NuclearScience Computing Center at CCNU

Correlated Dirac eigenvalues & QCD Chiral phase transition

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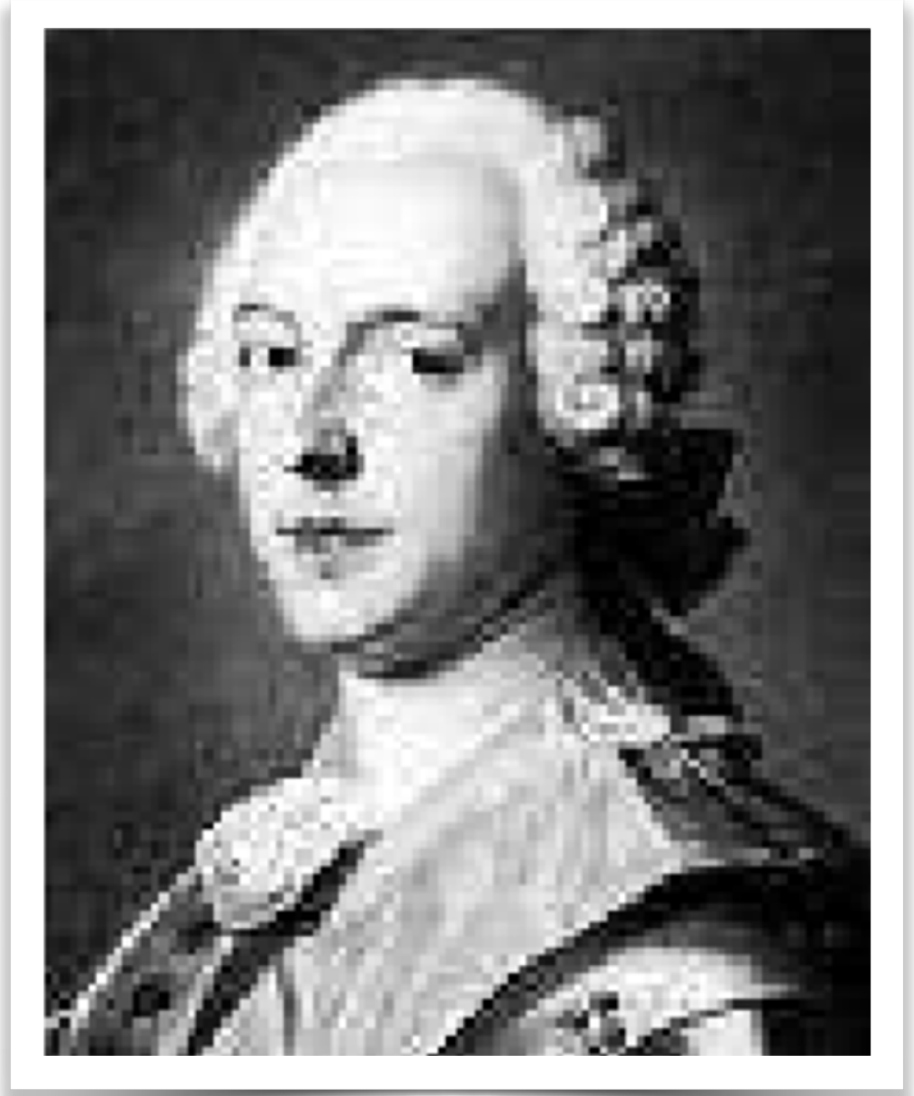
based on PRL 126 (2021) 082001, arXiv: 2112.00318, 2112.00465,
in collaboration with
Sheng-Tai Li, Wei-Ping Huang, Swagato Mukherjee, Peter Petreczky,
Akio Tomiya, Xiao-Dan Wang, Yu Zhang

New Trend in thermal phases of QCD@ Prague
April 14-17, 2023

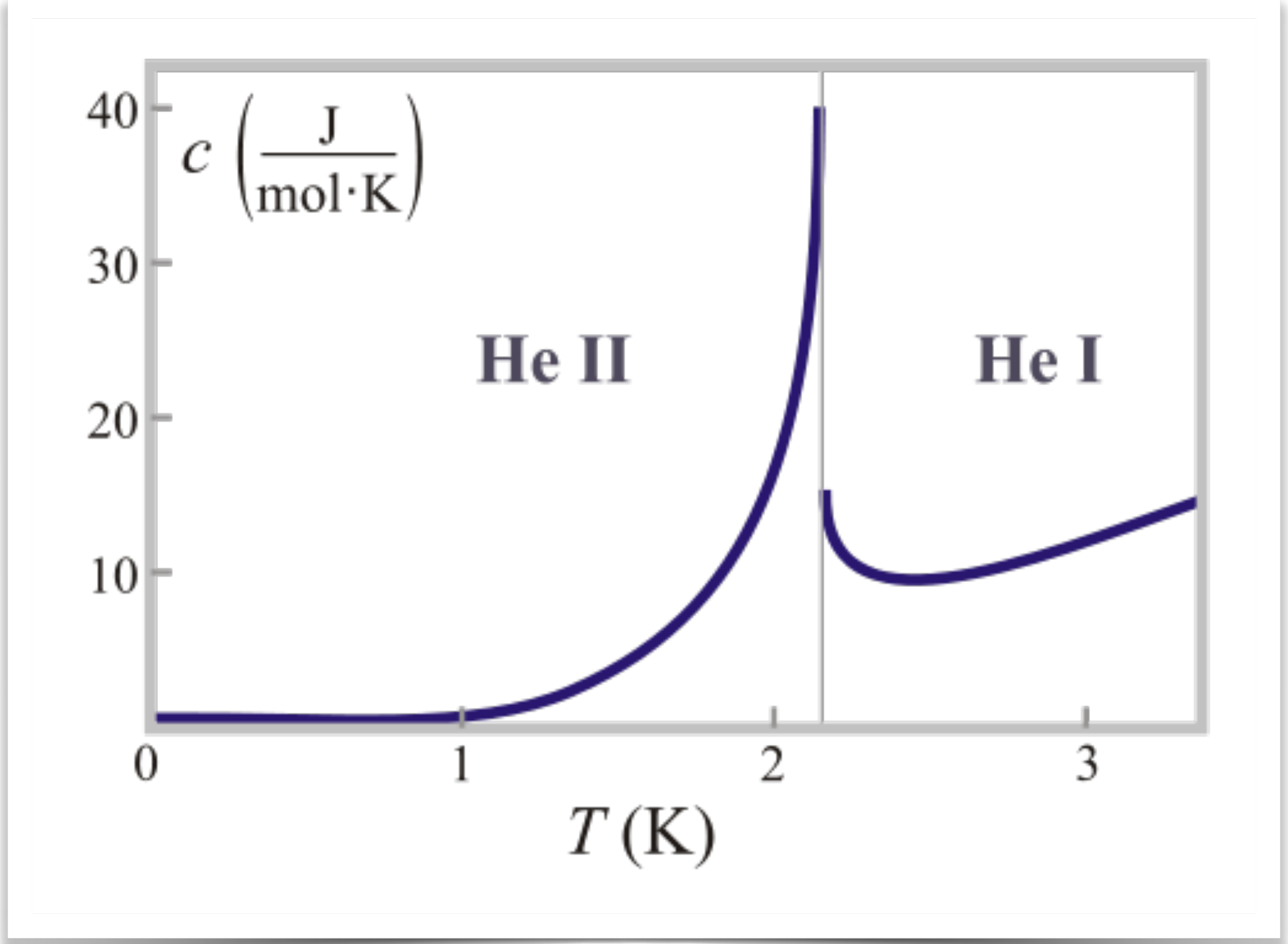
Critical phenomena and universality class

