Goldstone's theorem and localisation of Dirac eigenmodes in gauge theories

Matteo Giordano

Eötvös Loránd University (ELTE)
Budapest

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Localisation alternative to Goldstone's theorem

Localisation can invalidate the conclusions of Goldstone's theorem and remove massless excitations from the spectrum

- Known for a long time in condensed matter physics
  [McKane, Stone (1981)]
- Rediscovered in zero-temperature lattice QCD in an unphysical regime (Aoki phase of Wilson fermions in quenched QCD)
  [Golterman, Shamir (2003)]
- Zero and finite temperature QCD in the chiral limit: a finite density of near-zero Dirac modes breaks chiral symmetry but there are no massless excitations if they are localised
  [MG (2021), MG (2021), MG (2022)]

Goldstone's theorem

In a relativistic field theory, if there are

- charge Q generated by conserved local current J^μ

$$Q_V = \int_V d^3x \, J^0(x)$$

- spontaneous symmetry breaking due to a nonzero order parameter

$$i \lim_{V\to\infty} \langle 0|[Q_V, A]|0\rangle \neq 0$$

then there are massless particle in the spectrum:

$$\int d^4x \, e^{-ip\cdot x} \langle 0|J^0(x)A|0\rangle = a\delta(p^2) + \text{non-singular}, \quad a \neq 0$$

[Goldstone, Salam, Weinberg (1962), Strocchi (2008)]

Application: pions in QCD

Generalised QCD Lagrangian: N_g gluons (gauge fields) B_μ^a and N_f quarks (fermion fields) ψ_f, $\bar{\psi}_f$

$$\mathcal{L}_{\rm QCD} = -\tfrac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \bar{\psi}(i\not{D} - M)\psi \qquad M = {\rm diag}(m_1, \ldots, m_{N_f})$$

$$F^a_{\mu\nu} = \partial_\mu B^a_\nu - \partial_\nu B^a_\mu - g f^{abc} B^b_\mu B^c_\nu \qquad \not{D} = \gamma^\mu D_\mu = \gamma^\mu(\partial_\mu + igB^a_\mu T^a)$$

T^a: generators of gauge group
f^{abc}: structure constants

Vector flavour symmetry ${\rm SU}(N_f)_V$: for $M = m\mathbf{1}$, invariance under

$$\psi \to U\psi, \quad \bar{\psi} \to \bar{\psi}U^\dagger \qquad U \in {\rm SU}(N_f)$$

Chiral symmetry: for $m = 0$, ${\rm SU}(N_f)_V$ enhanced to ${\rm SU}(N_f)_L \times {\rm SU}(N_f)_R$

$$\bar{\psi}\not{D}\psi = \bar{\psi}_L\not{D}\psi_L + \bar{\psi}_R\not{D}\psi_R\,, \qquad \psi_{R,L} = \tfrac{1\pm\gamma^5}{2}\psi\,, \quad \bar{\psi}_{R,L} = \bar{\psi}\tfrac{1\mp\gamma^5}{2}$$

Application: pions in QCD II

Conserved currents: non-singlet vector and axial vector currents

$$V^{a\mu} = \bar{\psi}\gamma^\mu t^a \psi\,, \qquad A^{a\mu} = \bar{\psi}\gamma^\mu\gamma^5 t^a \psi\,, \qquad 1 \le a \le N_f^2 - 1$$

t^a : generators of $\mathrm{SU}(N_f)$, $2\mathrm{tr}\, t^a t^b = \delta^{ab}$

Spontaneous breaking of chiral symmetry

$$\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \to \mathrm{SU}(N_f)_V$$

Order parameter: (pseudo)scalar densities

$$P^a = \bar{\psi}\gamma^5 t^a \psi \qquad S = \bar{\psi}\psi$$

$$\langle 0|[Q_5^a, P^b]|0\rangle = \frac{1}{N_f}\delta^{ab}\langle 0|S|0\rangle = \Sigma\delta^{ab}, \quad \Sigma \neq 0$$

$$\Rightarrow \int d^4x\, e^{-ip\cdot x}\langle 0|A^{a0}(x)P^b(0)|0\rangle \propto \Sigma\delta^{ab}\delta(p^2)$$

