What can QCD practitioner learn from Anderson model and *vice versa*

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Motivation

- QCD at finite temperature: IR scale-invariant phase in QCD ?
  [A. Alexandru and I.H. PRL 2019]
- QCD: standard lore: no phase transition, only crossover
  [Aoki, *et al.*, Nature 2006]
- Is Anderson localization embedded in finite-temperature QCD ?
- Resolution requires reliable results from both AM and QCD
- Effective counting appears to be a relevant tool connecting the two

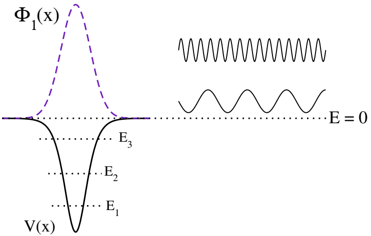
IH, PM, Super-Universality in Anderson Localization, PRL (2022)
IH, PM, Topological Dimension from Disorder and Quantum Mechanics?
arXiv:2212.09806

Anderson localization

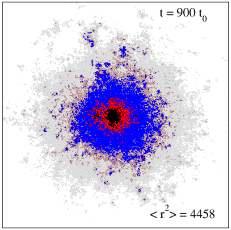
Quantum mechanics: eigenstates of quantum particle could be

bounded   extended   . . . and localized

[P. W. Anderson 1958]



$E < 0$ $E > 0$



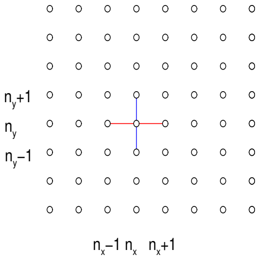
Localization is a consequence of

- disorder
- wave character of particle
- $\Phi(\vec{r}) \propto \exp\left[-r/\lambda\right]$

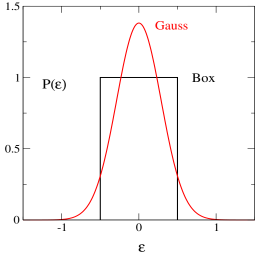
Anderson model

$$\mathcal{H} = W \sum_{n} \varepsilon_n c_n^\dagger c_n + t \sum_{[nn']} c_n^\dagger c_{n'}$$

random energies ε_n
disorder strength W
hopping $t = 1$,
dimension d ($= 1, 2, 2+\epsilon, \ 3, 4, 5, \dots$)
lattice size L, $N = L^d$.



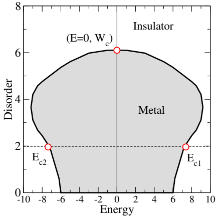
- orthogonal symmetry
  single spinless particle model
- unitary symmetry
  \+ magnetic field
- symplectic symmetry
  no magnetic field, fermion
  t is 2×2 matrix
- other seven symmetry classes (Wegner)

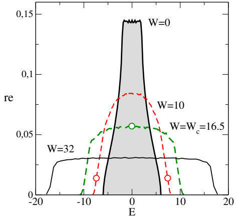


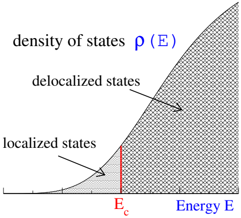
Localization-delocalization transition

- dimension (1d electron is always localized)
- symmetry (orthogonal, unitary, symplectic . . .)

3D Anderson model: phase diagram, density of state, mobility edge





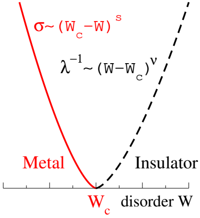


Critical exponents:

conductivity: $\sigma(E) \sim (E_c - E)^s$

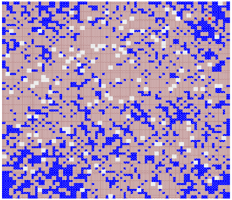
localization length: $\lambda(E) \sim (E - E_c)^{-\nu}$

$s = (d-2)\nu$ . . . critical exponents.