1822: discovered the critical point
of a substance
in his gun barrel experiments

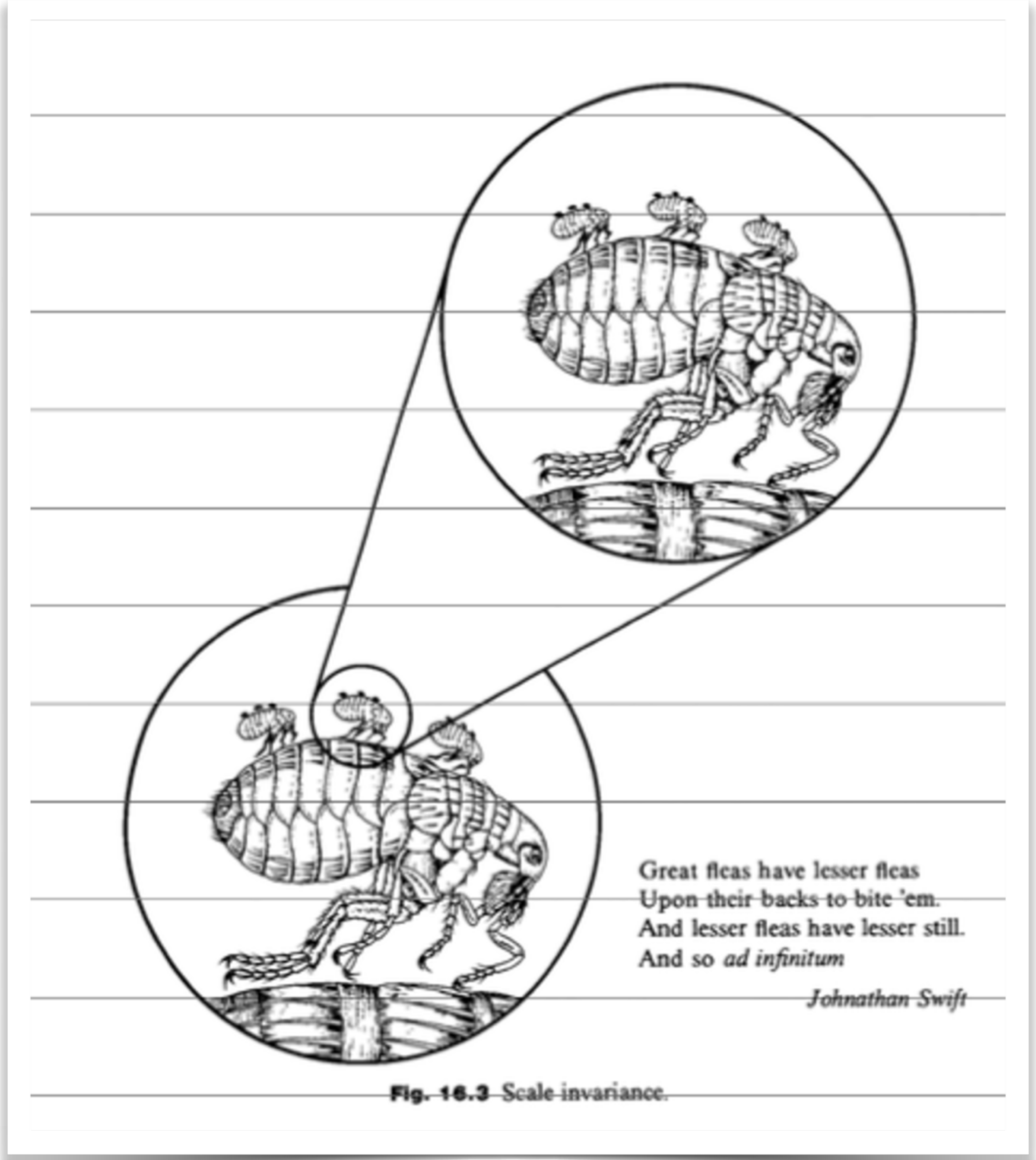


Charles Cagniard de la Tour
1777-1859

Superfluid transition: λ point
O(2) universality class



https://en.wikipedia.org/wiki/Lambda_point



Kerson Huang,
Statistical mechanics

Landau functional of QCD

Pisarski & Wilczek, PRD 84'

Symmetry: $SU(N_f)_L \times SU(N_f)_R \times U(1)_V \times U(1)_A$

Chiral field: $\Phi_{ij} \sim \frac{1}{2}\bar{q}^j(1-\gamma_5)q^i = \bar{q}^j_R q^i_L$ Chiral transformation: $\Phi \to e^{-2i\alpha_A} V_L \, \Phi \, V_R^\dagger$

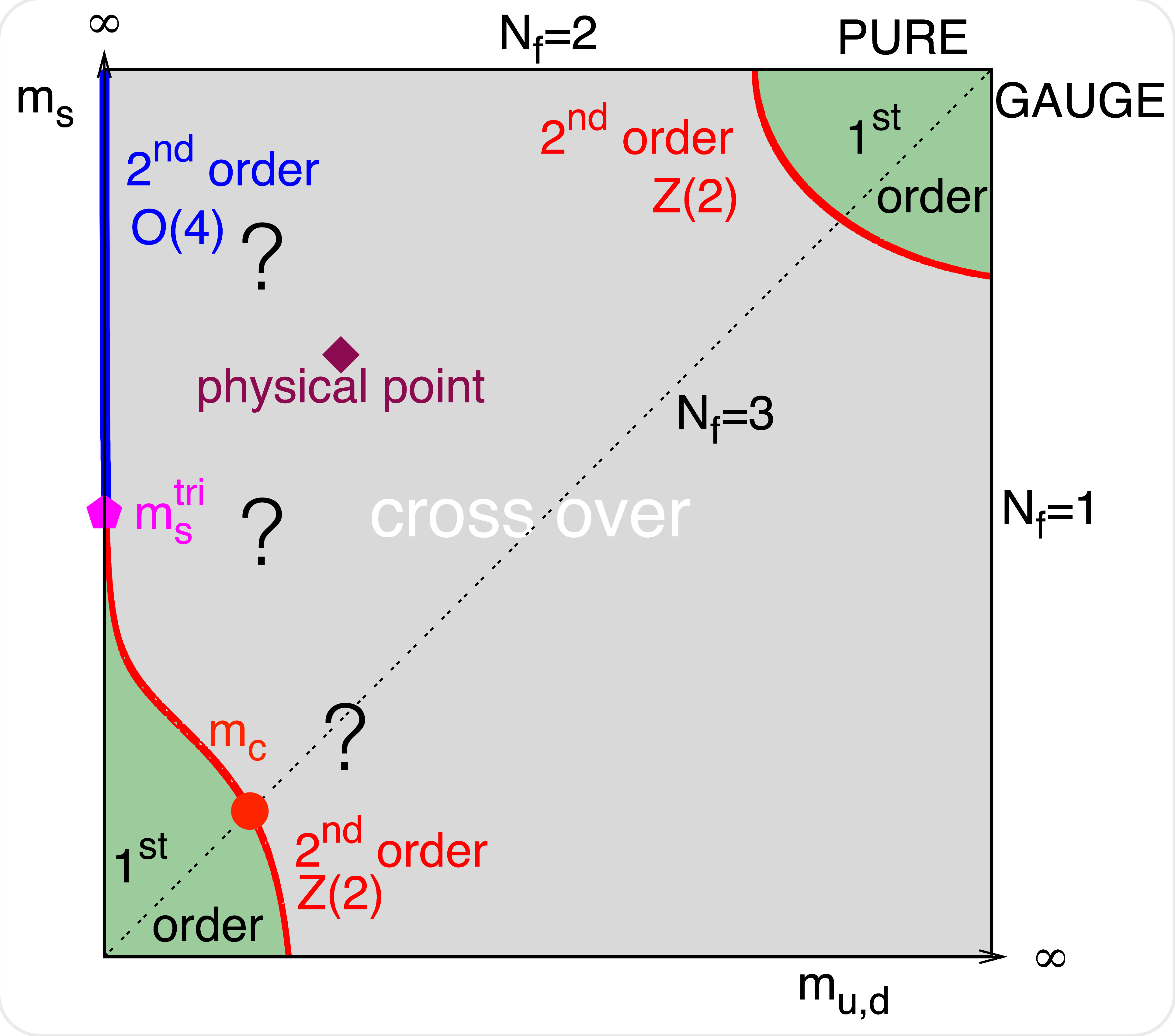
$$\mathcal{L}_{eff} = \frac{1}{2}\,\mathrm{tr}\,\partial\Phi^\dagger\partial\Phi + \frac{a}{2}\,\mathrm{tr}\,\Phi^\dagger\Phi + \frac{b_1}{4!}\left(\mathrm{tr}\,\Phi^\dagger\Phi\right)^2 + \frac{b_2}{4!}\,\mathrm{tr}\,\left(\Phi^\dagger\Phi\right)^2$$
→ $SU(N_f)_L \times SU(N_f)_R \times U(1)_A$

$$- \frac{c}{2}\left(\det\Phi + \det\Phi^\dagger\right)$$
→ $SU(N_f)_L \times SU(N_f)_R$

$$- \frac{d}{2}\mathrm{tr}\,h\left(\Phi + \Phi^\dagger\right).$$
→ Quark mass term