$Q_5^a = \int d^3x\, A^{a0}$

$\Rightarrow N_f^2 - 1$ massless pseudoscalar particles in the spectrum (pions)

Functional integral quantisation

- Nonperturbative physics requires nonperturbative approaches
- Explicit construction of quantum field operators generally unknown outside of perturbation theory
- Operators constructed implicitly through their correlation functions in the functional integral approach

Wick rotation to imaginary time: $x^0 \to -ix_{E4}$, $x^j \to x_{Ej}$ (same for B^μ, $i\gamma^\mu$)

$$S[B,\psi,\bar{\psi}] = \int d^4x \left\{ \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \bar{\psi}(\not{D} + M)\psi \right\} = S_G[B] + S_F[B,\psi,\bar{\psi}]$$

$$\begin{aligned} \langle \mathcal{O}[B,\psi,\bar{\psi}] \rangle &= Z^{-1} \int \mathcal{D}B\mathcal{D}\psi\mathcal{D}\bar{\psi}\, e^{-S[B,\psi,\bar{\psi}]} \mathcal{O}[B,\psi,\bar{\psi}] \\ &= Z^{-1} \int \mathcal{D}B\, e^{-S_G[B]} \det(\not{D}[B] + m)\, \tilde{\mathcal{O}}[B] \end{aligned}$$

$$Z = \int \mathcal{D}B\, e^{-S[B,\psi,\bar{\psi}]} \det(\not{D}[B] + m)$$

Functional-integral proof of Goldstone's theorem

Ward-Takahashi identities (WTI): making an infinitesimal symmetry transformation promoted from global to local, for any observable $\mathcal{O}$

$$\left\langle \frac{\delta S}{\delta\epsilon(x)}\mathcal{O} \right\rangle = \left\langle \frac{\delta \mathcal{O}}{\delta\epsilon(x)} \right\rangle$$

if integration measure is invariant

WTI for axial transformations (opposite rotation of L and R) at finite m

$$-\langle \partial_\mu A^a_\mu(x) P^b(0)\rangle + 2m\langle P^a(x)P^b(0)\rangle = \frac{1}{N_f}\delta^{(4)}(x)\delta^{ab}\langle S(0)\rangle$$

Symmetric, possibly spontaneously broken massless theory obtained as the limit $m \to 0$ of the explicitly broken massive one

Proof via Euclidean Ward-Takahashi identities

In momentum space, using unbroken $\mathrm{SU}(N_f)_V$

$$ip_\mu \mathcal{G}_{AP\,\mu}(p) + 2m\mathcal{G}_{PP}(p) = \Sigma$$

$$\mathcal{G}_{AP\,\mu}(p)\delta^{ab} = \int d^4x\, e^{-ip\cdot x}\langle A^a_\mu(x)P^b(0)\rangle$$
$$\mathcal{G}_{PP}(p)\delta^{ab} = \int d^4x\, e^{-ip\cdot x}\langle P^a(x)P^b(0)\rangle$$
$$\langle S(0)\rangle = N_f\langle(\bar{\psi}_f\psi_f)(0)\rangle = N_f\Sigma$$

Chiral limit $\Rightarrow \mathcal{G}_{PP}(p)$ term drops

$$ip_\mu \mathcal{G}_{AP\,\mu}(p) = \Sigma$$

$\mathrm{O}(4)$ invariance $\Rightarrow \mathcal{G}_{AP\,\mu}(p) = p_\mu g_{AP}(p^2)$

$$ip^2 g_{AP}(p^2) = \Sigma$$

$\Rightarrow$ massless pole in $\mathcal{G}_{AP\,\mu}(p)$ if $\Sigma \neq 0$
$\Rightarrow$ massless particle