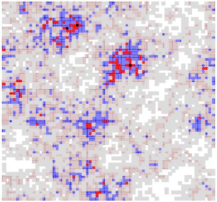


Spatial distribution of electron (2D lattice)

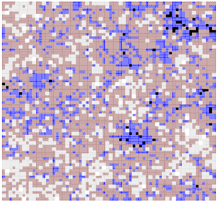
Metallic



Localized



Critical



Disorder changes the dimension of space occupied by electron:

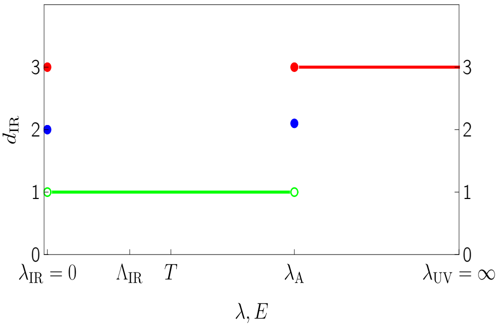
$d = 2$ $d = 0$ $0 < d < 2$

Various concepts of dimension:

- Inverse participation ratio
- Multifractals [Mirlin]
- Effective counting [I H]

Phase transitions (localization?) in QCD

A Alexandru and I H, PRL (2001)



Effective dimension of eigenstates changes at critical points $\lambda = 0$ and $\lambda = \lambda_A$ (Anderson transition?)

Can we observe similar effect in Anderson model?

Effective counting

Consider the eigenstate Φ_i $\quad i = 1, 2, \ldots L^d$ and define the site probabilities

$$p_i = |\Phi_i|^2$$

and new quantity $\mathcal{N}_*$:

$$\mathcal{N}_* = \sum_i \min(p_i L^d, 1)$$

$\mathcal{N}_*$ is averaged over statistical ensemble, typically of $N_{\mathrm{stat}} \sim 10^5$ samples.Size of samples: $6 \leq L \leq 160$

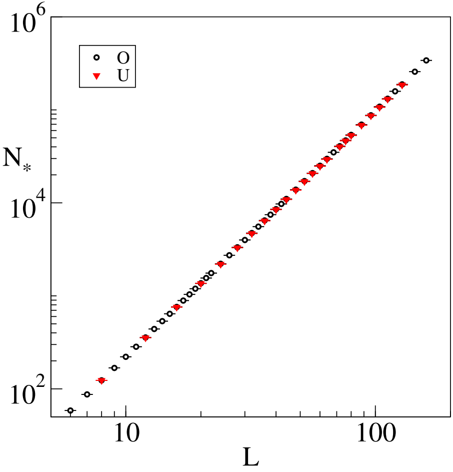
Conjecture:

$$\mathcal{N}_* \propto L^{d_{\mathrm{IR}}}, \qquad d_{\mathrm{IR}} = \begin{cases} d & \text{metal} \\ 0 & \text{insulator} \\ 0 < d_{\mathrm{IR}} < d & \text{critical} \end{cases}$$

Aim: numerical verification of the conjecture and calculation of d_{IR}

Numerical data for orthogonal and unitary symmetry

At the critical point:



Data indicate that the L-dependence of $\mathcal{N}_*$ is universal.

Superuniversality [IH, PM: PRL 2022]

Super-Universality ?

$$\mathcal{N}_* \propto L^{d_{\rm IR}}$$

Calculate for each size L

$$d_{\rm IR}(L,2) = \frac{1}{\ln 2} \ln \frac{\mathcal{N}_{*L}}{\mathcal{N}_{*L/2}}$$

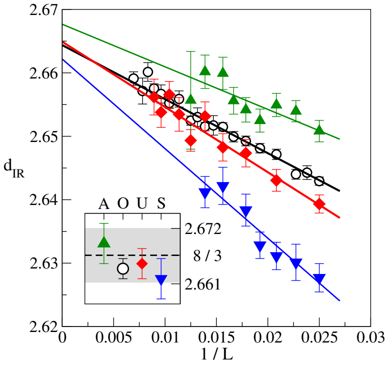
symmetry	$d_{\rm IR}$	ν
O	2.664(2)	1.572(5)
U	2.665(3)	1.43(6)
S	2.662(4)	1.360(6)
A	2.668(4)	1.071(4)