Nature of QCD phase transition

Columbia plot:
QCD phase diagram in quark mass plane



- At physical point $T_{pc} \approx 156$ MeV HotQCD, PLB 795 (2019) 15
WB, PRL125 (2020) 052001

- Chiral phase transition T_c =132(+3)(-6)MeV

HotQCD, PRL 123 (2019) 062002

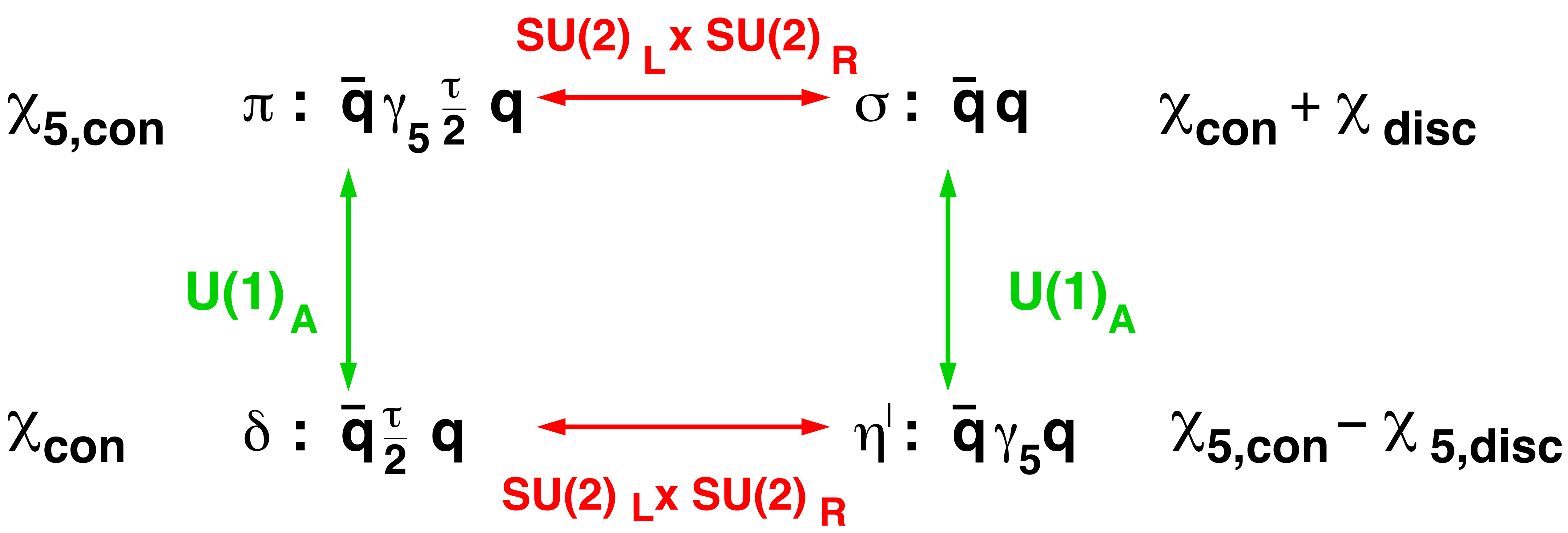
$U_A(1)$ symmetry:

Pisarski and Wilczek, PRD 29 (1984) 338
Butti, Pelissetto and Vicari, JHEP 08 (2003) 029
Pelissetto & Vicari, PRD 88 (2013) 105018
Grahl, PRD 90 (2014) 117904

- Broken, 2nd order (O(4)) phase transition
- Effectively restored, 1st or 2nd order ($U(2)_L \otimes U(2)_R / U(2)_V$)

Signatures of symmetry restorations

- Susceptibilities defined as integrated two point correlation functions of the local operators, e.g. $\chi_\pi = \int \mathrm{d}^4x \langle \pi^i(x)\pi^i(0)\rangle$ with $\pi^i(x) = i\bar{\psi}_l(x)\gamma_5\tau^i\psi_l(x)$



$$\chi_{\rm disc} = \frac{T}{V}\int \mathrm{d}^4x \left\langle \left[\bar{\psi}(x)\psi(x) - \langle\bar{\psi}(x)\psi(x)\rangle\right]^2 \right\rangle$$

Restoration of SU(2)$_L$xSU(2)$_R$:

$\chi_\pi - \chi_\sigma = 0$

$\chi_\delta - \chi_\eta = 0$

$\Rightarrow$ $\chi_\pi - \chi_\delta = \chi_{disc} = \chi_{5,disc}$

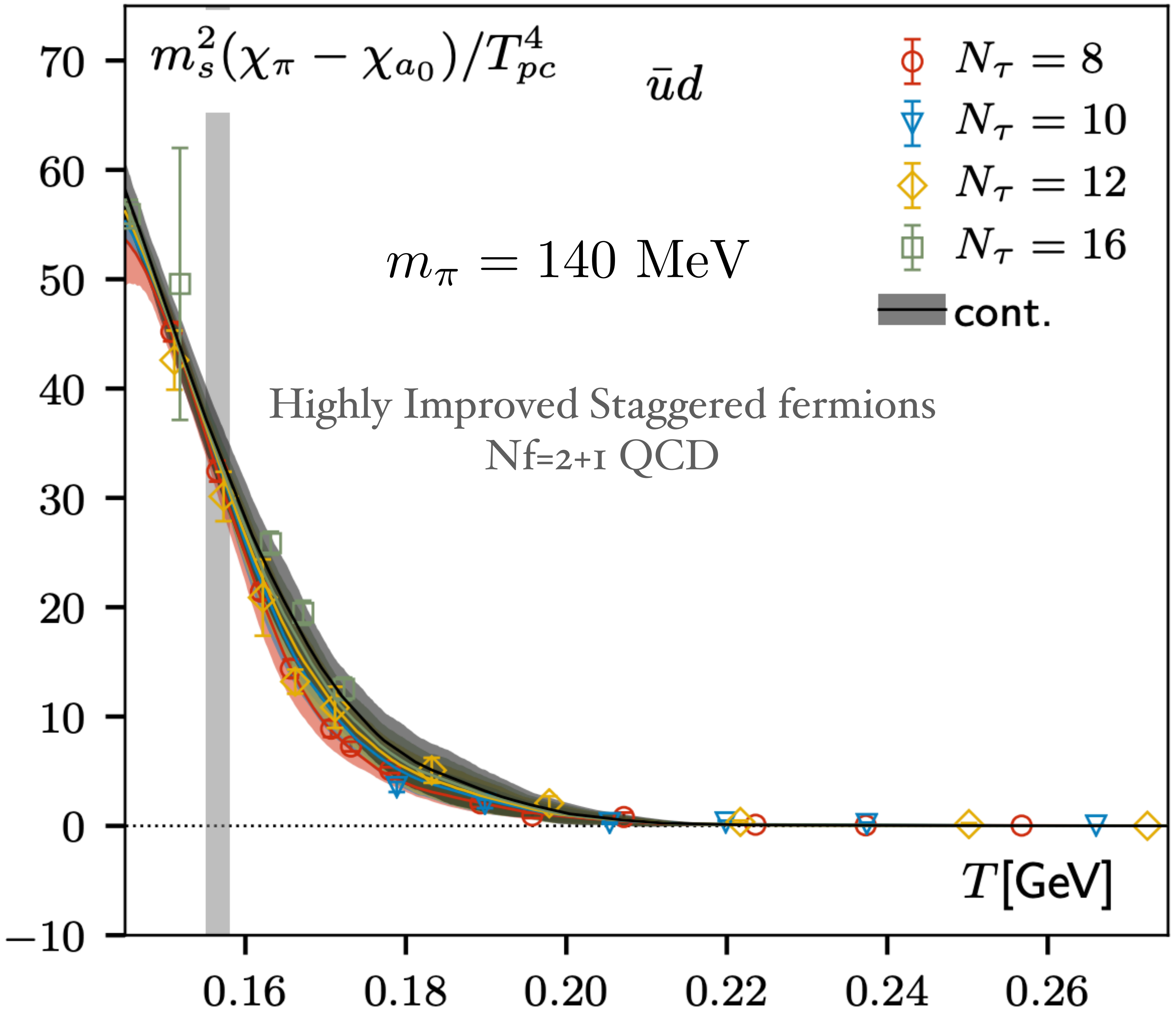
Effective restoration of U(1)$_A$:

