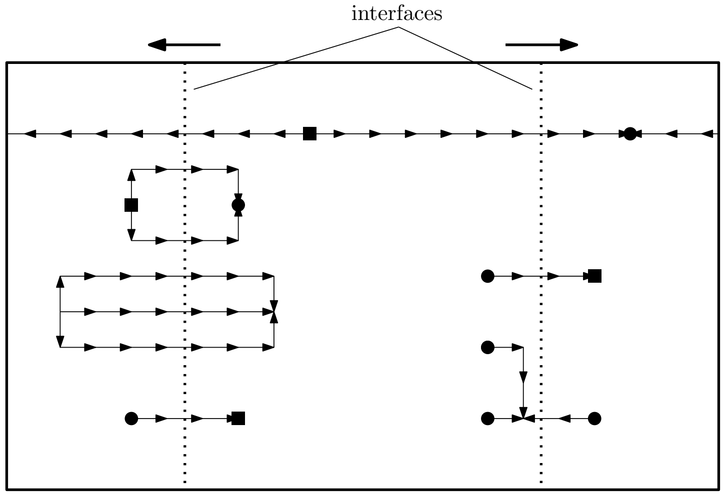
$$Z(q_V =_3 e) = \frac{1}{3} \sum_{z\in \mathbb{Z}_3} z^{-e}\, Z_S(z)$$

Z_3-flux ensembles **Z_3-interface ensembles**

- **interface ensembles**

$$Z_S(z) = \sum_{\{z_i \in \mathbb{Z}_3\}} \exp\Big\{ \sum_{\langle i,j\rangle} 2\gamma \,\mathrm{Re}\left(z^{-s_{\langle i,j\rangle}}\, z_i z_j^*\right)\Big\} \prod_i Q(z_i)$$





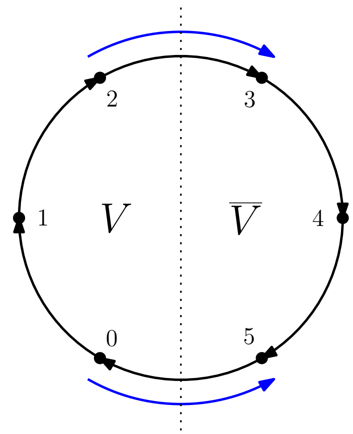
total flux e = 2 = q_V mod 3

Test in (1+1)d

- **Potts model representation**

total charge in sub-volume

$V: \; q_V \mod 3 = 1$



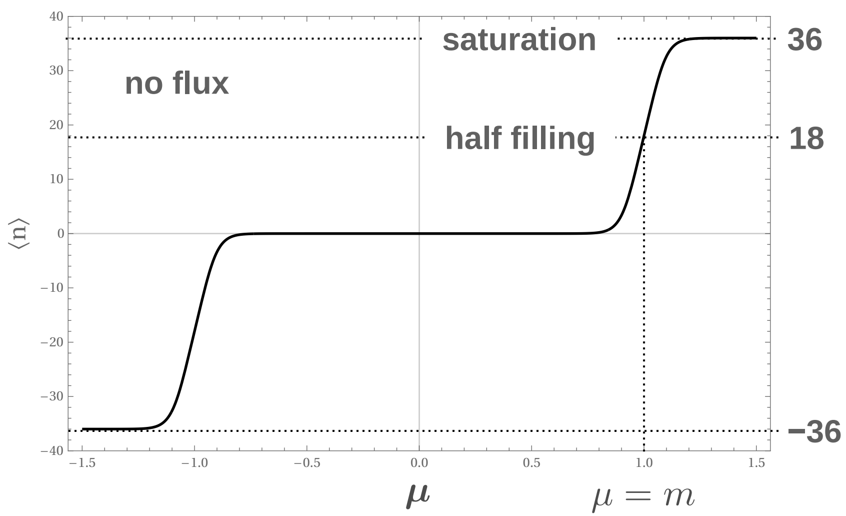
total charge in complement

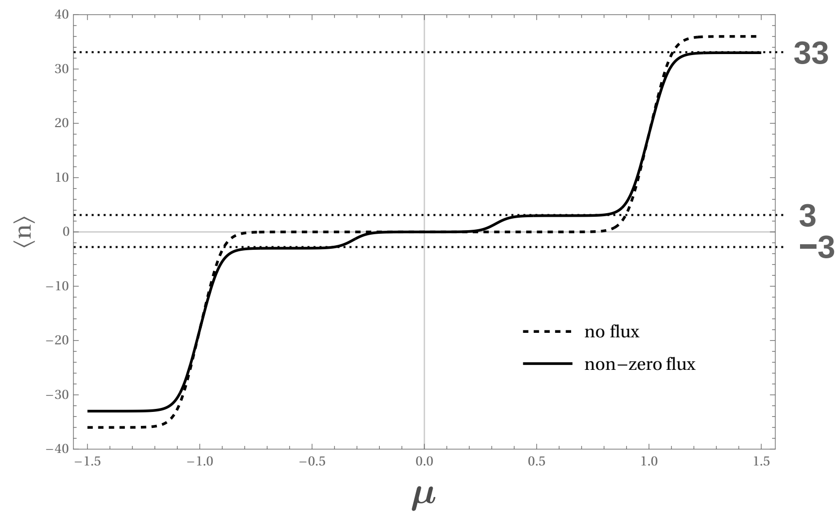
$\overline{V}: \; q_{\overline{V}} \mod 3 = 2$

total flux: $\phi_S \mod 3 = 1$

- **compare:**

$L = 6\,, \quad \sigma a/m = 0.3\,, \quad T/m = 0.1$

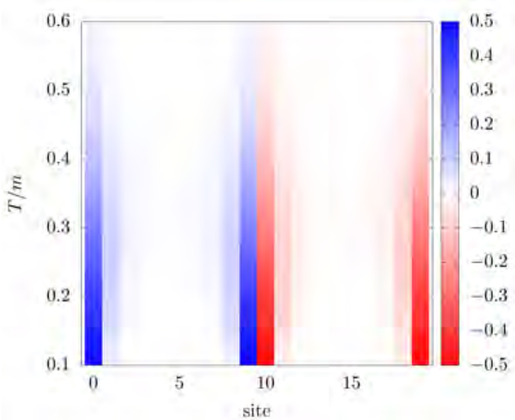




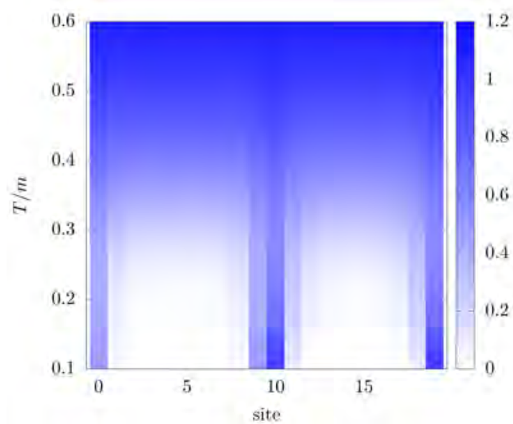
Test in (1+1)d

- **net quark number density**

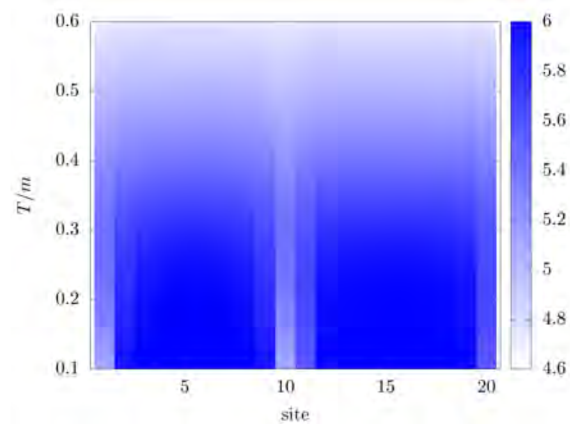
$$L = 20\,,\quad \sigma a/m = 0.3$$



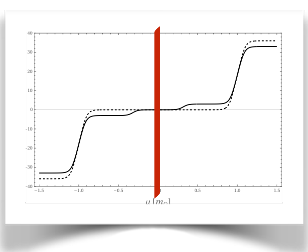
(a) $\mu/m = 0$ (mesonic)