Goldstone's theorem at finite temperature

Goldstone's theorem at finite T: if there are

- charge Q generated by conserved local current J^μ

$$Q_V = \int_V d^3x\, J^0(x)$$

- nonzero order parameter

$$i \lim_{V\to\infty} \langle\langle [Q_V, A] \rangle\rangle_\beta \neq 0\,, \qquad \langle\langle A \rangle\rangle_\beta = \frac{\mathrm{Tr}\, e^{-\beta H} A}{\mathrm{Tr}\, e^{-\beta H}}\,, \qquad \beta = \frac{1}{T}$$

then there are massless quasi-particles in the spectrum (excitations with vanishing energy and infinite lifetime at zero momentum):

$$\lim_{\vec{k}\to 0} i\varrho_{J^0 A}(\omega, \vec{k}) = a\delta(\omega) + \text{non-singular}, \quad a \neq 0$$

$$\varrho_{J^0 A}(\omega, \vec{k}) = \int d^4x\, e^{-i(\omega t - \vec{p}\cdot\vec{x})} \langle\langle | [J^0(t, \vec{x}), A] \rangle\rangle_\beta \quad \text{(spectral function)}$$

[Lange (1965), Strocchi (2008)]

Functional integral formulation of finite temperature QFT

Compactify temporal direction to size $\beta = 1/T$

$$S_\beta[B,\psi,\bar\psi] = \int_\beta d^4x \left\{ \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \bar\psi(\not{D} + M)\psi \right\} \qquad \int_\beta d^4x = \int_0^\beta dt \int d^3x$$

Impose periodic/antiperiodic BC on bosonic/fermionic fields

$$\int_\beta \mathcal{D}B\mathcal{D}\psi\mathcal{D}\bar\psi = \int_{B(\beta)=B(0)} \mathcal{D}B \int_{\psi(\beta)=-\psi(0)} \mathcal{D}\psi \int_{\bar\psi(\beta)=-\bar\psi(0)} \mathcal{D}\bar\psi$$

Replace expectation values $\langle \bullet \rangle \to$ thermal expectation values $\langle \bullet \rangle_\beta$

$$\langle\!\langle \mathrm{T}\left(\textstyle\prod_j \mathcal{O}_j(x_j)\right)\rangle\!\rangle_\beta = Z_\beta^{-1} \int_\beta \mathcal{D}B\mathcal{D}\psi\mathcal{D}\bar\psi\, e^{-S_\beta[B,\psi,\bar\psi]} \textstyle\prod_j \mathcal{O}_j[B,\psi,\bar\psi;x_j]$$

$$= Z_\beta^{-1} \int_\beta \mathcal{D}B\, e^{-S_{\beta G}[B]} \det(\not{D}[B] + m) \textstyle\prod_j \tilde{\mathcal{O}}_j[B;x_j] = \langle \textstyle\prod_j \mathcal{O}_j(x_j)\rangle_\beta$$

$$Z_\beta = \int_\beta \mathcal{D}B\, e^{-S_{\beta G}[B]} \det(\not{D}[B] + m)$$

Proof via Euclidean Ward-Takahashi identities at $T \neq 0$

WTI still hold at finite T

$$-\langle \partial_\mu A^a_\mu(x) P^b(0)\rangle_\beta + 2m\langle P^a(x)P^b(0)\rangle_\beta = \frac{1}{N_f}\delta^{(4)}(x)\delta^{ab}\langle S(0)\rangle_\beta$$

Temporal momentum quantised to Matsubara frequencies $\omega_n = \frac{2\pi n}{\beta}$

$$\begin{aligned} i\vec{p}\cdot\vec{\mathcal{G}}_{AP}(0,\vec{p}) + 2m\mathcal{G}_{PP}(0,\vec{p}) &= \Sigma \qquad n = 0 \\ i\omega_n\mathcal{G}_{AP4}(\omega_n,\vec{p}) + i\vec{p}\cdot\vec{\mathcal{G}}_{AP}(\omega_n,\vec{p}) + 2m\mathcal{G}_{PP}(\omega_n,\vec{p}) &= \Sigma \qquad n \neq 0 \end{aligned}$$

$$\mathcal{G}_{PP}(\omega_n,\vec{p})\delta^{ab} = \int_\beta d^4x\, e^{-i(\omega_n t - \vec{p}\cdot\vec{x})}\langle P^a(x)P^b(0)\rangle_\beta \text{ etc.}$$

