Effective Counting Entropy (μ-Entropy)

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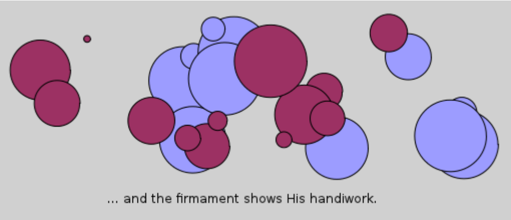

Effective-Number Theory, brief review

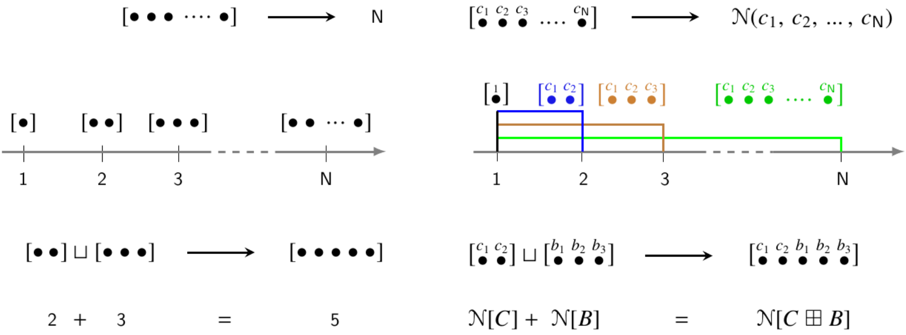

Figure: Comparison of countings, (S), (B), (A)

! Our counting is realized by a **funcion** on vectors, and is a generalization of the "+" binary operation !

Comparison of standard counting (left) and effective counting (right).
(S) - Symmetry - count independent of the order of counting.
(B) - Boundary conditions - if the weights are same, then both countings give the same results.
(A) - Additivity (bottom two lines).

Definitions

Definition. *Set of counting vectors:*

$\mathcal{C} = \cup\, \mathcal{C}_N$, where
$\mathcal{C}_N = \{(c_1, c_2, ..., c_N) : \quad \sum c_i = N, \quad c_i \geq 0\}$

Example.

N ones: $(1, ..., 1) \in \mathcal{C}, \quad (N-1)$ zeros: $(0, ..., 0, N) \in \mathcal{C}$

Effective numbers modeled by functions:

$\mathcal{N} : \mathcal{C} \to \mathbb{R}$

Example.

$\mathcal{N}(1, ..., 1) = N, \quad \mathcal{N}(0, ..., 0, N) = 1$

The bottom Example is actually the boundary conditions (B).

Definition. *Effective numbers.*

$\mathfrak{N}$ is the set of effective number functions $\mathcal{N}$,
where $\mathcal{N} : \mathcal{C} \to \mathbb{R}$ have the following properties.
For all M, N, for all $1 \leq i, j \leq N$, $i \neq j$,
for all $C = (c_1, ..., c_N) \in \mathcal{C}_N$, and for all $B \in \mathcal{C}_M$

(S) *Symmetry:* $\mathcal{N}(..., c_i, ..., c_j, ...) = \mathcal{N}(..., c_j, ..., c_i, ...)$

(B2) *Boundary values:* $\mathcal{N}(0, ..., 0, N) = 1$, *in* $\mathcal{C}_N$

(A) *Additivity:* $\mathcal{N}[C \boxplus B] = \mathcal{N}[C] + \mathcal{N}[B]$

(C) *Continuity of* $\mathcal{N}$ *restricted to* $\mathcal{C}_N$ *with topology from* $\mathbb{R}^N$

(M^-) *Monotonicity:* $0 < \varepsilon \leq \min\{c_i, N - c_j\}, \;\; c_i \leq c_j \;\Rightarrow$

$$\mathcal{N}(..., c_i - \varepsilon, ..., c_j + \varepsilon, ...) \leq \mathcal{N}(..., c_i, ..., c_j, ...)$$

These properties are independent.
The notation is consistent with the paper [3].

(M-) - the left side of the inequality is illustrated on the next page.

Illustration of Monotonicity (M-)

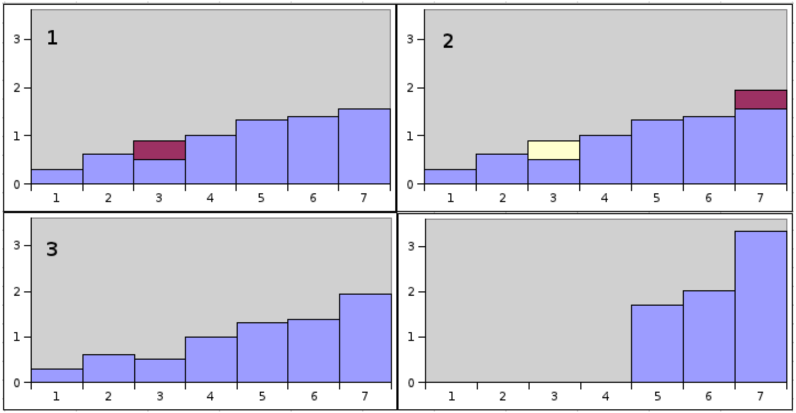

Figure: Cumulation, (M), (C) (graphed c_i vs. i)

Cumulation and Continuity - these properties do not have a standard analog.
(M) - Monotonicity - the state with more cumulated weights has lower count (compare top left vs. top right).
(C) - Continuity - the count depends continuously on the weight change.
The count for the cumulated state at bottom right is 3.