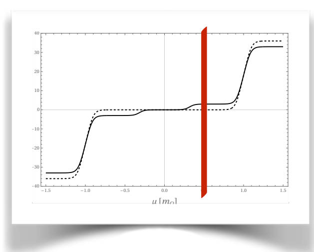


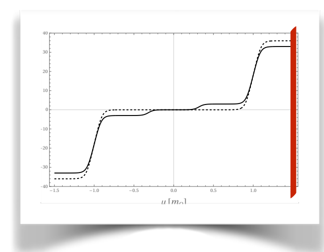
(b) $\mu/m = 0.5$ (baryonic)



(c) $\mu/m = 1.5$ (saturation)







Quark Free Energies

- **define**

$$\Delta F_\infty = \lim_{L_1\to\infty}\left[-\frac{1}{\beta}\ln\left(\frac{Z(q_V =_3 1)}{Z(q_V =_3 0)}\right)\right]$$

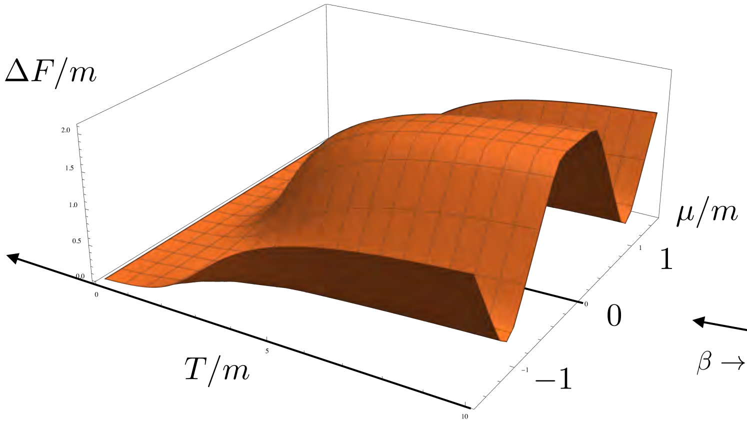
- **still (1+1)-dim**

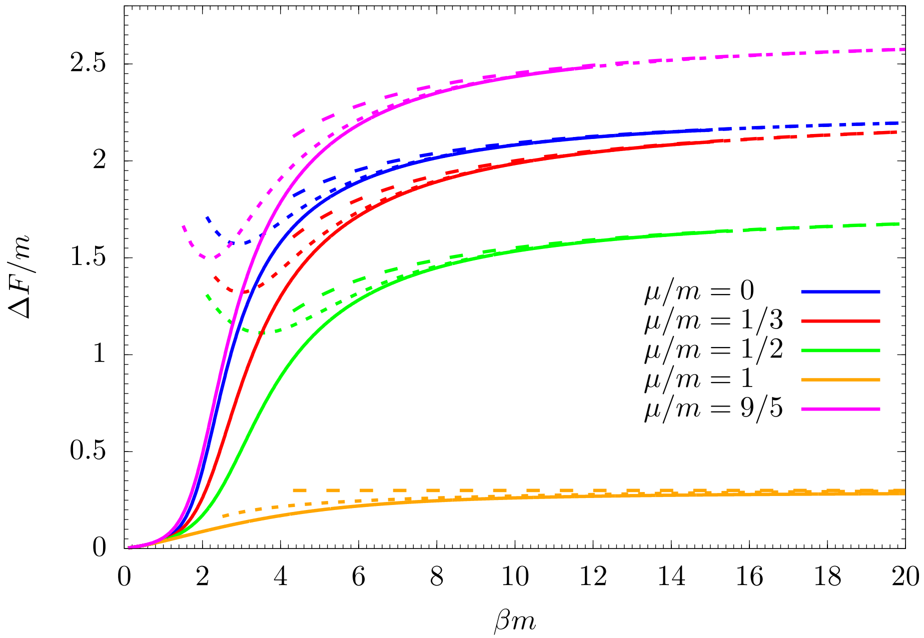
  Potts model representation, flux-tube model parameters ($V = L/2$)

- **independent of L**

  no asymptotic string tension

$L = 64\,,\quad \sigma a/m = 0.3$





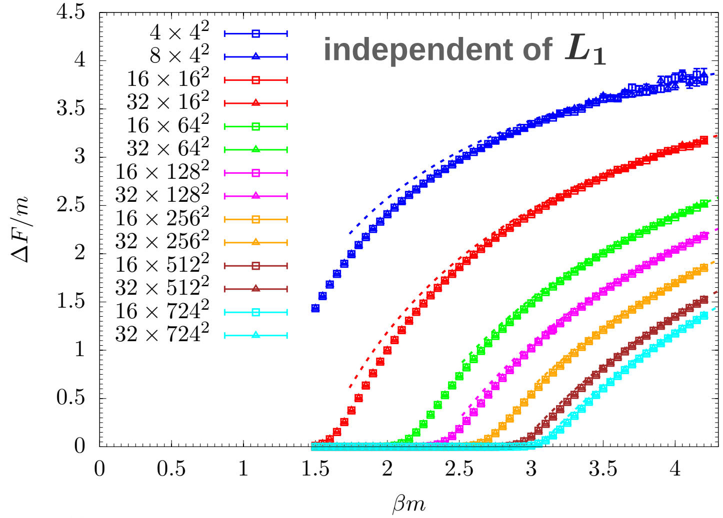
$\beta\to\infty$

$$\Delta_1^{(0)} - \Delta_0^{(0)} = \begin{cases} 2m+\sigma a\,, & \mu\le m/3 \\ 3|m-\mu|+\sigma a\,, & \mu > m/3 \end{cases}$$