$\mathrm{O}(3)$ invariance $\vec{\mathcal{G}}_{AP}(\omega_n,\vec{p}) = \vec{p}g_n(\vec{p}^{\,2})$, taking the chiral limit

$$i\vec{p}^{\,2}g_0(\vec{p}^{\,2}) = \Sigma \Rightarrow \text{massless pole in } \vec{\mathcal{G}}_{AP}(0,\vec{p}) \text{ if } \Sigma \neq 0$$

…but this $\not\Rightarrow$ massless particle *a priori*

Proof via Euclidean Ward-Takahashi identities at $T \neq 0$ II

Minkowskian theory (real time) reconstructed from Euclidean theory (imaginary time) by reversing Wick's rotation

In Minkowski space $[A^a_\mu(x), P^b(0)] = 0$ if $x^2 < 0$ (relativistic locality)

Necessary condition on Euclidean correlators for relativistic locality in the reconstructed Minkowskian theory ("regularity condition"):

$$\lim_{\vec{p}\to 0} \vec{p}\cdot\vec{\mathcal{G}}_{AP}(\omega_n,\vec{p}) = 0 \qquad \text{for} \quad n \neq 0$$

From WTI in the chiral limit

$$i\omega_n \mathcal{G}_{AP4}(\omega_n,\vec{p}) + i\vec{p}\cdot\vec{\mathcal{G}}_{AP}(\omega_n,\vec{p}) = \Sigma$$

$+$ regularity condition

$$\mathrm{G}(\omega_n) \equiv \lim_{\vec{p}\to 0}\lim_{m\to 0} \mathcal{G}_{AP4}(\omega_n,\vec{p}) = \frac{\Sigma}{i\omega_n} \qquad \text{for } n \neq 0$$

$\mathrm{G}(0) = 0$ since $\mathrm{G}(-\omega_n) = -\mathrm{G}(\omega_n)$
from time reflection symmetry

Proof via Euclidean Ward-Takahashi identities at $T \neq 0$ III

Spectral function from Euclidean correlators via analytic interpolation (unique in upper/lower complex half-plane by Carlson's theorem)

$$i\varrho_{A^{a0}P^a}(\omega,\vec{p}) = \overline{\mathcal{G}}_{AP4}(\omega_n \to \epsilon - i\omega, \vec{p}) - \overline{\mathcal{G}}_{AP4}(\omega_n \to -\epsilon - i\omega, \vec{p})$$

$$\lim_{\vec{p}\to 0}\lim_{m\to 0} i\varrho_{A^{a0}P^a}(\omega,\vec{p}) = \overline{\mathrm{G}}(\omega_n \to \epsilon - i\omega) - \overline{\mathrm{G}}(\omega_n \to -\epsilon - i\omega)$$

$$= \Sigma\left(\frac{1}{\omega + i\epsilon} - \frac{1}{\omega - i\epsilon}\right) = -2\pi i \Sigma \delta(\omega)$$

Massless pseudoparticles if $\Sigma \neq 0$ (coincides with presence of a pole at zero in $\vec{\mathcal{G}}_{AP}(0,\vec{p})$, after all) [MG (2020), MG (2022)]

- Proof in the Euclidean functional-integral formalism bypasses explicit reconstruction of operators and check of current conservation
- Massless theory approached from $m \neq 0$ (selects vacuum in case of χSSB), proof can be adapted if something goes wrong as $m \to 0$

Refinement of Goldstone's theorem

What could possibly go wrong? Implicit assumption: $\mathcal{G}_{PP}$ regular in the chiral limit, $2m\mathcal{G}_{PP} \to 0$, but what if $\mathcal{G}_{PP} \propto 1/m$, and so $2m\mathcal{G}_{PP} \not\to 0$?