Effective Counting Theory - Results

Theorem 1. *Separability.*

$$\mathcal{N}[(c_1, c_2, \ldots, c_N)] = \sum \mathfrak{n}(c_i) \quad \text{for some } \mathfrak{n} : [0, \infty) \to \mathbb{R}$$

The function $\mathfrak{n}(x)$ is called generating function for $\mathcal{N}[C]$.

Theorem 2. *Unique continuous.*

(**a**) $\forall t \quad \exists$ unique continuous $\mathfrak{n}(x)$ with $\mathfrak{n}(0) = t$

(**b**) All continuous $\mathfrak{n}(x)$ are concave.

Theorem 3. *Unique bounded.*

(**a**) $\exists$ unique bounded $\mathfrak{n}(x)$

(**b**) This bounded $\mathfrak{n}(x)$ is continuous.

Theorem 4. *Minimum exists.*

$$\exists \mathcal{N}_\star \quad \forall \mathcal{N} \in \mathfrak{N} \quad \forall C \in \mathcal{C} \quad \mathcal{N}_\star[C] \leq \mathcal{N}[C]$$

Applications

$$\text{weights} \quad c_i \longrightarrow p_i = \frac{c_i}{N} \quad \text{probabilities}$$

$$\mathcal{N}(\ldots, c_i, \ldots) \longrightarrow \mathcal{N}(\ldots, p_i, \ldots)$$

In the past *(ad hoc) Bell and Dean [1]*

[Q] *"How many atoms do vibrations effectively spread over?"*

Participation number: $\frac{1}{\mathcal{N}_p[C]} = \frac{1}{N^2} \sum c_i^2$

Pros

(S), (B), (C), (M^-)

Cons

(A) is not satisfied

Moreover, it is also not multiplicative and so it doesn't scale well.

Quantum Mechanics

- effective count of quantum states, [3]:

  [Q] *"How many basis states* $|\, i \,\rangle$ *is the system described by* $|\, \psi \,\rangle$ *effectively in?"*, see [3]

  [A] *If* $P = (p_1, p_2, \ldots, p_N)$, $p_i = |\langle\, i \,|\, \psi \,\rangle|^2$, *is the probability vector assigned to state* $|\, \psi \,\rangle$ *and basis* $\{\, |\, i \,\rangle \}$ *by quantum mechanics, then the system described by* $|\, \psi \,\rangle$ *is effectively in at least* $\mathcal{N}_\star[C]$ *states from* $\{\, |\, i \,\rangle \}$, *where* $C = NP = (c_1, c_2, \ldots, c_N)$ *and*
  $\mathcal{N}_\star[C] = \sum \mathfrak{n}_\star(c_i)$, $\mathfrak{n}_\star(c) = \min\{c, 1\}$.
- new measure of uncertainty, [5]
- new measure of entanglement, [5]
- quantum computing - decoherence

Statistical Physics *(entropy, [here])*

Fractals *(dimension, multidimensionality, [6])*

Transport phenomena *(Anderson localization, [7])*

Biological Sciences *(diversity - counting species)*

References

for Effective Number Theory:

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[2] I. Horváth and R. Mendris,
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[3] I. Horváth and R. Mendris,
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[4] A.W. Marshall, I. Olkin, B.C. Arnold,
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[5] I. Horváth, *"The Measure Aspect of Quantum Uncertainty, of Entanglement, and the Associated Entropies"*,
Quantum Rep. 3(3) (2021), 534-548, https://arxiv.org/abs/1809.07249

[6] Ivan Horváth, Peter Markoš and Robert Mendris
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[7] Ivan Horváth and Peter Markoš
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Question

Since this was very brief,

are there any questions before we continue?

Entropy

Boltzmann entropy has been used with a great success but there are situations where it doesn't work.

Here are some examples in high energy physics.

• In high energy collisions of an electron with a positron, annihilation occurs and, immediately after, typically two or three hadronic jets are produced. The probability distribution of their transverse momenta is non-Boltzmannian. This phenomenon has defied theoreticians since several decades, particularly since Hagedorn [5][6].

• The distribution of energies E of cosmic rays arriving on Earth has been measured for decades. This distribution is very far from exponential [5].

• Solar neutrino problem can be caused in part by the Boltzmann statistics used in Solar Standard Model (SSM). There is no good reason why it should be applicable there [5][7].

• The anomalous diffusion of a charm quark in a quark-gluon plasma has been analyzed by Walton and Rafelski [5][8] through both nonextensive statistical mechanical arguments and quantum chromodynamics. The results coincide for Tsallis entropy S_q with q $= 1.114$.

Question

What is Entropy?

Similarity of Entropy and Effective numbers:

(S) *Symmetry:* $\mathcal{N}(..., c_i, ..., c_j, ...) = \mathcal{N}(..., c_j, ..., c_i, ...)$

(B) *Boundary values:* $\mathcal{N}(0, ..., 0, N) = 1$, *in* $\mathcal{C}_N$

?? **(A)** Additivity: $\mathcal{N}[C \boxplus B] = \mathcal{N}[C] + \mathcal{N}[B]$??