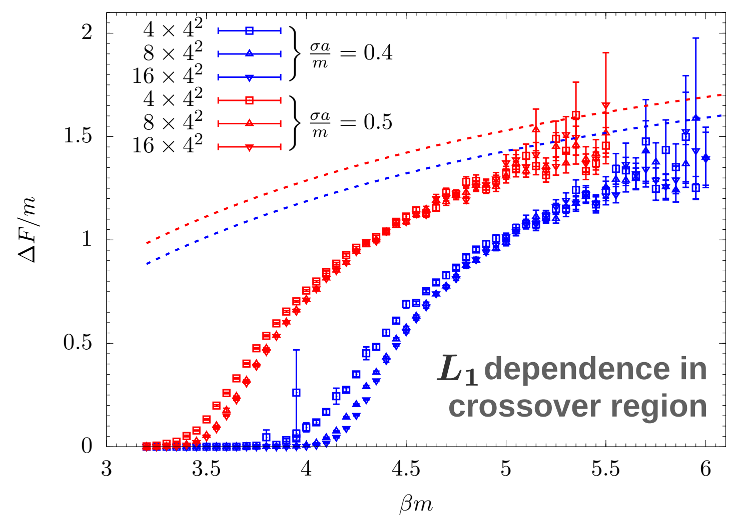
$\beta\to\infty$

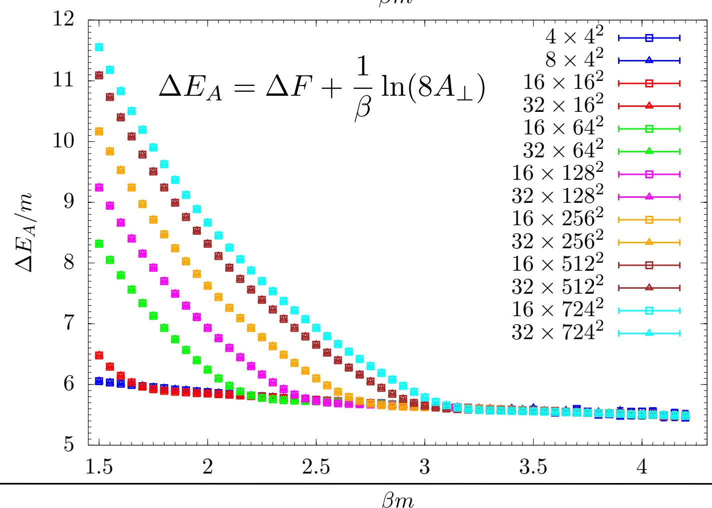
3+1 Dimensions

$\mu = 0\,,\;\; \sigma a/m = 3$



$\sigma a/m = 0.4,\, 0.5$





- **leading entropy contribution**

meson with spin *s* = 0, 1 at either of two interfaces with area $A_\perp = L_2 L_3$

$$\Delta F \simeq \underbrace{(\Delta_1^{(0)} - \Delta_0^{(0)})}_{2m+\sigma a} - \frac{1}{\beta}\ln(8A_\perp)$$

Percolation in Potts Model

- ***q*-state Potts, Boltzmann factor:**

$$\omega(\{s,b\}) = \prod_{\langle i,j\rangle} \left(e^{-K}\delta_{b_{\langle i,j\rangle},0} + (1-e^{-K})\delta_{b_{\langle i,j\rangle},1}\delta_{s_i,s_j}\right) \prod_i e^{h\delta_{s_i,0}}$$

site-bond representation

Edwards & Sokal, PRD 38 (1988) 2009

- **place bond:** $b_{\langle i,j\rangle} \in \{0,1\}$ **with probability** $1-e^{-K}$

  between like nearest-neighbor spins $s_i \in \{0,1,\ldots q-1\}$

- **infinite external field:** $h \to \infty$ $\rightsquigarrow$ **bond percolation**

  with bond probability $p = 1-e^{-K}$, $K = J/T$ **controlled by temperature**

- **vanishing external field:** $h \to 0$,

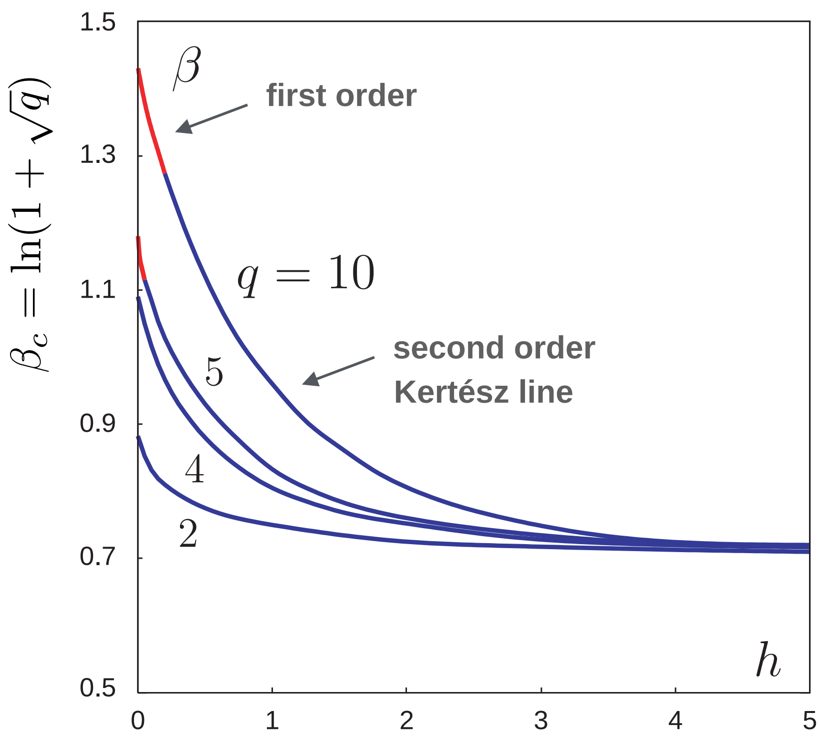
  if $p = p_c$ **at** $T = T_b > T_c$ $\rightsquigarrow$ **bond percolation in ordered phase below** T_c

  lose at Curie temperature T_c

Percolation in Potts Model

- ***q*-state Potts, 2 dimensions:**

Blanchard, Gandolfo, Laanait, Ruiz, Satz, J. Phys. A 41 (2008) 085001



$$\beta_b = 1/T_b = \ln 2$$

$$(p_c = 1/2)$$

Percolation in Flux-Tube Model

- spanning probability:

$$R(T,\mu,L) = \frac{1}{Z_{\text{flux}}} \sum_{\{n,l\}\in\mathcal{R}} \exp\left\{ -\beta\big(H(n,l)-\mu q\big)\right\}$$

set of percolating configs $\mathcal{R}$: contain at least one cluster of bond configurations spanning the entire volume in at least one direction

- simulate with worm algorithm

Prokof'ev & Svistunov, PRL 87 (2001) 160601
Korzec & Vierhaus, 2011, CPC 182 (2011) 1477
Delgado, Evertz, Gattringer, CPC 183 (2012) 1920
Rindlisbacher, Akerlund, de Forcrand, NPB (2016) 542