Define:

$$\text{zero } T: \qquad \lim_{m\to 0} 2m\mathcal{G}_{PP}(p) = \mathcal{R}(p)$$

$$\text{finite } T: \qquad \lim_{\vec{p}\to 0} \lim_{m\to 0} 2m\mathcal{G}_{PP}(\omega_n, \vec{p}) = \mathrm{R}(\omega_n)$$

Generalised Goldstone's theorem:

$$T = 0 \qquad ip_\mu \mathcal{G}_{AP\mu}(p) = [\Sigma - \mathcal{R}(0)]$$

$$T \neq 0 \qquad \lim_{\vec{p}\to 0} \lim_{m\to 0} -\frac{\varrho_{A^{a0}P^a}(\omega, \vec{p})}{2\pi} = [\Sigma - \overline{\mathrm{R}}(0^+)]\delta(\omega) = [\Sigma - \mathrm{R}(0)]\delta(\omega)$$

No transport peak expected in the pseudoscalar channel, i.e., no term $\propto \omega\delta(\omega)$ in ϱ_{PP} [Burnier *et al.* (2017)] $\Rightarrow \overline{\mathrm{R}}(0^+) = \mathrm{R}(0)$

Massless excitations present only if $\Sigma - \mathcal{R}(0)$ or $\Sigma - \mathrm{R}(0)$ are nonzero

Refinement of Goldstone's theorem II

- Goldstone theorem evaded: non-singlet axial current not conserved, "anomalous" remnant ($\mathcal{R}, \mathrm{R} \sim m \cdot \frac{1}{m}$) not a contact term, breaks symmetry explicitly also in the chiral limit
- Fate of Goldstone excitations depends on $\Sigma - \mathcal{R}(0)$ or $\Sigma - \mathrm{R}(0)$, balance between spontaneous and explicit breaking

Is it possible to have nonzero $\mathcal{R}(0)$ or $\mathrm{R}(0)$?

Yes, in the presence of a finite density of localised near-zero Dirac modes

Not happening at $T = 0$ and low T, near-zero Dirac modes are dense but delocalised

Crossover to approximately chirally restored phase at $T_c \approx 155\,\mathrm{MeV}$ [Borsányi *et al.* (2010), Bazavov *et al.* (2016)], what happens above T_c?

Localised and delocalised Modes

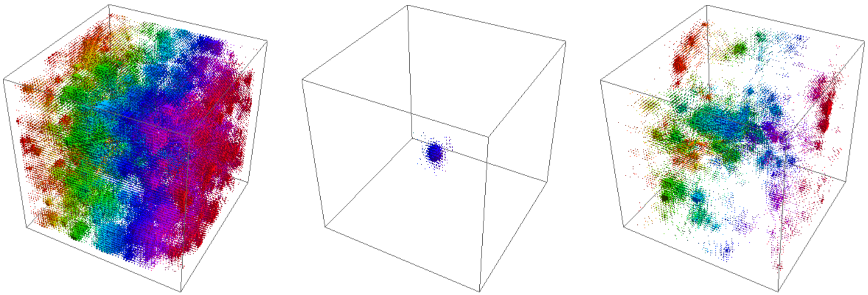


Figure from [Ujfalusi *et al.* (2015)]

Eigenmodes ψ of some Hamiltonian

- Delocalised mode: extends throughout the system $\rightarrow$ $\|\psi(x)\|^2 \sim 1/V$, supported everywhere
- Localised mode: is confined in a finite region R_0 of size V_0 $\rightarrow$ $\|\psi(x)\|^2 \sim 1/V_0$ within R_0, zero outside
- Critical mode: nontrivial fractal dimension $\rightarrow$ $\|\psi(x)\|^2 \sim 1/V^{\alpha}$, support has size V^{α}

IPR

Inverse participation ratio measures the inverse of the size of a mode

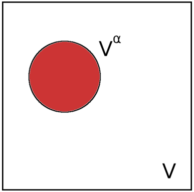
$$\mathrm{IPR}_n = \int d^4x \|\psi_n(x)\|^4 \qquad \mathrm{PR}_n = \frac{\mathrm{IPR}_n^{-1}}{V}$$