(C) *Continuity of* $\mathcal{N}$ *restricted to* $\mathcal{C}_N$ *with topology from* $\mathbb{R}^N$

(M) *Monotonicity:* $0 < \varepsilon \leq \min\{c_i, N - c_j\}, \;\; c_i \leq c_j \;\Rightarrow$
$\mathcal{N}(..., c_i - \varepsilon, ..., c_j + \varepsilon, ...) \leq \mathcal{N}(..., c_i, ..., c_j, ...)$

Boltzmann Entropy

$S(P) = -\sum_i p_i \ln(p_i)$, where $P = (p_1, p_2, ..., p_N)$

Its additivity differs from **(A)**.

If p_i-s are constant $S = k \ln(W)$.
Need to transfer from $-\sum_i p_i \ln(p_i)$ to $-\int_0^L p \ln(p) dx$.

Note: Additivity vs. Extensivity.

Maximum Entropy Principle

Equilibrium states are those with maximal achievable entropy.

In standard statistical mechanics:

Step 1. *Let $S(P)$ is the Boltzmann-Gibbs Entropy.*

Step 2. *Find P_0 that maximizes $S(P)$ and identifies the equilibrum state.*

Step 3. *Use $S(P_0)$ to find other thermodynamic parameters, e.g. free energy F, internal energy U, specific heat C, ...*

We will concentrate on Step 2.

Boltzmann Entropy Equilibria - simple case

Typically we have a system with constraints.
We start with the necessary constraint:

Maximize $S(P) = -\int_0^L P \ln(P) dx$

Constraint $w(P) = \int_0^L P dx = 1$

No physical constraint yet.

$P(x) = ?$

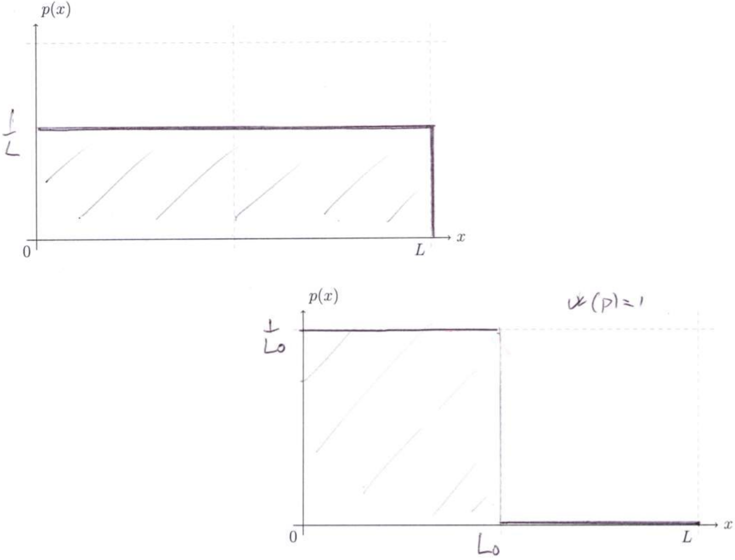


Figure: 101, 104

Note, that there cannot be any δ-functions in $P(x)$ since $-\int_0^L \delta(x-x_0)\ln(\delta(x-x_0))dx = -\infty$
On graphs, we denote the part of $P(x)$ without δ-functions as $p(x)$ and δ-functions are marked separately.

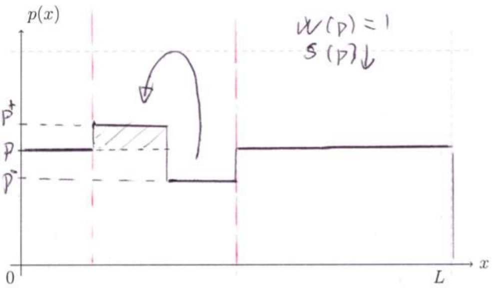


Figure: 103

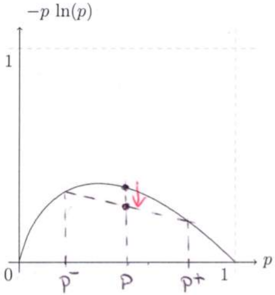


Figure: 105

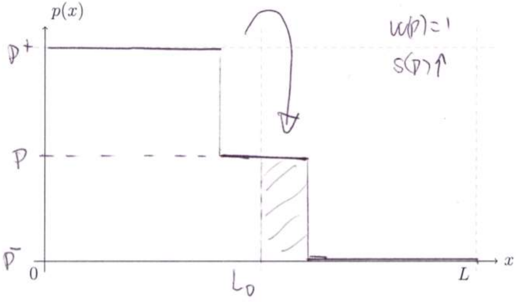


Figure: 106

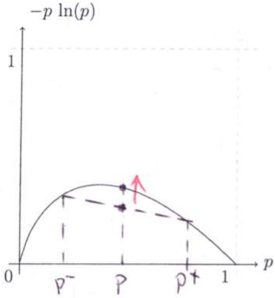


Figure: 127

We arrived to the standard result - the uniform distribution.

Boltzmann Entropy Equilibra - simple case - Result

Maximize $S(P) = -\int_0^L P \ln(P) dx$

Constraint $w(P) = \int_0^L P dx = 1$

No physical constraint yet.