- measure with fully-dynamic connectivity algorithm

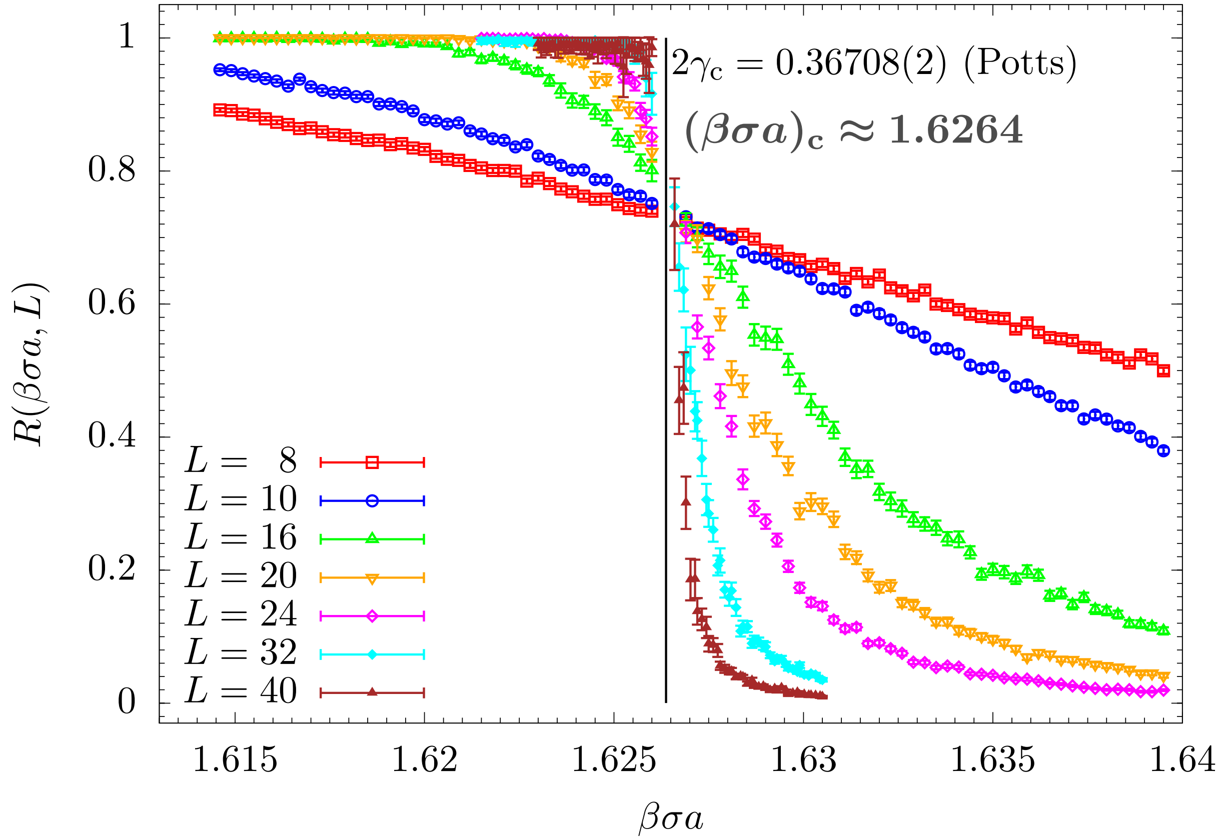
Holm, Lichtenberg, Thorup, J. ACM 48 (2001) 723
Alexandru, Bergner, Schaich, Wenger, PRD 97 (2018) 114503

Percolation in Flux-Tube Model

- **infinitely heavy quarks**

  Z_3-Potts (1st order transition)