$\chi_\pi - \chi_\delta = 0$

$\chi_\sigma - \chi_\eta = 0$

$\Rightarrow$ $\chi_\pi - \chi_\delta = \chi_{disc} = \chi_{5,disc} = 0$

Status of lattice studies on axial anomaly



HotQCD, Phys.Rev.D 100 (2019) 094510

See also in O. Kaczmarek, R. Shanker, S. Sharma, arXiv: 2301.11610

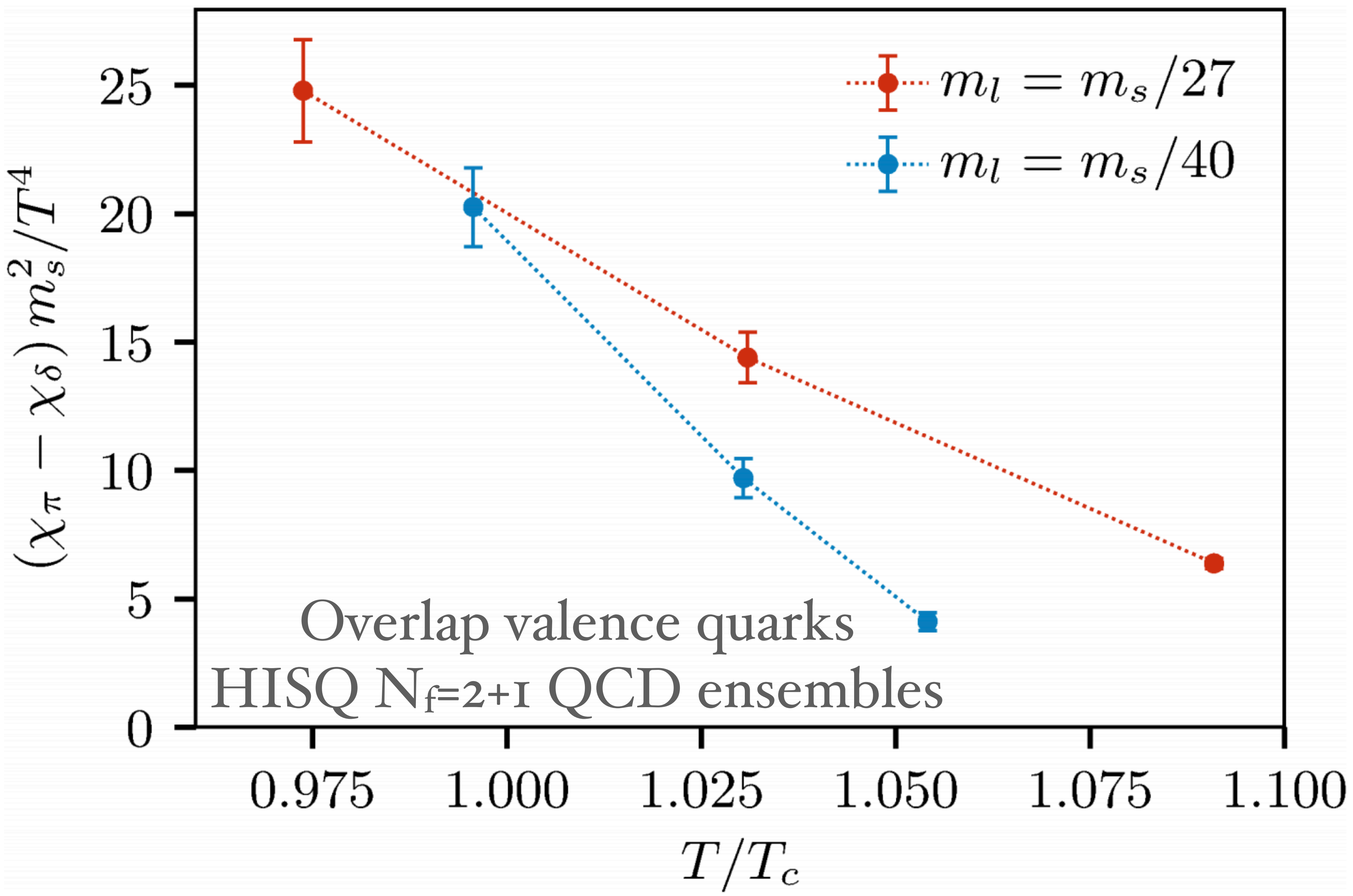
- ☑ At physical pion mass at $T \lesssim T_{pc}$ axial anomaly remains manifested in $\chi_\pi - \chi_\delta$

See similar conclusions obtained using chiral fermions:
HotQCD, PRL 113(2014)082001, PRD 89 (2014)054514
JLQCD, arXiv: 2011.01499, ...

- ☐ How about the case in the chiral limit ?

Axial anomaly towards chiral limit

Nt=8, lattice spacing a≈0.15 fm

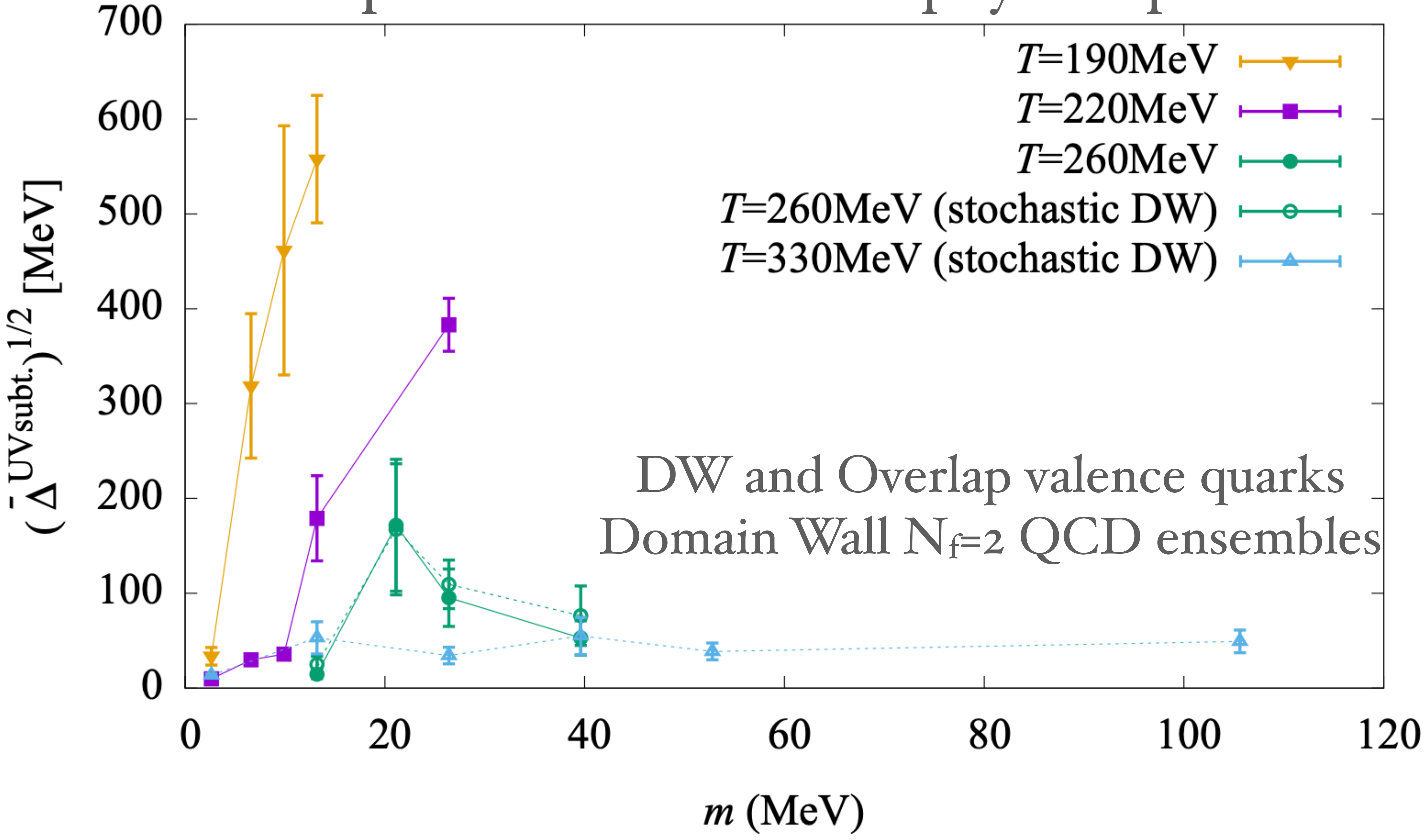


Sharma, Lattice 2018 Review talk, 1901.07190

Remains manifested at m_π=110 MeV and T<1.1T_c

Similar conclusions from
Dick et al., PRD 91(2015)094504, Ohno et al., PoS Lattice 2012(2012)095, Mazur et al., 1811.08222,...