Random Hamiltonians, average over disorder locally in the spectrum

$$\overline{\mathrm{IPR}}(\lambda, V) = \frac{\langle \sum_n \delta(\lambda - \lambda_n) \mathrm{IPR}_n \rangle}{\langle \sum_n \delta(\lambda - \lambda_n) \rangle}$$

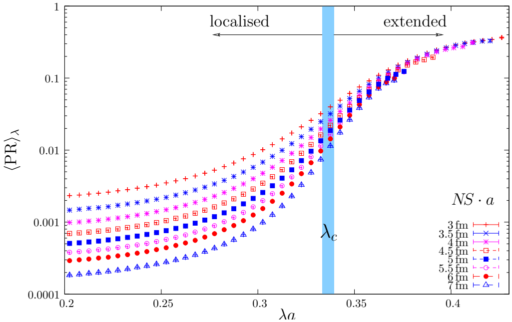
Typical mode: $\mathrm{IPR}_n \sim \underbrace{V^{-2\alpha}}_{\|\psi_n(0)\|^4} \times \underbrace{V^{\alpha}}_{\text{size of support}} \sim V^{-\alpha}$

$$\lim_{V\to\infty} \overline{\mathrm{IPR}}(\lambda, V) \to \begin{cases} \frac{1}{V} \to 0 & \text{delocalised} \\ \frac{1}{V^{\alpha}} \to 0 & \text{critical} \\ \neq 0 & \text{localised} \end{cases}$$



Localised Dirac eigenmodes at finite temperature

$-i\not{D}$ in a gauge background is a random Hamiltonian



2+1 QCD at $T = 394\,\mathrm{MeV}$
Data from [MG *et al.* (2014)]

- low Dirac modes localised in deconfined phase, delocalised in confined phase [García-García, Osborn (2007), Kovács, Pittler (2012), . . . , Giordano, Kovács (2021)]
- features analogous to condensed matter systems: Anderson transition, multifractality at mobility edge λ_c [MG *et al.* (2014), Ujfalusi *et al.* (2015)]

Localised near-zero modes

Density of near-zero modes $\rho(0^+)$ = order parameter for chiral symmetry in the chiral limit, approximate order parameter at finite mass

$$-\Sigma = \int_0^\infty d\lambda\, \rho(\lambda) \frac{2m}{\lambda^2 + m^2} \underset{m\to 0}{\to} \pi\rho(0^+)\,, \quad \rho(\lambda) = \lim_{V\to\infty} \underbrace{\tfrac{1}{V}\left\langle \textstyle\sum_n \delta(\lambda - \lambda_n)\right\rangle}_{\rho_V(\lambda)}$$

Density of low modes drops above T_c ≈ approximate chiral restoration

Localised modes appear at T_c when transition is sharp: pure gauge [Kovács, Vig (2018, 2020), MG (2019), Bonati *et al.* (2021), Baranka, MG (2021, 2022)], QCD at $\mu = i\pi$ [Cardinali *et al.* (2022)] ≈ "order parameter" for deconfinement

Suggests that low Dirac modes connect chiral restoration ($\sim$ decrease in spectral density) and deconfinement ($\sim$ localisation)

With dynamical quarks the transition is generally a crossover, becomes sharp in the chiral limit - what happens to localised near-zero modes?