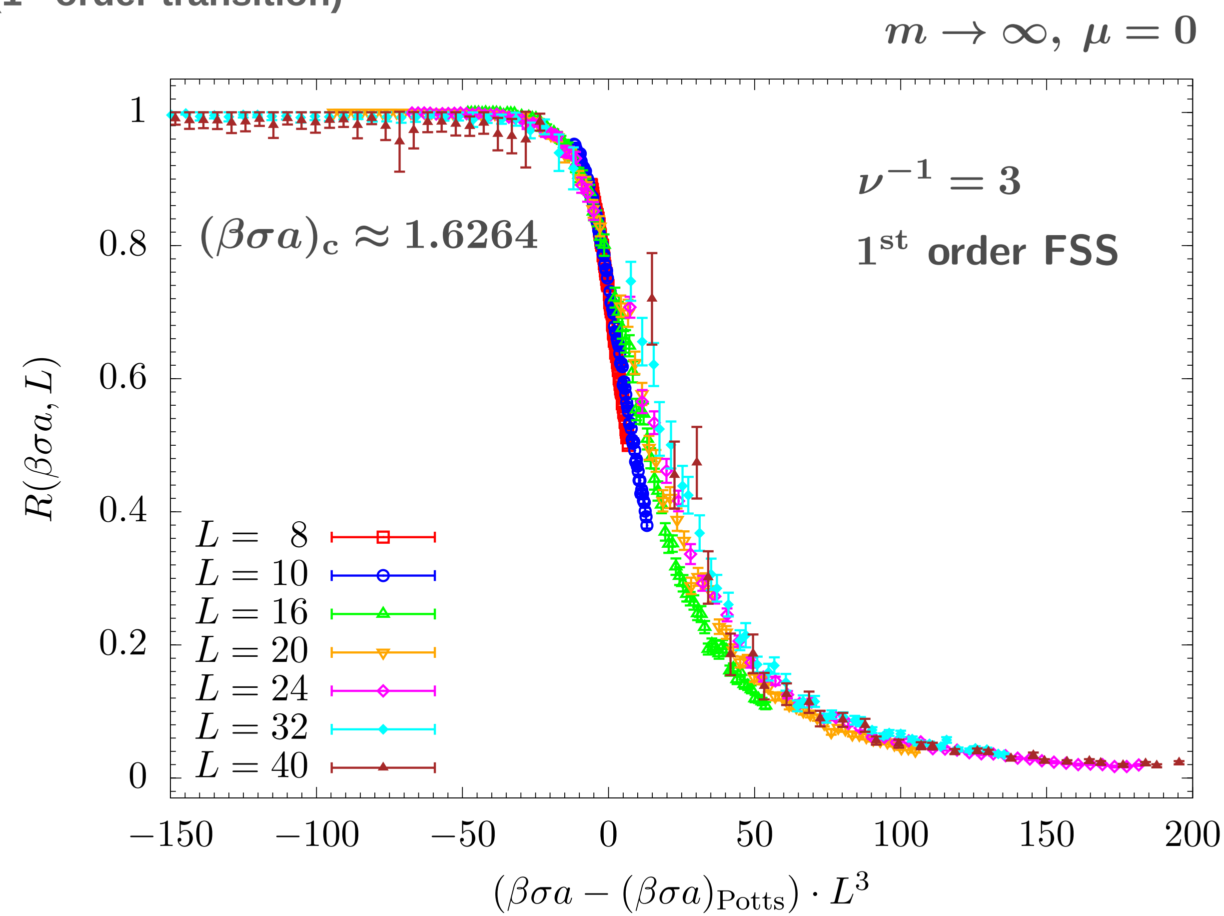
$m \to \infty, \ \mu = 0$



Percolation in Flux-Tube Model

- **infinitely heavy quarks**

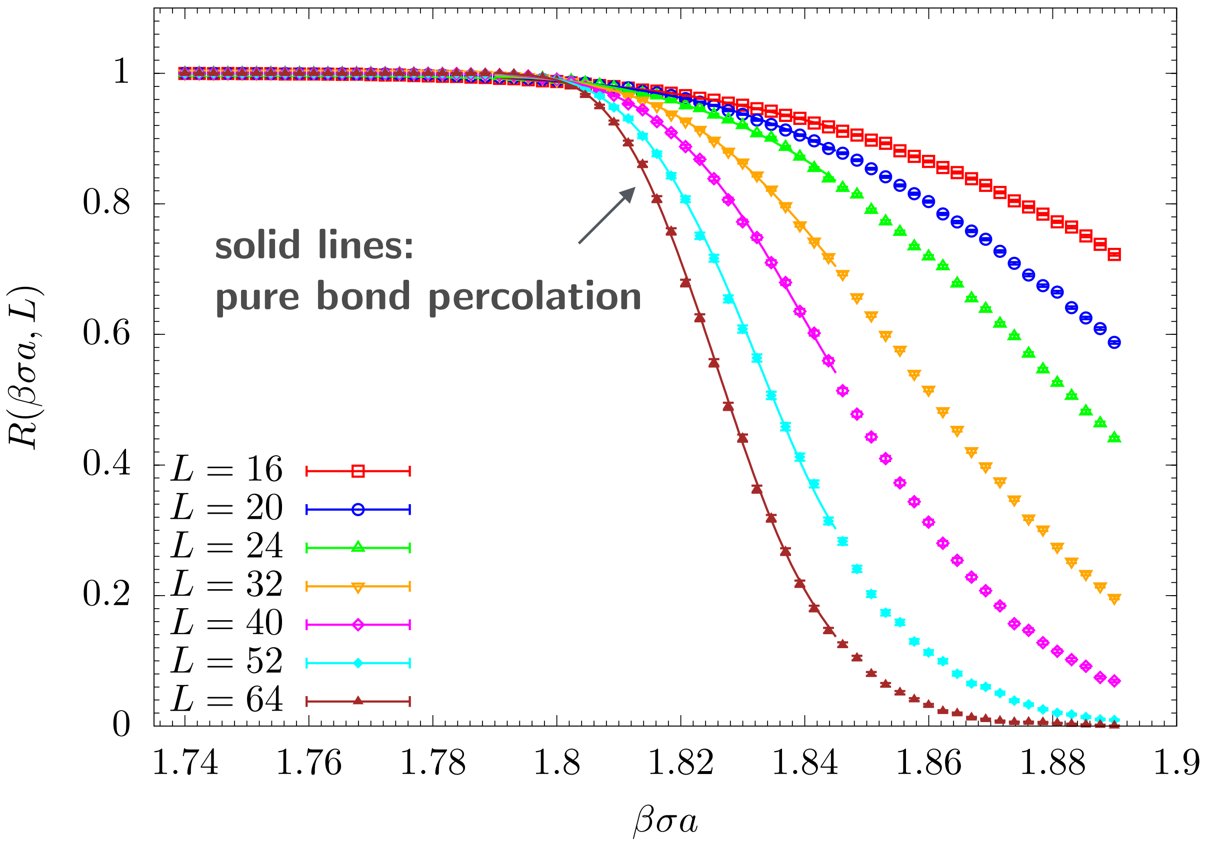
  Z_3-Potts (1st order transition)



- **massless limit**

  bond percolation (2nd order)

$m = 0,\ \mu = 0$

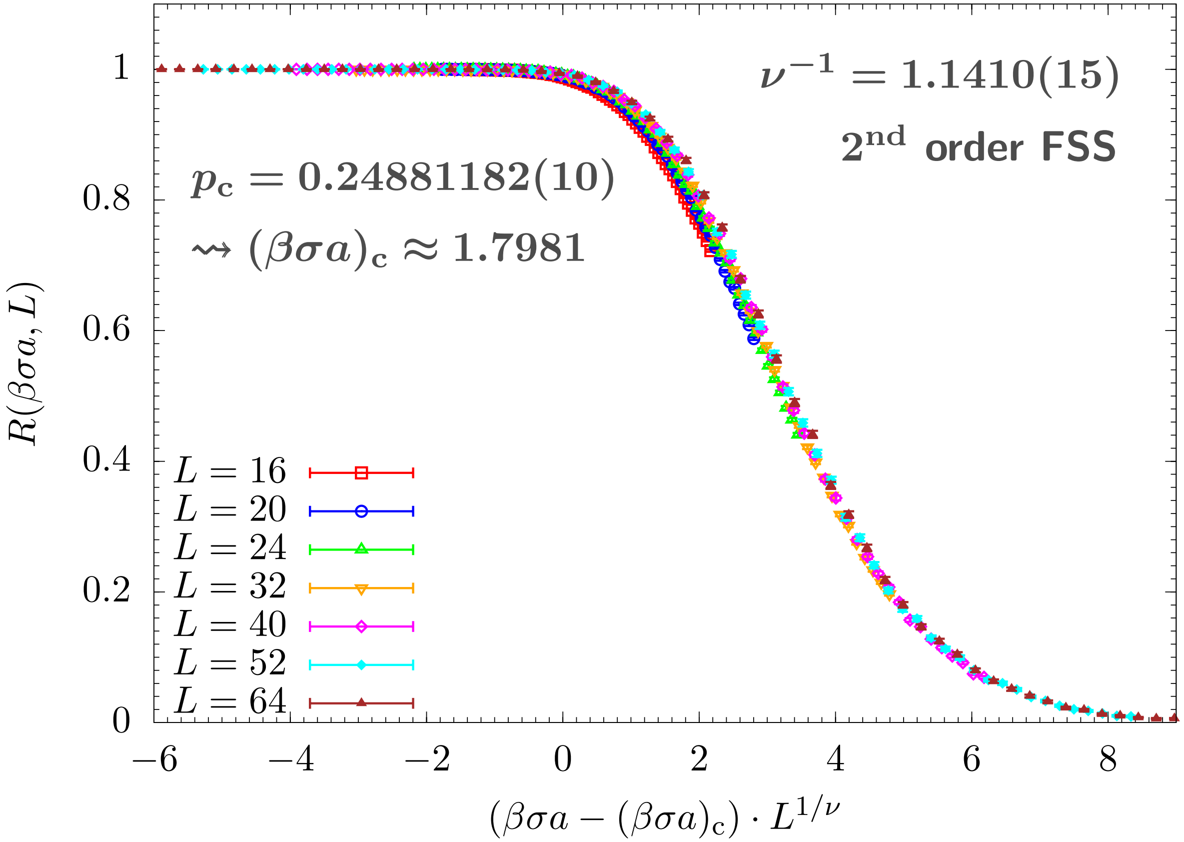


- **massless limit**

  bond percolation (2nd order)