Fixed scale approach, lattice spacing a=0.074 fm
One quark mass below the physical point



JLQCD, arXiv: 2011.01499

Seems to disappear at T ≳ 220 MeV

Similar conclusions from
Chiu et al.,PoS Lattice 2013 (2014)165,
Tomiya et al.,[JLQCD] PRD 96(2017)079902,
Brandt et al., JHEP 12 (2016) 158,..

Axial anomaly towards chiral limit

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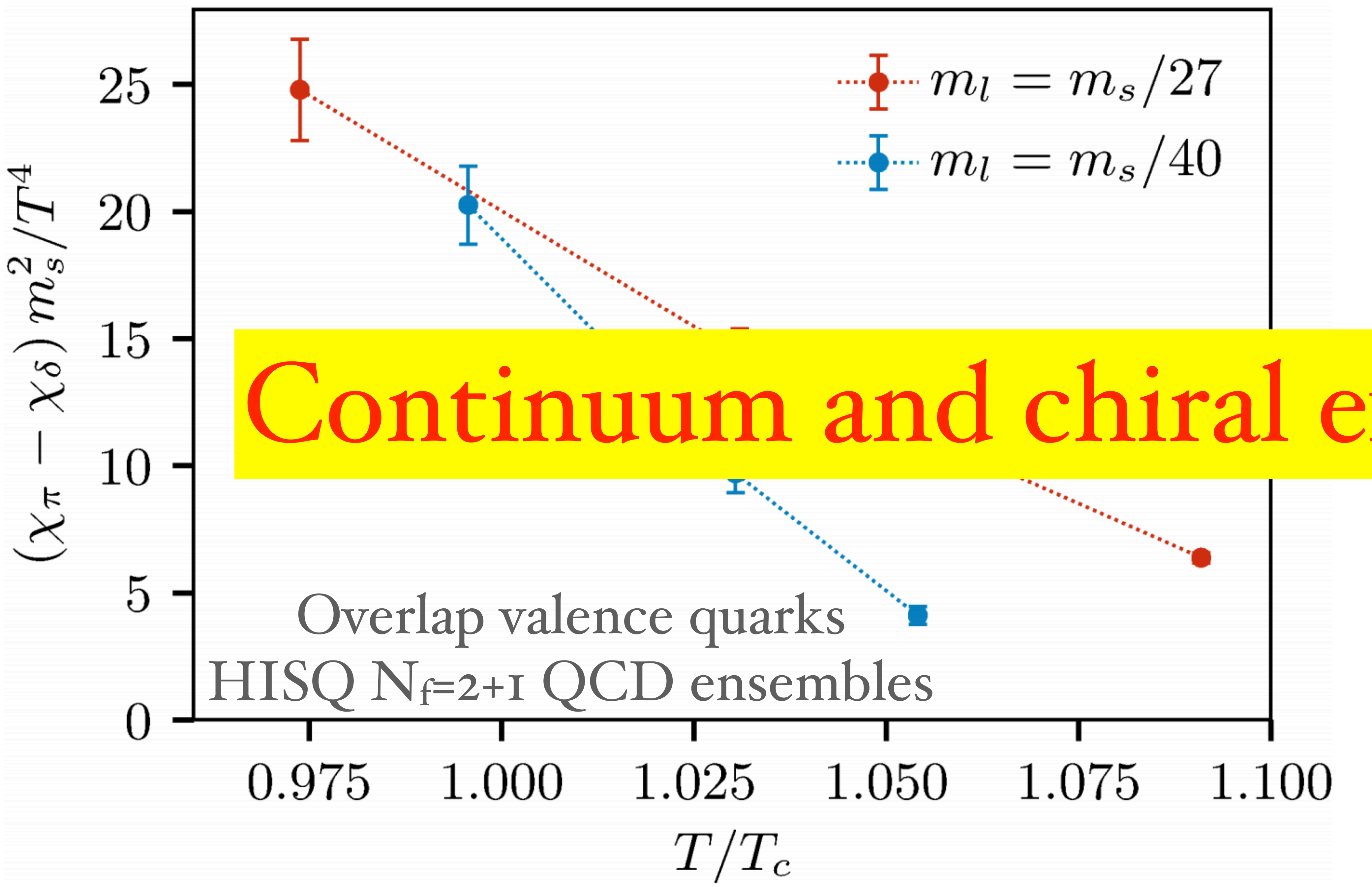

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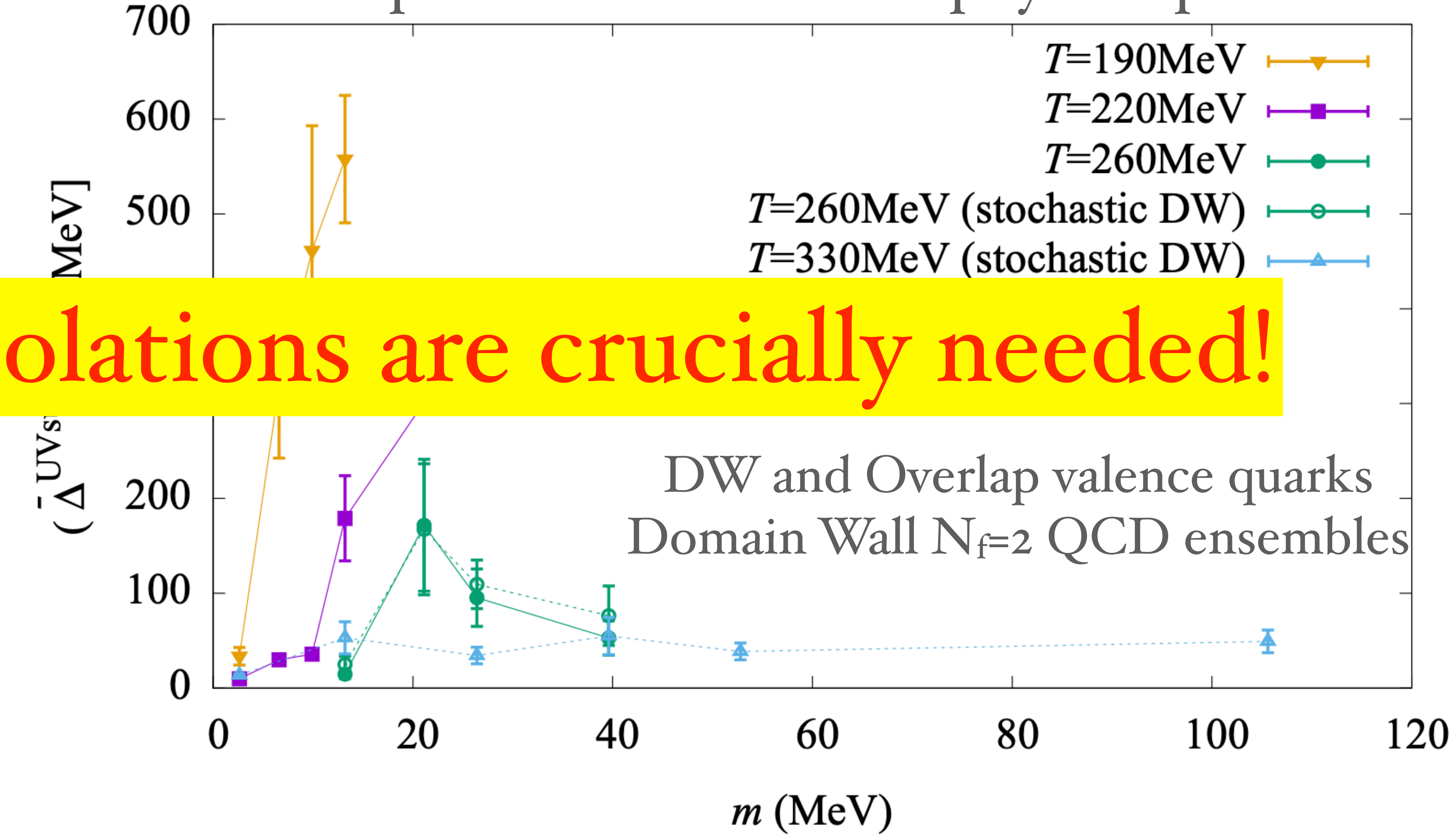

Continuum and chiral extrapolations are crucially needed!