If $\rho_{\rm loc}(0) \neq 0$ when $m \to 0 \Rightarrow \mathrm{R}(0) \neq 0$

Localised near-zero modes II

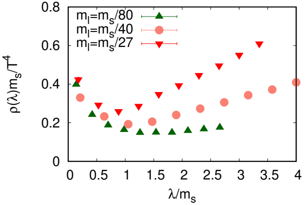
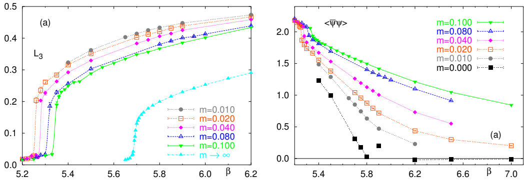


Figure from [Kaczmarek *et al.* (2021)]



Figures from [Karsch and Lütgemeier (1999)]

$N_f = 2+1$ **QCD** (overlap on HISQ): peak of localised near-zero modes for $m_l \gtrsim m_l^{\mathrm{ph}}$ at $1.05\,T_{pc}$ [Dick *et al.* (2015)]

$m_l < m_l^{\mathrm{ph}}$, ov/HISQ [Kaczmarek *et al.* (2021)]; $m_l \gtrless m_l^{\mathrm{ph}}$, $1.3\,T_{pc}$, HISQ [Ding *et al.* (2021)]; $m_l = m_l^{\mathrm{ph}}$, $1.07, 1.1\,T_{pc}$, HISQ [Kaczmarek *et al.* (2023)]; $m_l = m_l^{\mathrm{ph}}$, $1.5\,T_{pc}$, stout [C. Bonanno];

localisation properties unknown

$\Rightarrow$ Does the peak survive $m_l \to 0$? Are peak modes localised?

$N_f = 2$ **massless adjoint quarks**: intermediate chirally broken ($\rho(0^+) \neq 0$) but deconfined phase [Karsch and Lütgemeier (1999)]

$\Rightarrow$ Are near-zero modes localised?

Pseudoscalar correlator

IR, $1/m$ divergence in pseudoscalar correlator (continuum):

- expand bare correlator in Dirac modes $\not{D}\psi_n = i\lambda_n\psi_n$

$$-\langle P_B^a(x) P_B^b(0)\rangle = \frac{\delta^{ab}}{2}\left\langle \sum_{n,n'} \frac{\mathcal{O}^{\gamma_5}_{n'n}(x)\mathcal{O}^{\gamma_5}_{nn'}(0)}{(i\lambda_n + m_B)(i\lambda_{n'} + m_B)} \right\rangle = \delta^{ab}\Pi_B(x)$$

$$\mathcal{O}^{\Gamma}_{nn'}(x) \equiv (\psi_n(x), \Gamma\psi_{n'}(x))$$

- renormalise

$$\Pi(x) = Z_m^2[\Pi_B(x) - \Pi^{\mathrm{add.\,div.}}(x)] \qquad m = Z_m^{-1} m_B \qquad \lambda_n^R = Z_m^{-1}\lambda_n$$

- correlation functions of eigenmodes are renormalised, finite quantities

$$C^{\Gamma}(\lambda; m; x) \equiv \lim_{V\to\infty} \left\langle {\sum}'_{n,n'} \delta(\lambda - \lambda_n^R)\mathcal{O}^{\Gamma}_{nn}(x)\mathcal{O}^{\Gamma}_{nn}(0) \right\rangle$$

$$C_2^{\Gamma}(\lambda, \lambda'; m; x) \equiv \lim_{V\to\infty} \left\langle {\sum}'_{n,n'} \delta(\lambda - \lambda_n^R)\delta(\lambda' - \lambda_{n'}^R)\mathcal{O}^{\Gamma}_{n'n}(x)\mathcal{O}^{\Gamma}_{nn'}(0) \right\rangle$$

${\sum}'$: nonzero modes

$\Gamma = 1, \gamma_5$

Large V

$$\lim_{m\to 0} 2m\Pi(x) = 2\lim_{m\to 0}\int_0^{\frac{\mu}{m}} dz \left(\frac{C^1(mz;m;x)}{z^2+1} + \frac{(1-z^2)C^{\gamma_5}(mz;m;x)}{(z^2+1)^2}\right)$$

Zero modes negligible as $V\to\infty$
μ arbitrary, should drop from final result
$|\lambda^R| > \mu$ and C_2^Γ negligible as $m\to 0$