Wang, Zhou, Zhang et al., PRE 87 (2013) 052107

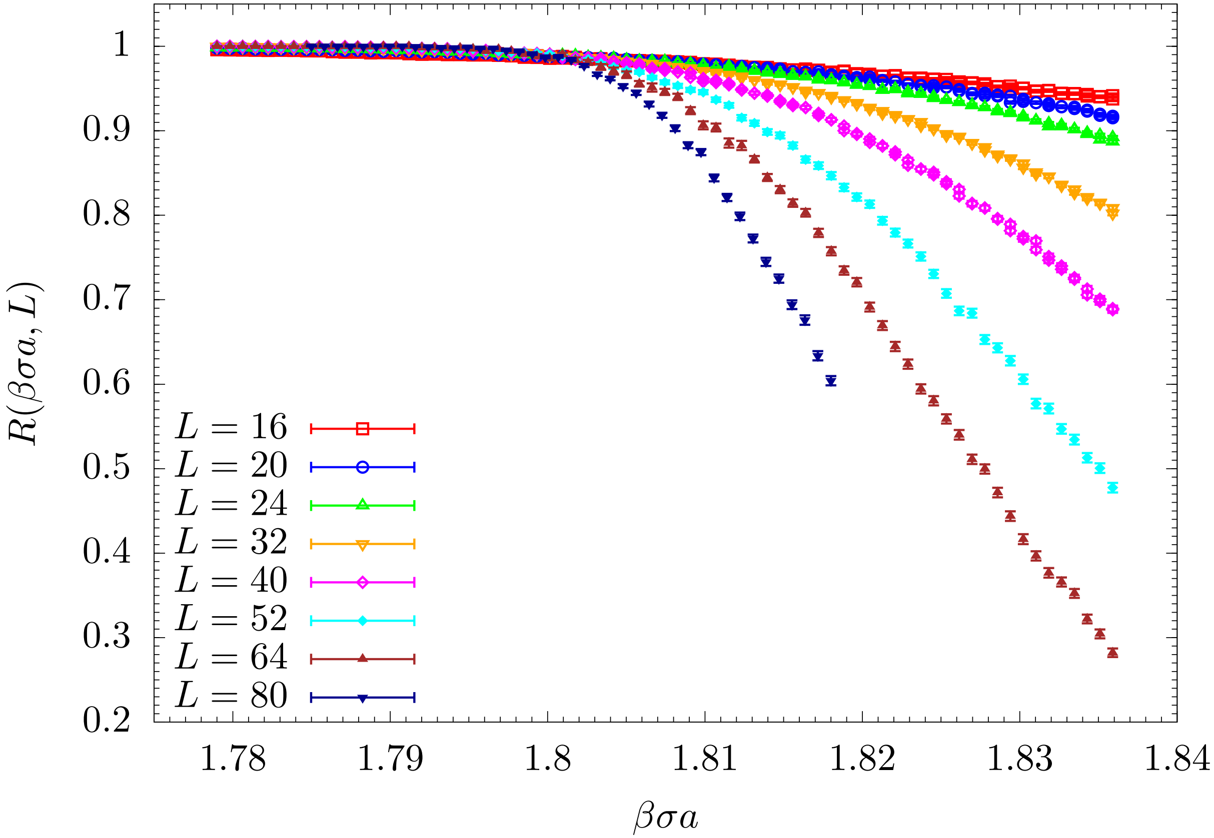
$m = 0, \ \mu = 0$



- **fairly light quarks**

  smooth Z_3-Potts crossover

$m = \sigma a/6$

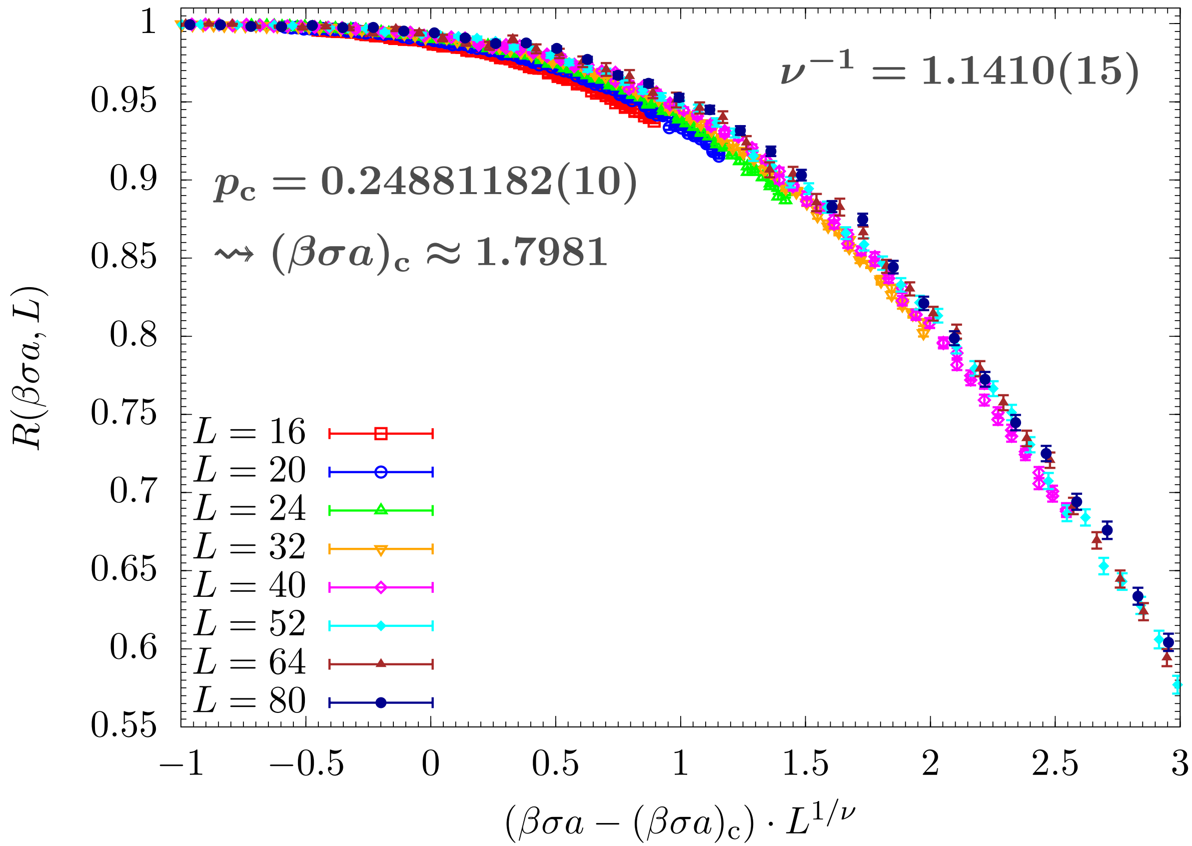


Percolation in Flux-Tube Model

- **fairly light quarks**

  smooth Z_3-Potts crossover

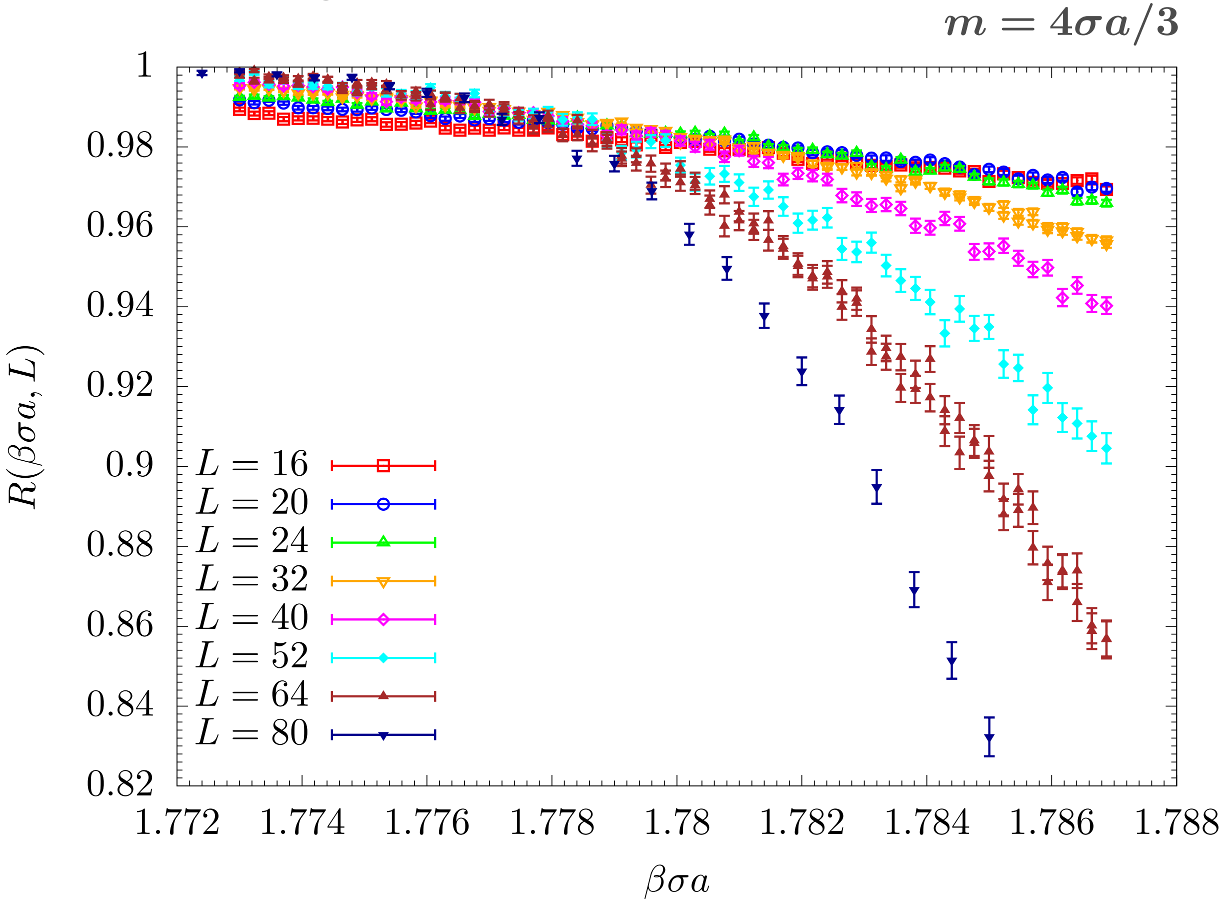
$m = \sigma a/6$



Percolation in Flux-Tube Model

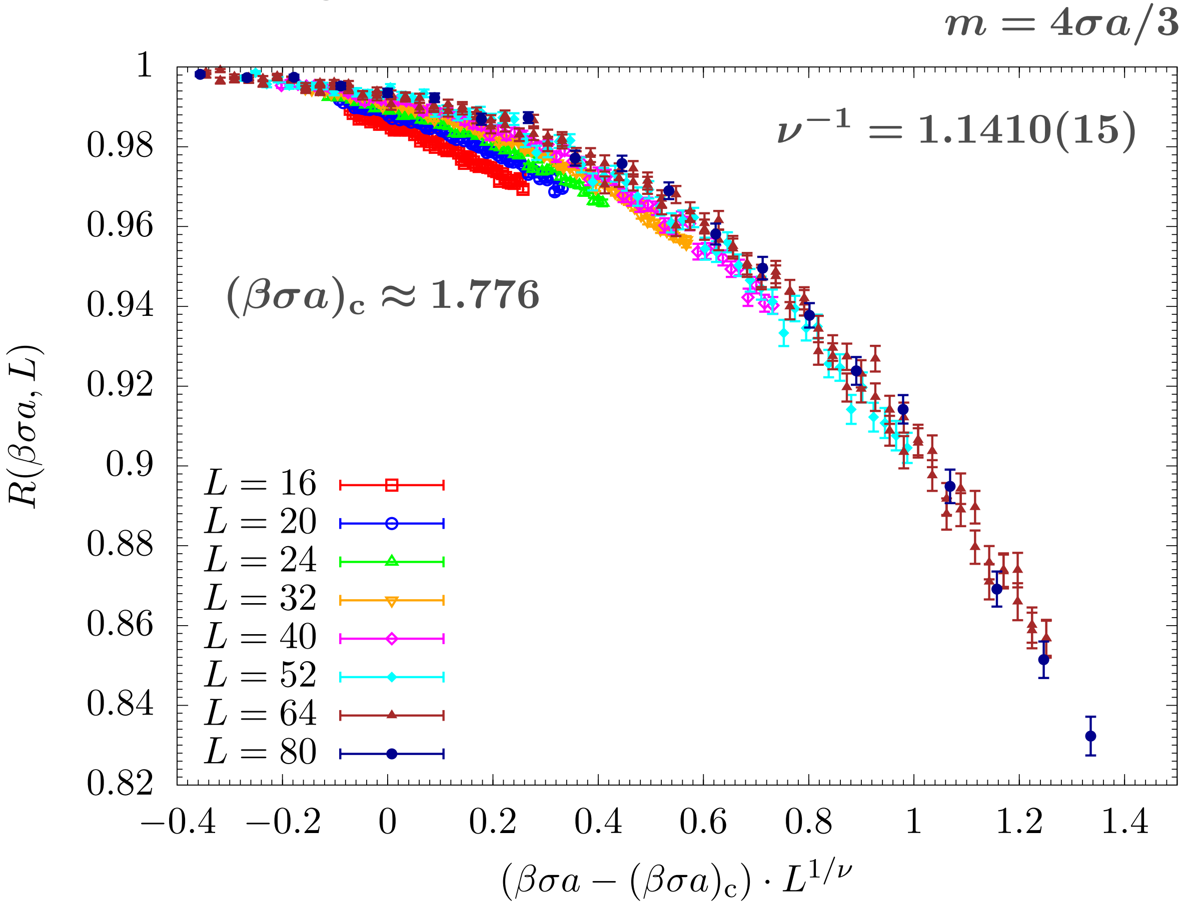
- **medium heavy quarks**

  still in Z_3-Potts crossover region



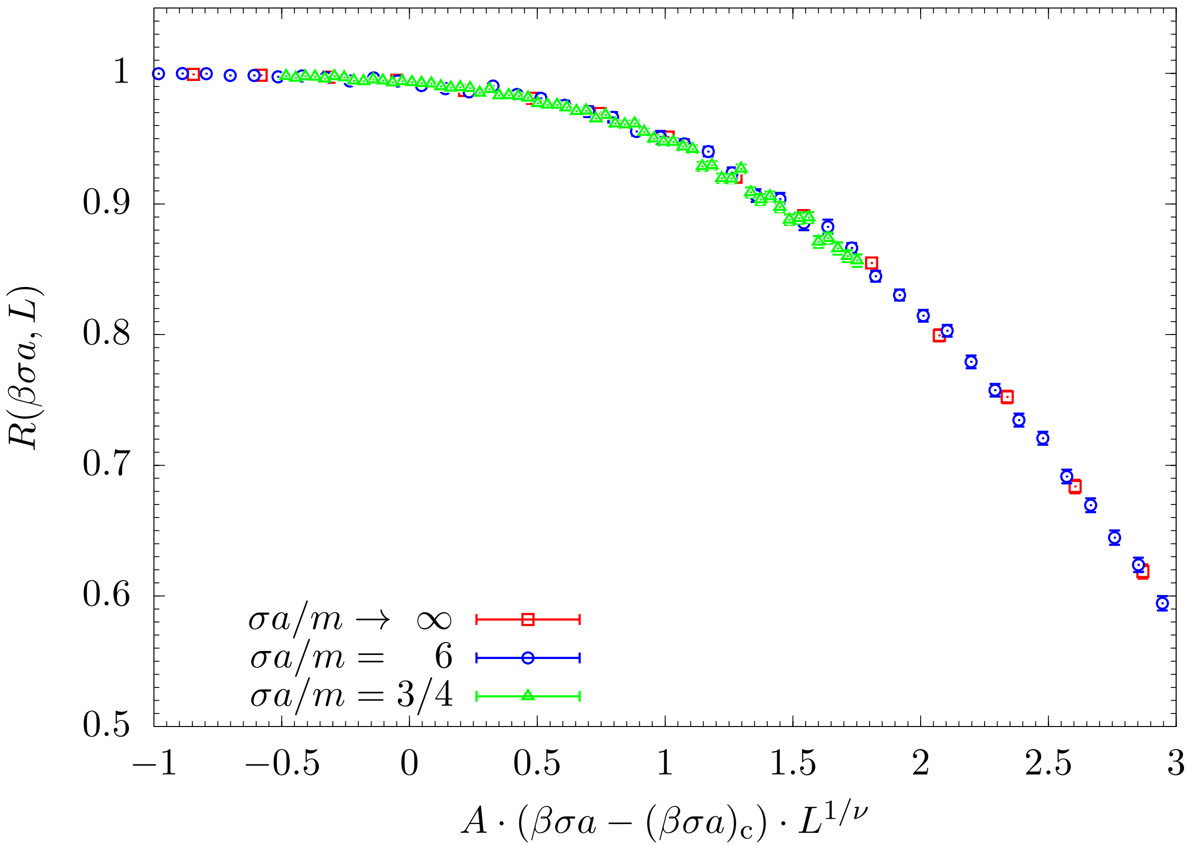
- **medium heavy quarks**

  still in Z_3-Potts crossover region



- **universal scaling function**

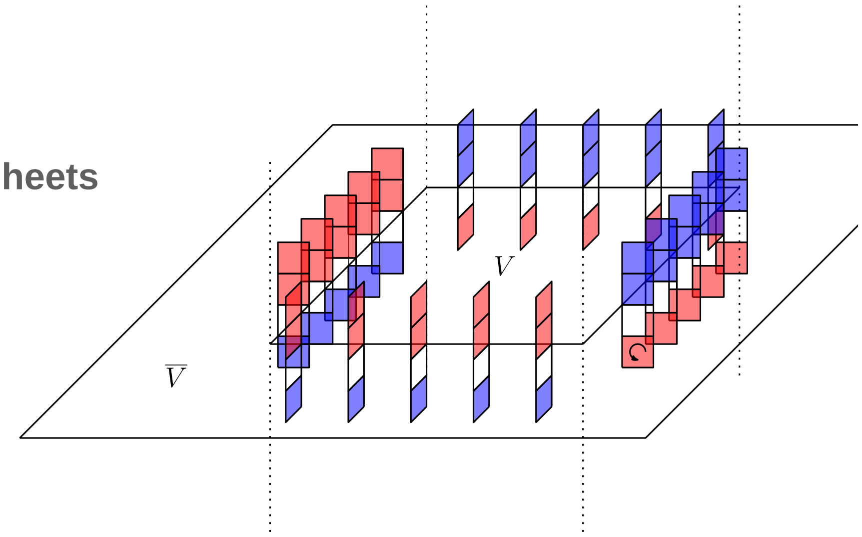