JLQCD, arXiv: 2011.01499

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Signature of restorations in Dirac Eigenvalue Spectrum

$$\langle\bar{\psi}\psi\rangle = \int_0^\infty \frac{4m_l\,\rho}{\lambda^2+m_l^2}\,\mathrm{d}\lambda\ , \qquad \chi_\pi - \chi_\delta = \int_0^\infty \mathrm{d}\lambda\,\frac{8m_l^2\,\rho}{\left(\lambda^2+m_l^2\right)^2}$$

- Restoration of $SU(2)_L x SU(2)_R$ symmetry
  - $\rho(0)=0$ as from Banks-Casher formula: $\lim_{m_l\to 0}\langle\bar{\psi}\psi\rangle = \pi\rho(0)$ Banks and Casher, NPB 169 (1980) 103
  - Partition function is an even function in quark mass due to the Z(2) subgroup
- Absence of manifestation of $U(1)_A$ symmetry in χ_π-χ_δ
  - a sizable gap from zero, i.e. $\rho(\lambda<\lambda_c)=0$ Cohen, arXiv:nucl-th/9801061

⚠ if $\rho(\lambda)$ is analytic in m^2, NOT be manifested in differences of up to 6 point correlation functions Aoki, Fukaya and Taniguchi, PRD86 (2012) 114512

Possible behavior of $\rho(\lambda)$ making SU(2)xSU(2) restored but NOT $U(1)_A$

$$\rho(\lambda,m) = c_0 + c_1\lambda + c_2 m^2 \delta(\lambda) + c_3 m + c_4 m^2 + O(\lambda,m)$$

$$\langle\bar\psi\psi\rangle = 2c_0\pi - 4c_1 m\ln(m) + 2c_2 m + 2\pi c_3 + 2\pi c_4 m^2$$

$$\chi_\pi - \chi_\delta = 2c_0\pi/m + 4c_1 + 4c_2 + 2\pi c_3 + 2\pi c_4 m$$

Ansatz	$\langle\bar\psi\psi\rangle$	χ_π	χ_δ	$\chi_\pi - \chi_\delta$	χ_{disc}
c	$2c\pi$	$2c\pi/m$	0	$2c\pi/m$	0
λ	$-4m\ln(m)$	$-4\ln(m)$	$-4\ln(m)$	4	0
$m^2\delta(\lambda)$	$2m$	2	-2	4	4
m	$2\pi m$	2π	0	2π	2π
m^2	$2\pi m^2$	$2\pi m$	0	$2\pi m$	$2\pi m$

HotQCD, PRD86(2012)094503

c_0 & c_1 terms: break both symmetries Smilga & Stern, PLB 93'

c_2: near zero mode contribution Gross, Yaffe & Pisarski, RMP 81'

c_3: another $U(1)_A$ breaking term

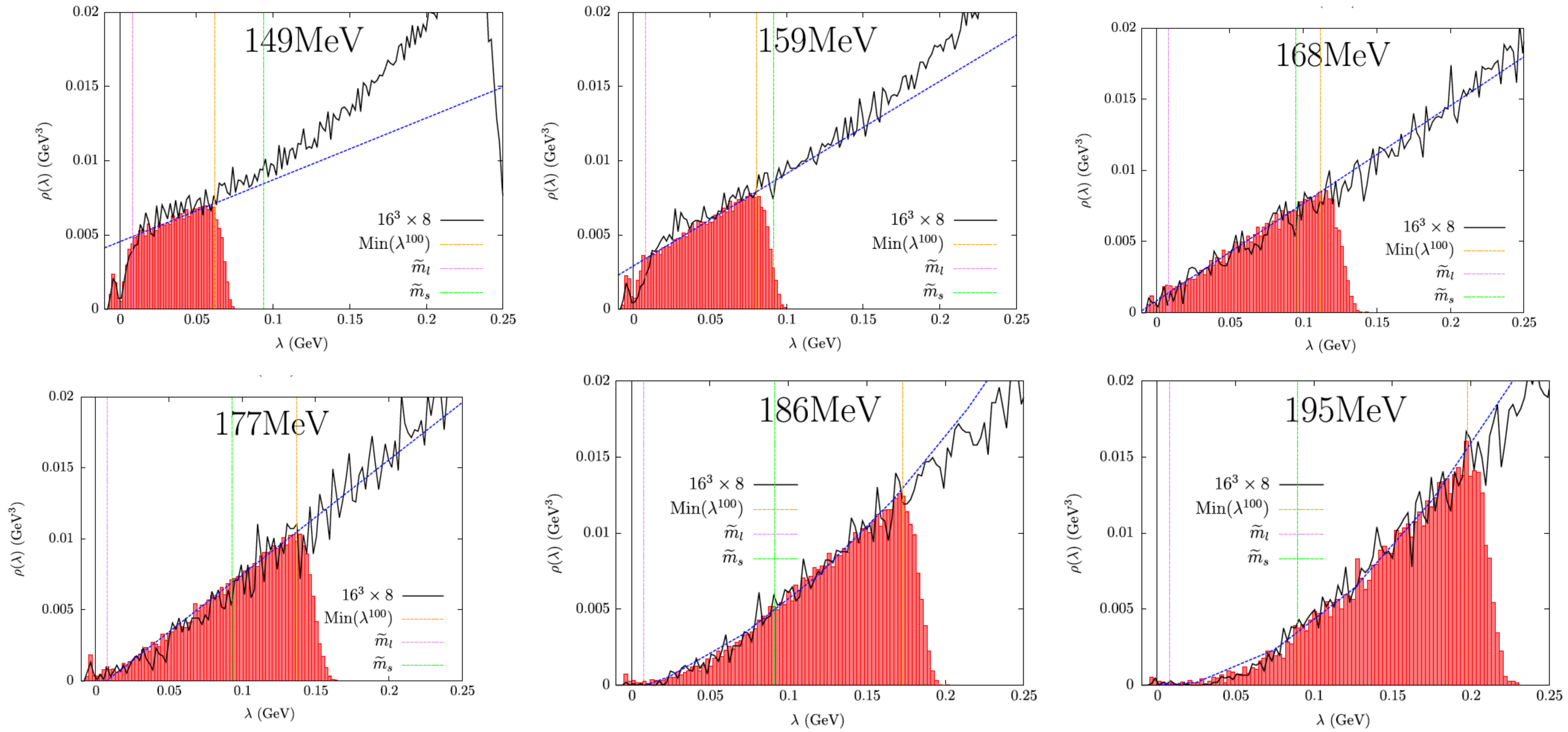
c_4: Not manifested in 2-pt correlators Aoki, Fukaya & Taniguchi, PRD12'

- LQCD: At high T for physical *m*, the T dependence of χ_t follows dilute instanton gas approximation prediction See recent review: Lombardo & Trunin, IJMPA 35(2020)2030010

Due to $\rho(\lambda,m) \propto m^2 \delta(\lambda)$? What happens for $m \to 0$?