Exponent $d_{\rm IR}$: is super-universal

$$d_{\rm IR} \approx \frac{8}{3}$$

for four universality classes

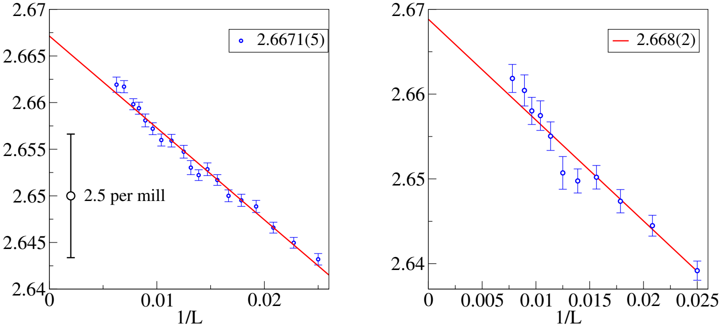
The same exponent for all universality classes ?



Not everybody accepts this:
[I S Burmistrov, arXiv:2210.10539v3]

Quantitative analysis - a year after

for Orthogonal and Unitary symmetry



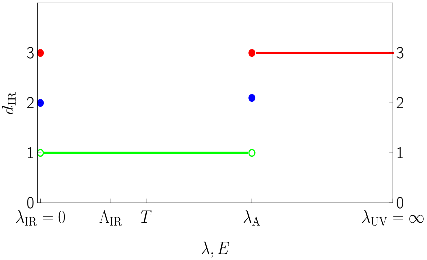
Note that the range on the vertical axis is $\approx 1\%$ of typical values of $d_{\mathrm{IR}}(L, 2)$

Physical meaning of super-Universality

Super-Universality would mean that there is something still unknown in Anderson model.

Geometry? Does a new dimension arise from the interplay of disorder and quantum mechanics?

Anderson - QCD Super-Universality? Can we expect $d_{\mathrm{IR}} \approx 8/3$ also in QCD ?



It seems that dominant dimension is close to 2 in QCD,

Low-dimensional life of critical electron

For a given critical state Φ, order set of probabilities $\{p_i\}$, $i = 1, 2, \dots L^d$:

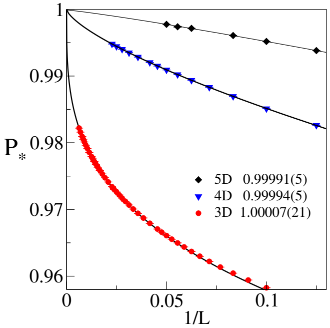
Order all probabilities

$$p_1 > p_2 > \dots$$

and calculate

$$P_* = \sum_{i=1}^{\mathcal{N}_*} p_i$$

Average over N_{stat} samples for each L
Find that $P_* \to 1$ when $L \to \infty$



In the limit $L \to \infty$ critical electron occupies subspace of dimension d_{IR}.

Are any other dimensions present with similar physical interpretation in Anderson model?

Multidimensionality

For a given critical state Φ, order set of probabilities $\{p_i\}$, $i = 1, 2, \ldots N = L^d$:

$$p_1 > p_2 > \cdots > p_N$$

Consider cumulative probabilities

$$q_j = j \times \Delta, \qquad j = 1, 2, \ldots n = [\Delta^{-1}]$$

Calculate N_j such that

$$q_j = \sum_{i=1}^{N_j} p_i$$

Then

$$\mathcal{N}_j = N_j - N_{j-1}$$

determines the number of sites which together have occupancy Δ.

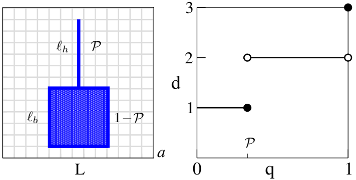
Multidimensionality

Conjecture:

$$\mathcal{N}_L(q = q_j) = v(q) L^{\,d(q)}$$

An (infinite) set of dimensions d_q, each of them might be important.