The solution is uniform on all the available space, $P(x) = \frac{1}{L}$.

Question

What is the solution for Effective Counting Entropy?

Effective Counting Entropy

Effective number of states is $\mathcal{N}_*[P] = \sum_i \min\{p_i, 1\}$,
where $P = (p_1, ..., p_N)$ *in the discrete case, [4]*
and the Effective Volume is $\mathcal{V}_*[P] = \int_\Omega \min\{V\, P(\mathbf{x}), 1\} d\mathbf{x}$,
where $P = P(\mathbf{x})$ *in the continuous case, [3][10].*

Since we can count states with different probabilities, we define the Entropy directly as follows:

Definition. *Effective Counting Entropy:*

$$\mathcal{S}_*[P] = \ln(\mathcal{N}_*[P])$$

$$\mathcal{S}_*[P] = \ln(\mathcal{V}_*[P])$$

Theorem. *Super-additivity over product of independent sets of states* ($p_{AB,i,j} = p_{A,i}\, p_{B,j}$): $\mathcal{S}_*[A \times B] \geq \mathcal{S}_*[A] + \mathcal{S}_*[B]$

Effective Counting Entropy Equilibra - simple case

Maximize $\mathcal{V}[P] = \int_0^L \min\{L\, P(x), 1\} dx$

Constraint $w(P) = \int_0^L P(x) dx = 1$

No physical constraint yet.

$P(x) = ?$

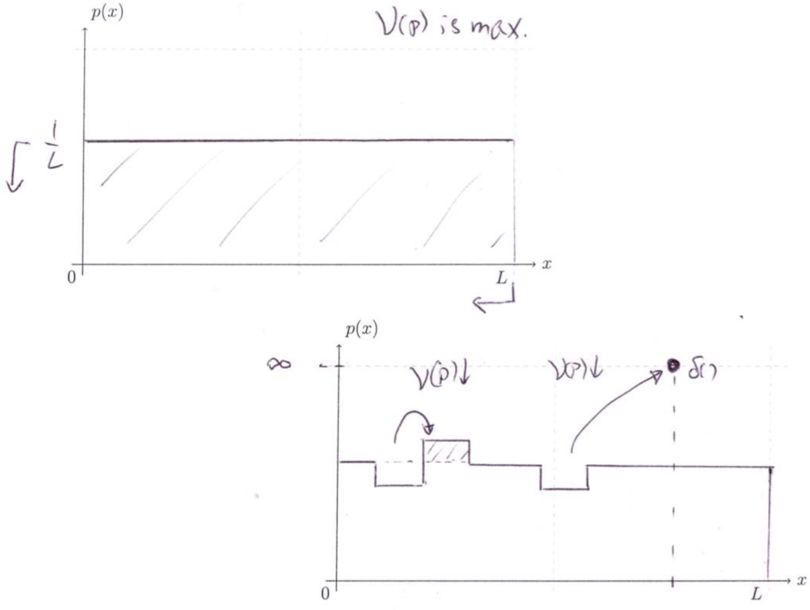


Figure: 114, 115

Effective Counting Entropy - simple case - Result

We obtained the same result as in the case of Boltzmann Entropy - the uniform distribution:

Maximize $S(P) = -\int_0^L P \ln(P) dx$

Constraint $w(P) = \int_0^L P dx = 1$

No physical constraint yet.

The solution is uniform on all the available space, $P(x) = \frac{1}{L}$.

Boltzmann Entropy Equilibra - generic case

Maximize $S[P] = -\int_0^L P(x)\ln(P(x))dx$

Constraint $w(P) = \int_0^L P(x)dx = 1$

Physical Constraint $y(P) = \int_0^L x\,P(x)dx = y_0$

$P(x) = ?$

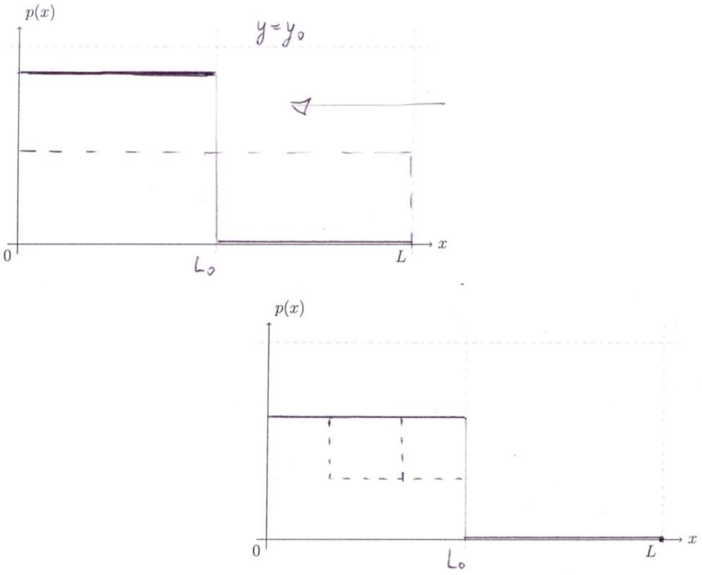


Figure: 107, 108

Note, that there cannot be any delta functions in $P(x)$ since $-\int_0^L \delta(x - x_0) \ln(\delta(x - x_0)) dx = -\infty$.