λ behavior in ρ

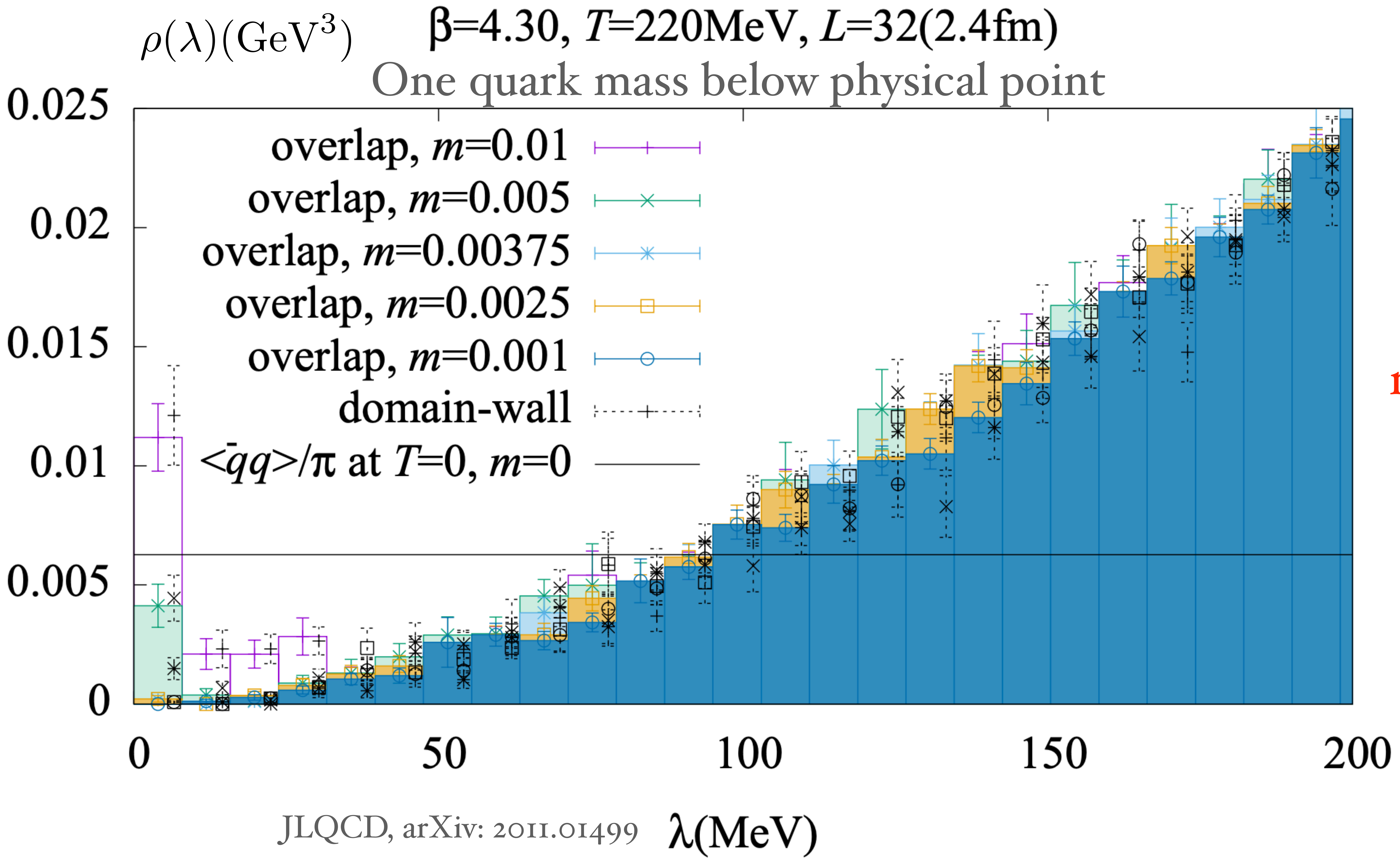
LQCD simulations of N_f=2+1 QCD using Domain Wall fermions, m_π=200 MeV



The m_l^2 dependence is not demonstrated as $m_l \to 0$

HotQCD, PRD 89 (2014)054514

No infrared enhancement in ρ



JLQCD, arXiv: 2011.01499

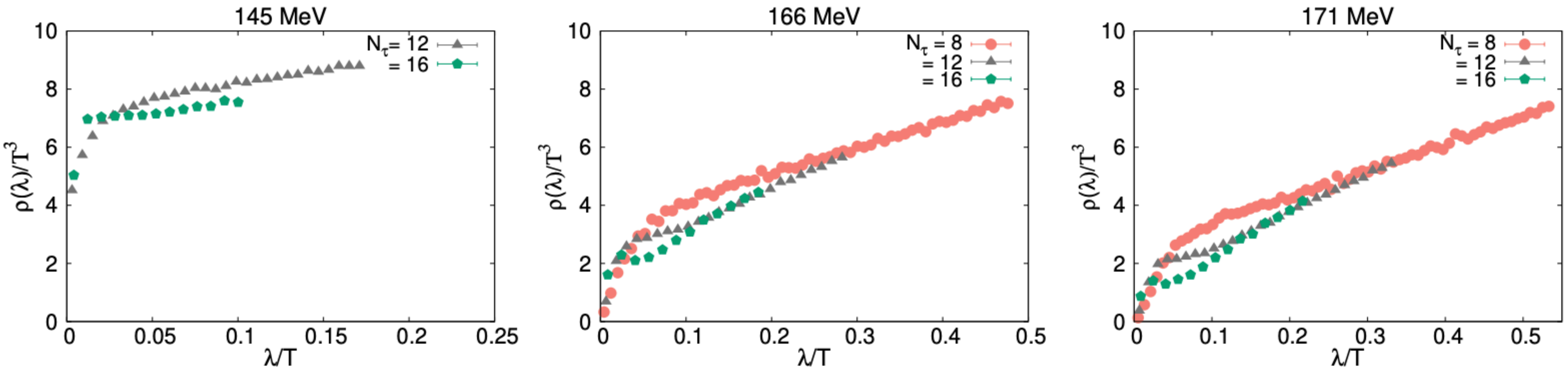
No clear gap

At m<0.01 and λ>0, m dependence can be hardly seen

Continuum extrapolation is important

Infrared behavior of ρ

LQCD simulations of N_f=2+1 QCD using HISQ, m_π=140 MeV



Raiv Shanker, talk in this workshop

So far…

- Mass dependence of ϱ ?

- Continuum and chiral extrapolations ?

Novel relation: Light quark mass derivative of ρ and C_n

$$\rho(\lambda, m_l) = \frac{T}{VZ[\mathcal{U}]}\int \mathcal{D}[\mathcal{U}]e^{-S_G[\mathcal{U}]}\det\left[\not{D}[\mathcal{U}]+m_s\right]\left(\det\left[\not{D}[\mathcal{U}]+m_l\right]\right)^2\rho_U(\lambda)$$

Partition function $$Z[\mathcal{U}] = \int \mathcal{D}[\mathcal{U}]e^{-S_G[\mathcal{U}]}\det\left[\not{D}[\mathcal{U}]+m_s\right]\left(\det\left[\not{D}[\mathcal{U}]+m_l\right]\right)^2$$

Eigenvalue spectrum per ensemble $$\rho_U(\lambda) = \sum_j \delta(\lambda-\lambda_j)$$

Quark mass dependence of $\boldsymbol{\rho}$ is enclosed in

$$\det\left[\not{D}[\mathcal{U}]+m_l\right] = \prod_j(+\mathrm{i}\,\lambda_j+m_l)(-\mathrm{i}\,\lambda_j+m_l) = \exp\left(\int_0^\infty \mathrm{d}\lambda\,\rho_U(\lambda)\ln\left[\lambda^2+m_l^2\right]\right)$$

$$\Rightarrow \quad \frac{V}{T}\frac{\partial\rho}{\partial m_l} = \int_0^\infty \mathrm{d}\lambda_2\,\frac{4m_l C_2}{\lambda_2^2+m_l^2}$$

$$C_2(\lambda,\lambda_2) = \left\langle\rho_U(\lambda)\rho_U(\lambda_2)\right\rangle - \left\langle\rho_U(\lambda)\right\rangle\left\langle\rho_U(\lambda_2)\right\rangle$$

Relation between ρ derivatives and C_{n+1}

$$\frac{V}{T}\frac{\partial^2\rho}{\partial m_l^2} = \int_0^\infty \mathrm{d}\lambda_2\, \frac{4(\lambda_2^2 - m_l^2)\, C_2}{\left(\lambda_2^2 + m_l^2\right)^2} + \int_0^\infty \mathrm{d}\lambda_2\, \mathrm{d}\lambda_3\, \frac{(4m_l)^2\, C_3}{(\lambda_2^2 + m_l^2)(\lambda_3^2 + m_l^2)}$$