combine $m = \{0,\, \sigma a/6,\, 4\,\sigma a/3\}$



Summary

- **Quarks and triality in a finite volume**

  from FTs over stacks of closed center vortex sheets



- **Proof in two ways:**

  [see Ghanbarpour, LvS, arXiv:2206.11697 [hep-lat]]

  1. **dualization of quark action**

     Gattringer & Marchis, NPB 916 (2017) 627

     Marchis & Gattringer, PRD 97 (2018) 034508

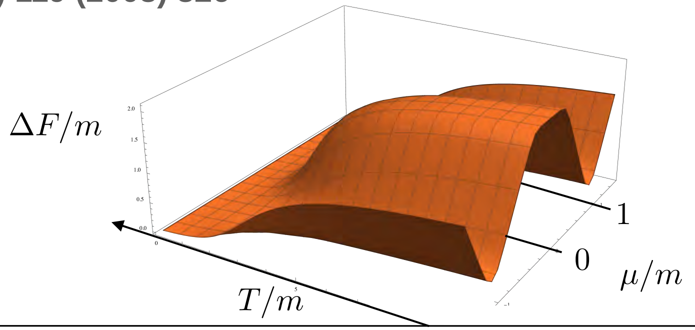
  2. **transfer matrix approach**

     Lüscher, Com. Math. Phys. 54 (1977) 283, Borgs & Seiler, Com. Math. Phys. 91 (1983) 329

     Palumbo, NPB 645 (2002) 309, Mitrjushkin, NPB (PS) 119 (2003) 326

- **Illustration: heavy-dense QCD**