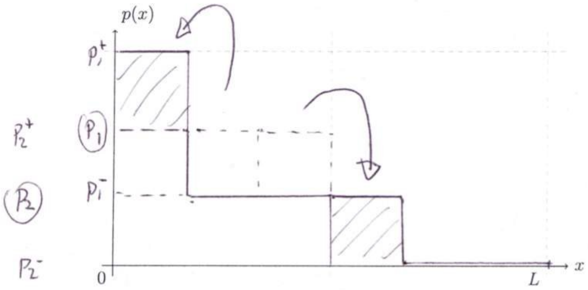
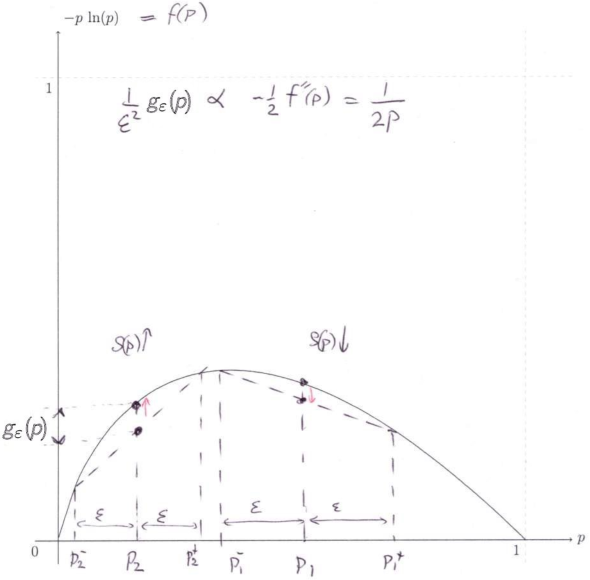


Figure: 109



$-p\ln(p) = f(p)$
1
$\frac{1}{\varepsilon^2} g_\varepsilon(p) \propto -\frac{1}{2} f''(p) = \frac{1}{2p}$
$S(p)\uparrow$
$S(p)\downarrow$
$g_\varepsilon(p)$
ε
ε
ε
ε
0
p_2^-
p_2
p_2^+
p_1^-
p_1
p_1^+
1
p

Figure: 110

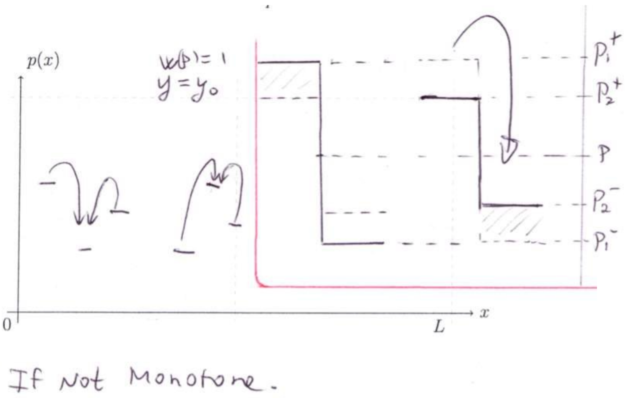


Figure: 111

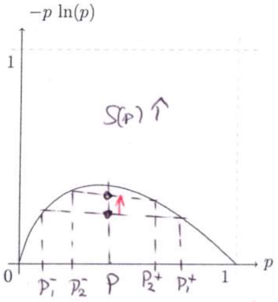
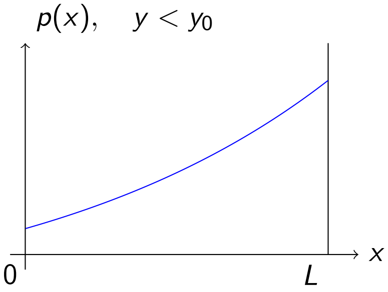


Figure: 112

Discontinuities are similar.
Consequently if $S[P]$ is maximized,
then $P(x)$ is monotone and continuous.

Boltzmann Entropy Equilibria - generic - Result

To finish the solution faster we can use variations and find the standard exponential results:



Truncated exponentials.

Question

What is the solution for Effective Counting Entropy?

Effective Counting Entropy Equilibria - generic

Maximize $\mathcal{V}[P] = \int_0^L \min\{L\, P(x), 1\} dx$

Constraint $w(P) = \int_0^L P(x) dx = 1$

Physical Constraint $y(P) = \int_0^L x\, P(x) dx = y_0$

$P(x) = ?$

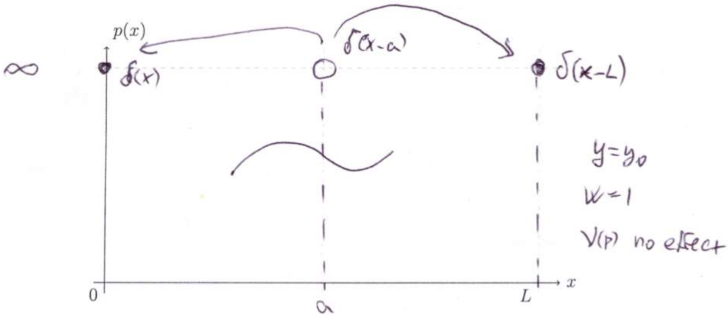


Figure: 116

If $y(P) = \int_0^L x\,P(x)dx = y > y_0$, *then we need to change* $P(x)$ *so that* y *is lowered down to* y_0.