Large-V behaviour of $C_V^\Gamma = \left\langle {\sum_n}' \delta(\lambda-\lambda_n^R)\mathcal{O}_{nn}^\Gamma(x)\mathcal{O}_{nn}^\Gamma(0)\right\rangle$

$$\begin{aligned}|\langle\mathcal{O}_{nn}^\Gamma(x)\mathcal{O}_{nn}^\Gamma(0)\rangle| &\le \langle\|\psi_n(x)\|^2\|\psi_n(0)\|^2\rangle \le \langle\|\psi_n(0)\|^4\rangle \\ &= \frac{T}{V}\left\langle\int_\beta d^4x\,\|\psi_n(x)\|^4\right\rangle = \frac{T}{V}\langle\mathrm{IPR}_n\rangle\end{aligned}$$

$$|C_V^\Gamma(\lambda;m;x)| \le \frac{T}{V}{\sum_n}'\langle\delta(\lambda-\lambda_n^R)\,\mathrm{IPR}_n\rangle = \rho_V(\lambda)\overline{\mathrm{IPR}}(\lambda,V) \sim \rho(\lambda)V^{-\alpha(\lambda)}$$

$C_V^\Gamma \to 0$ unless $\alpha = 0 \Rightarrow$ only localised modes contribute

$$C^\Gamma = \lim_{V\to\infty} C_V^\Gamma = C_{\mathrm{loc}}^\Gamma$$

Anomalous remnant

Assume near-zero localised modes in $\lambda \in [0, \lambda_c(m)] \Rightarrow C^\Gamma_{\mathrm{loc}}, \rho_{\mathrm{loc}} \neq 0$

For localised modes $\int_\beta d^4x$ and the various limits are expected to commute

$$-\mathrm{R}(0) = \int_\beta d^4x \lim_{m\to 0} 2m\Pi(x) = \pi\xi\, \rho_{\mathrm{loc}}(0) \qquad \xi \equiv \lim_{m\to 0} \frac{2}{\pi} \arctan \frac{\lambda_c(m)}{m}$$

Ratio $\frac{\lambda_c(m)}{m}$ is renormalisation group invariant [Kovács, Pittler (2012), MG (2022)]

Localised and delocalised modes usually do not coexist, $\rho_{\mathrm{loc}} = \rho$ or 0

$$\varrho_{A^{a0}P^a}(\omega, \vec{p} = 0)|_{\mathrm{sing}} = -2\pi[\Sigma - \mathrm{R}(0)]\delta(\omega) = 2\pi^2 \left[\rho(0) - \xi\rho_{\mathrm{loc}}(0)\right] \delta(\omega)$$

1. no near-zero localised modes, $\rho_{\mathrm{loc}}(0) = 0$, or $\rho_{\mathrm{loc}}(0) \neq 0$ but $\xi = 0$
   $\Rightarrow$ Goldstone excitations present if $\rho(0) \neq 0$ (standard scenario)
2. near-zero localised modes, $\rho_{\mathrm{loc}}(0) \neq 0$, and $0 < \xi < 1$
   $\Rightarrow$ Goldstone excitations present (but modified "residue")
3. near-zero localised modes, $\rho_{\mathrm{loc}}(0) \neq 0$, and $\xi = 1$ (e.g., $\lambda_c(0) \neq 0$)
   $\Rightarrow$ Goldstone excitations disappear

Summary and outlook

In the presence of a finite density of near-zero **localised** Dirac modes

- pseudoscalar correlator diverges $\sim 1/m$ (unless $\frac{\lambda_c}{m} \to 0$)
- axial nonsinglet WTI in the chiral limit modified by "anomalous" remnant, current **not** conserved $\Rightarrow$ Goldstone's theorem evaded
- spectral function modified, Goldstone excitations disappear if $\frac{\lambda_c}{m} \to \infty$ (modified residue if $\frac{\lambda_c}{m} \to$ finite)

Open issues:

- Is there an explicit realisation of the non-standard scenarios?
  - Intermediate phase of $N_f = 2$ massless adjoint quarks?
  - Peak in 2+1 QCD in the chiral limit?
- Phenomenological consequences?

References

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