  effective theory dual to flux-tube model



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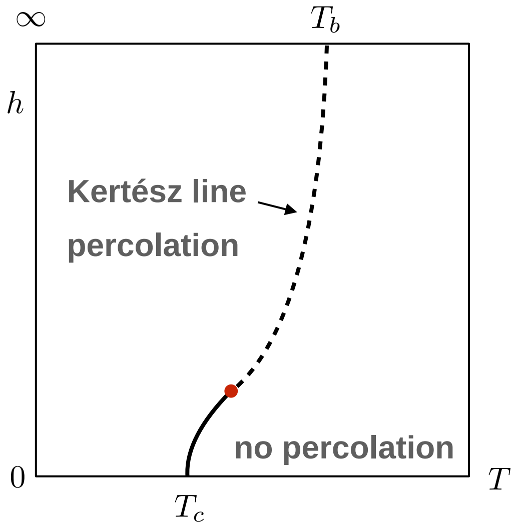
Summary

- **Percolation of electric fluxes in effective theory**

  geometric deconfinement phase transition
  at strong coupling with static fermion determinant



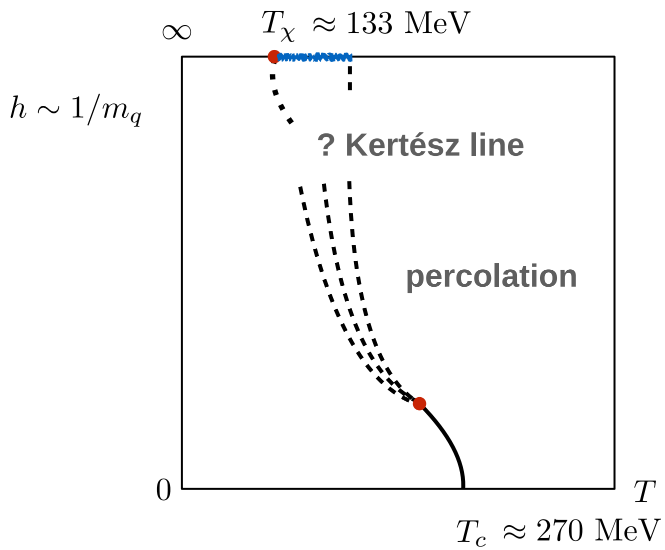
Potts model

Summary & Outlook

- **Percolation of electric fluxes in QCD**

  expect: geometric deconfinement phase transition

  have: gauge invariant definition of fluxes
  and spanning probability



- **Entanglement entropy in QCD**

  so far only in pure gauge theory

  Buividovich, Polikarpov, NPB 802 (2008) 458

Thank you for your attention!