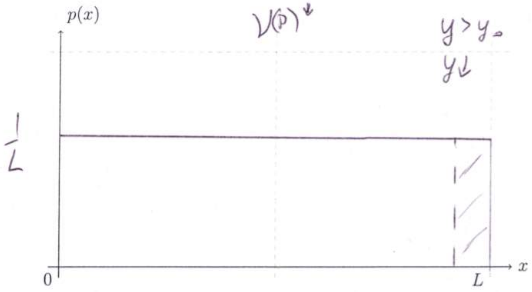


Figure: 117

To better see what is happening here, let's use an analogy.

Suppose $\mathbf{p}(\mathbf{x})$ = #of the items with a price equal to $\mathbf{x}$.
Then $\mathbf{x}\,\mathbf{p}(\mathbf{x})$ = the cost of the items with the price equal to $\mathbf{x}$.
And $\int_{\mathbf{0}}^{\mathbf{L}} \mathbf{x}\,\mathbf{p}(\mathbf{x})\mathbf{dx}$ = the total cost of all items (max. price is $\mathbf{L}$).

Then to lower the cost we need to exchange the most expensive items for free ones.

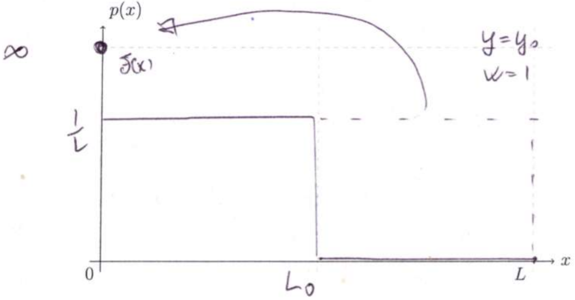


Figure: 118

Comparison of generic Equilibria for Effective Counting and Boltzmann Entropies:

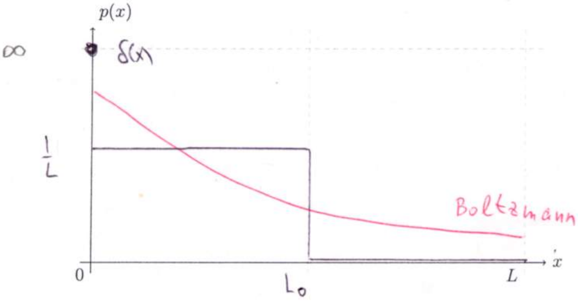


Figure: 119

Case $y_0 < \frac{L}{2}$.

Comparison of generic Equilibria for Effective Counting and Boltzmann Entropies:

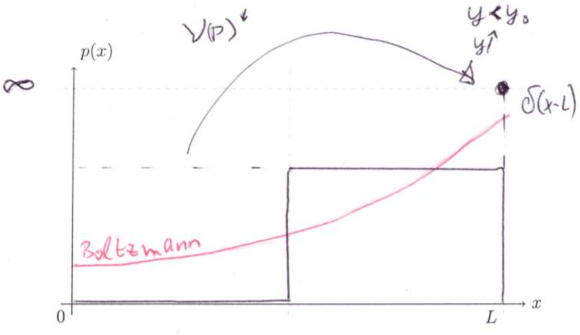


Figure: 125

Case $y_0 > \frac{L}{2}$, and for $y_0 = \frac{L}{2}$ it is the uniform solution, which is same for both.

On the next slide, we give the solution algebraically.

Effective Counting Entropy - generic - Result

Maximize $\mathcal{V}[P] = \int_0^L \min\{L\,P(x), 1\} dx$

Constraint $w(P) = \int_0^L P(x) dx = 1$

Physical Constraint $y(P) = \int_0^L x\,P(x) dx = y_0$

For $y_0 \leq \frac{L}{2}$ the solution is $P(x) = p(x) + b_0 \delta(x)$, see slide 34, where

$$p(x) = \begin{cases} \frac{1}{L}, & x \in [0, L_0] \\ 0, & x \in (L_0, L] \end{cases}, \; L_0 = \sqrt{2Ly_0}, \text{ and } b_0 = 1 - \sqrt{\frac{2y_0}{L}}.$$

For $y_0 > \frac{L}{2}$ the solution is flipped around $x = \frac{L}{2}$, see slide 35,

Effective Counting Entropy in dimension d

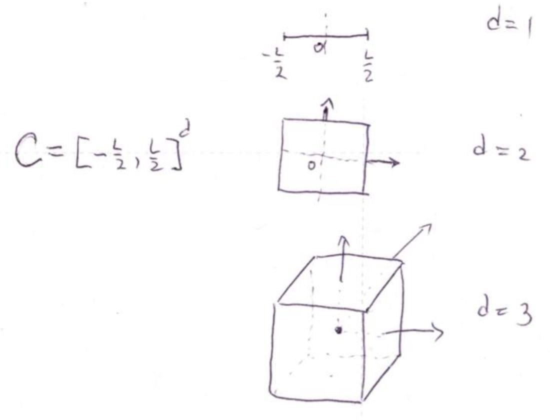


Figure: 120

d-dimensional cubes.

Effective Counting Entropy in dimension d

$C = [-\frac{L}{2}, \frac{L}{2}]^d$

We can glue the $(d-1)$-dimensional sides of this cube to get flat d-torus or flat d-sphere and the calculations will be same.