Example: shovel has two dimensions



both are important for its complete description.
Probability to get dimension d is

$$P(d) = \mathcal{P}\delta(d-2) + (1-\mathcal{P})\delta(d-1)$$

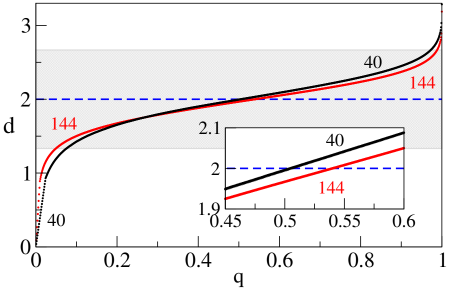
Similar picture for Anderson model ?

Multidimensionality

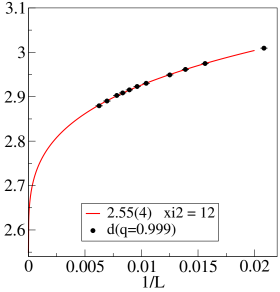
$$d_q = \ln \frac{\mathcal{N}_L(q)}{\mathcal{N}_{L/2}(q)} \frac{1}{\ln 2}$$

$d(q)$ ... dimension of subspace created by sites which increase the total probability from q_j to q_{j+1}

Spectrum of d_q for $L = 40$ and $L = 140$



$d(q = 0.999)$

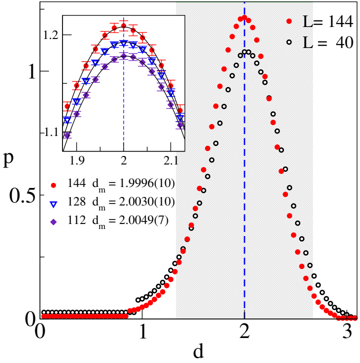


Multidimensionality

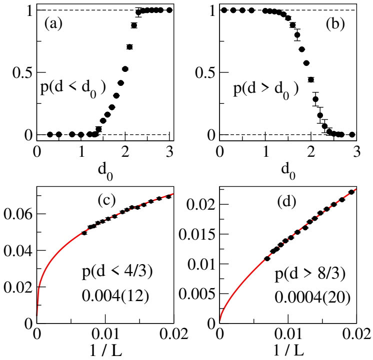
Probability $P(d)$ to find $d_q = d$

$$P(d) = \left[\frac{\partial d(q)}{\partial q}\right]^{-1}$$

Maximum at $d = 2$



Lower and higher cut-off



$P(d) \neq 0$ only in the interval

$$d_{\min} = 4/3 < d < 8/3 = d_{\max}$$

Multidimensionality

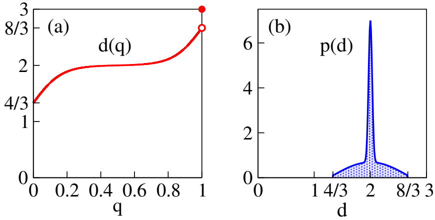
There is a (continuous) set of dimensions,

$$d_{\min} \leq d \leq d_{\max} \leq d_{\mathrm{IR}}$$

which define fractal manifolds occupied by critical electron.

The distribution $P(d)$ possesses maximum at $d_2 \approx 2$

Conjecture: in the limit $L \to \infty$, maximum at $d = 2$ increases;



Does dimension $d = 2$ correspond to dimension observed in QCD ?

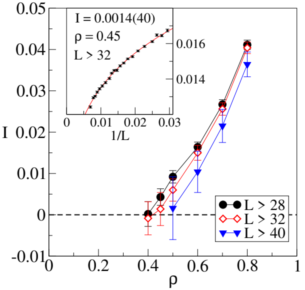
Appendix

$$I(\rho, L) = \int_{q_2-\rho/2}^{q_2+\rho/2} \mathrm{d}q \, [2 - d(q, L)]^2$$

and

$$I(\rho) = \lim_{L\to\infty} I(\rho, L)$$

$I(\rho)$ measures the deviation of $d(q, L)$ from $d = 2$ when $L \to \infty$.



Conclusion

- Universal dimension of "critical subspace"
- Continuous set of dimensions which characterize the critical state
- Disorder-induced topology of Anderson model?