$\Delta(\mathbf{x}) =$ countable sum of $\delta-$functions

Consider probability distribution $P(\mathbf{x}) = p(\mathbf{x}) + \Delta(\mathbf{x})$ on C

Maximize $\mathcal{V}[P] = \int_C \min\{L^d\, P(\mathbf{x}), 1\}\, d\mathbf{x} \leq L^d$

Constraint $w(P) = \int_C P(\mathbf{x}) d\mathbf{x} = 1$

Physical Constraint $y(P) = \int_C |\mathbf{x}|\, P(\mathbf{x})\, d\mathbf{x} = y_0$

$P(\mathbf{x}) = ?$

Figure A. (2D flat torus)
Any δ-function can be 'moved' to A and B as it was done before, see Fig. 116. on slide 31 (identification of edges and vertices as depicted).

$$R = \frac{\sqrt{d}L}{2}$$

$$r = \frac{L}{2}$$

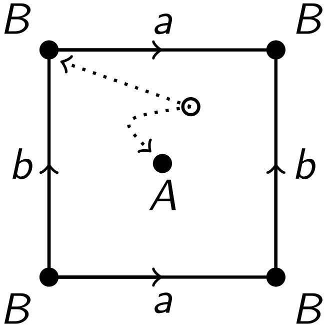


Figure: A

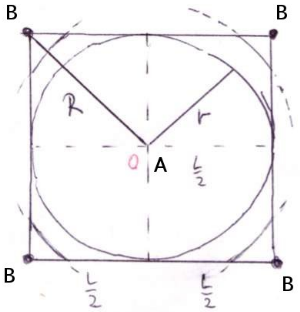


Figure: 121

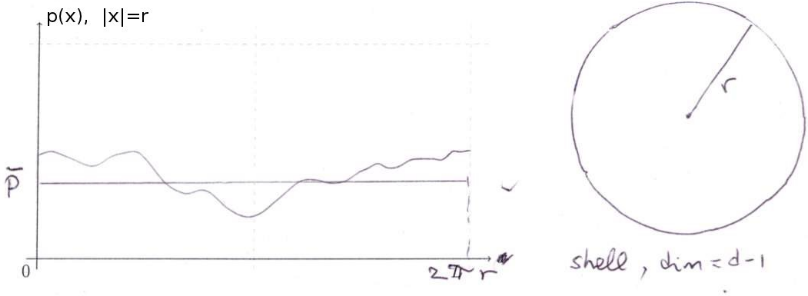


Figure: 126

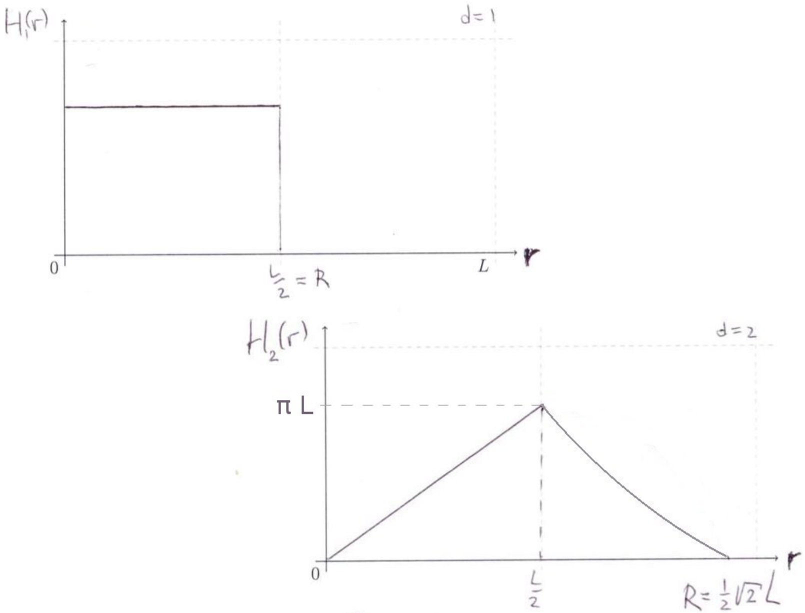


Figure: 122, 123

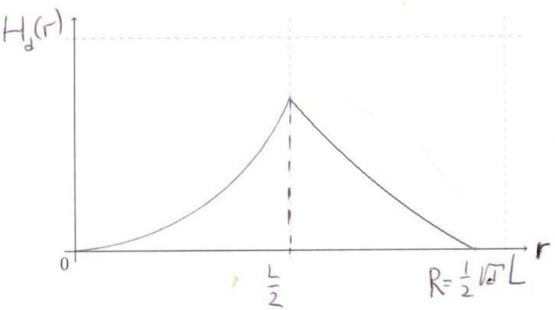


Figure: 124

Rotationally Symmetric Solution

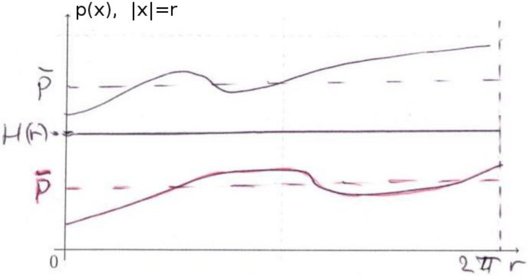
$$Q(r) = q(r) + a_0\delta(r) + b_0\delta(r-R), \quad R = \frac{\sqrt{d}L}{2}$$

$$\textbf{Maximize} \quad \mathcal{V}_d(Q) := \int_0^R \min\{Q(r), H_d(r)\}\, dr \le L^d$$

$$\text{Constraint} \quad w_d(Q) := \int_0^R Q(r)dr = L^d$$

$$\textbf{Physical Constraint} \quad y_d(Q) := \frac{1}{L^d}\int_0^R r\, Q(r)\, dr$$

$Q(r) = ?$



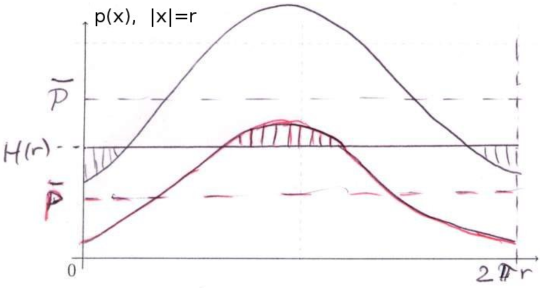


Figure: 128, 129

Effective Counting Entropy in dim d - Results

The solution is $P(\mathbf{x}) = \frac{Q(|\mathbf{x}|)}{L^d H_d(|\mathbf{x}|)}$, *where* $Q(r)$ *is depicted below.*

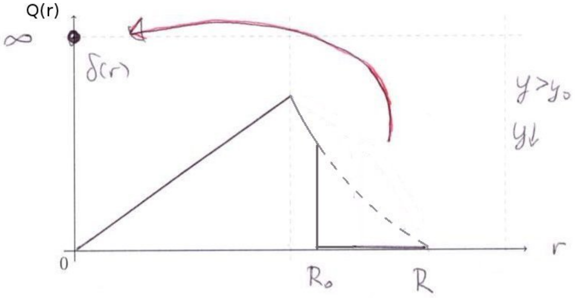


Figure: 130

Case when $y_0 < \int_0^R r\, H_d(r)\, dr$.

Effective Counting Entropy in dim d - Results

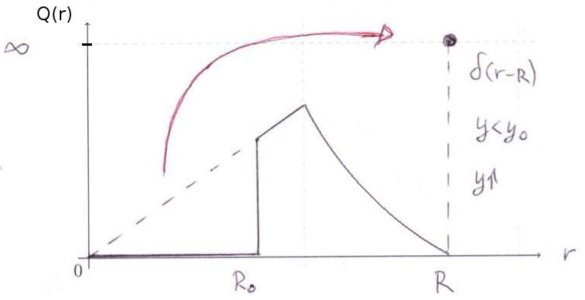


Figure: 131

Case when $y_0 > \int_0^R r\, H_d(r)\, dr$.

If $y_0 = \int_0^R r\, H_d(r)\, dr$, then $Q(r) = H_d(r)$, which means that the solution is uniform $P(\mathbf{x}) = \frac{1}{L^d}$.

Figure 45. 2D flat torus, a solution for small y_0 has a δ-function at A and a step function at the gray circle (identification of edges and vertices as depicted).

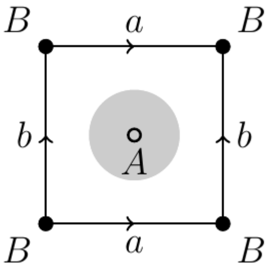


Figure: 45

Figure 46. 2D flat torus, a solution for large y_0 has a δ-function at A and a step function at the gray area.

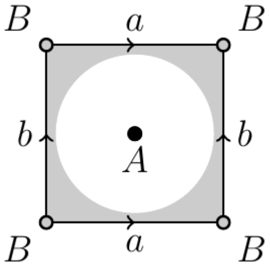


Figure: 46

Figure 47. 2D flat sphere, a solution for small y_0 has a δ-function at B and a step function at the gray circle.

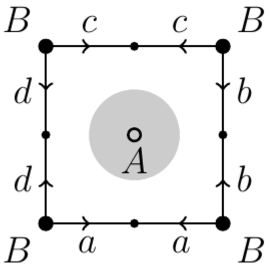


Figure: 47

Figure 48. 2D flat sphere, a solution for large y_0 has a δ-function at B and a step function at the gray area.

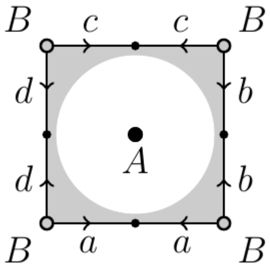


Figure: 48

Conclusion.

Boltzmann		Effective Counting
$S(P) = \sum_i p_i \ln(p_i)$		$\mathcal{N}_*[P] = \sum_i \min\{N\, p_i, 1\}$
$S(P) = -\int_0^L P \ln(P) dx$	**Maximize**	$\mathcal{S}_*[P] = \ln(\int_0^L \min\{L\, P, 1\} dx)$
$< x >$ is fixed.	**if**	$< x^k >$ is fixed.
$P(x) =$ truncated exponential	**Results**	$P(x) =$ step function $+ \delta(\,)$

We have another Tool

in our Toolbox of Entropies!

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Thank you! v7