••• •••

••• •••

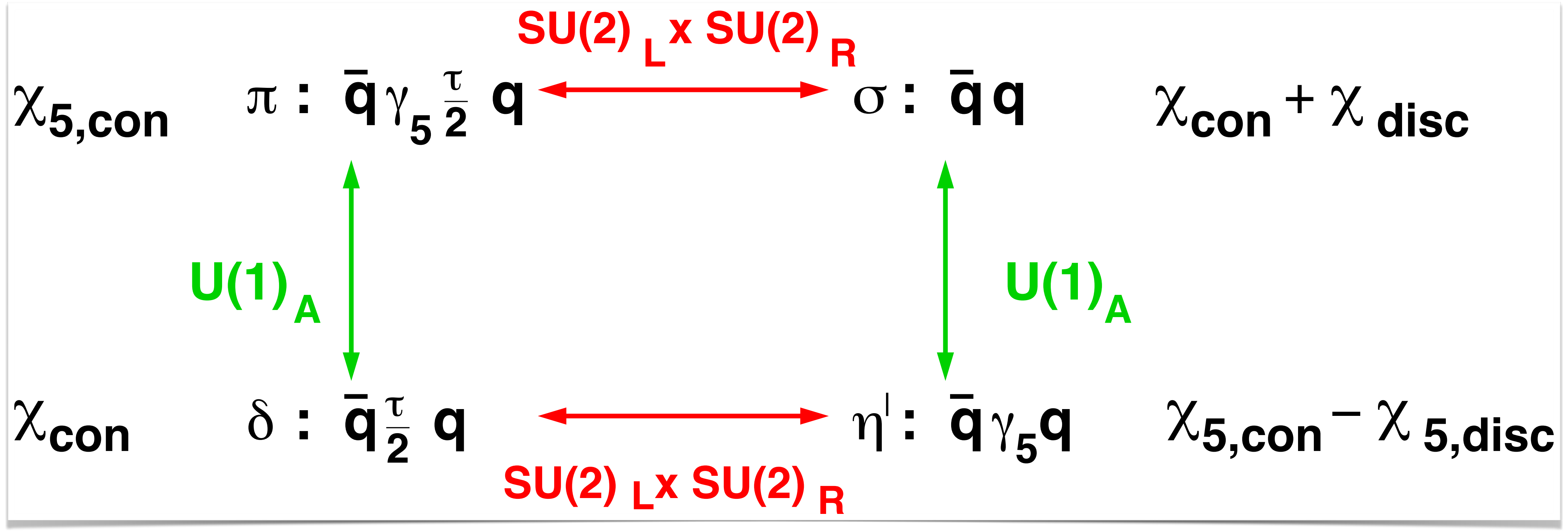
$$C_n(\lambda_1, \cdots, \lambda_n; m_l) = \left\langle \prod_{i=1}^{n} \left[\rho_U(\lambda_i) - \langle \rho_U(\lambda_i) \rangle\right] \right\rangle$$

Signatures of symmetry restorations

- Chiral symmetry restoration: $\chi_\pi - \chi_\delta = \chi_{disc}$

$$\chi_\pi - \chi_\delta = \int_0^\infty d\lambda \frac{8m_l^2\,\rho}{(\lambda^2 + m_l^2)^2} \,,$$

$$\chi_{\rm disc} = \int_0^\infty d\lambda \frac{4m_l \; \partial\rho/\partial m_l}{\lambda^2 + m_l^2}$$



Toublan and Verbaarschot, NPB603 (2001) 343

HotQCD, PRD90 (2014) 094503

Kanazawa & Yamamoto, JHEP 01(2016)141

- If eigenvalues are uncorrelated $C_n^{\rm Po}(\lambda_1, \cdots, \lambda_n) = \delta(\lambda_1 - \lambda_2)\cdots\delta(\lambda_n - \lambda_{n-1})\,\langle\rho_U(\lambda_1)\rangle + \mathcal{O}(1/N)$

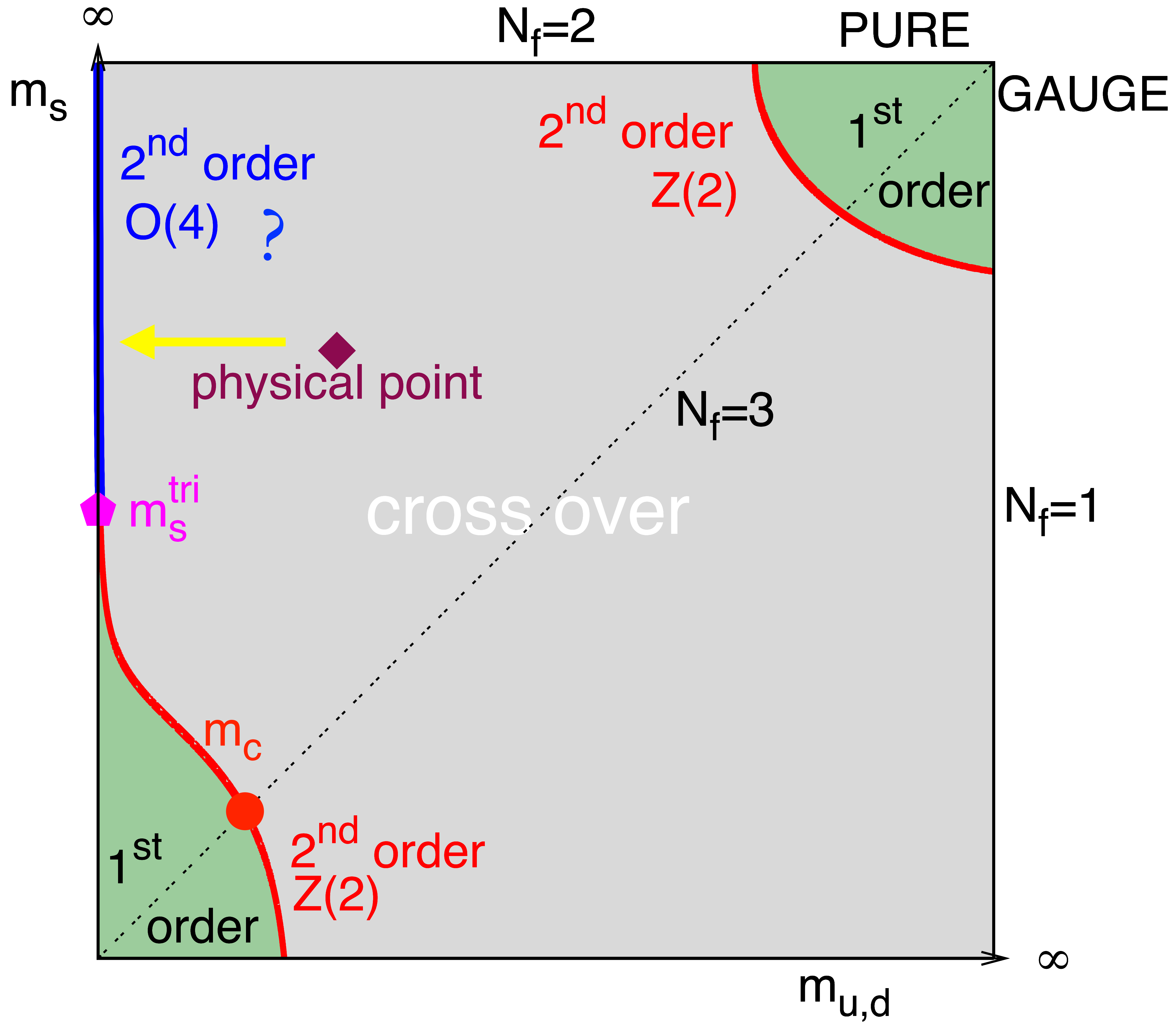
$$\left(\frac{\partial\rho}{\partial m_l}\right)^{\rm Po} = \frac{4m_l\rho}{\lambda^2 + m_l^2} - \frac{V\rho}{TN}\langle\bar{\psi}\psi\rangle \;\Rightarrow\; \chi_{\rm disc}^{\rm Po} = 2(\chi_\pi - \chi_\delta)$$

Non-Poisson correlation among eigenvalues:
needed for chiral symmetry restoration if $\chi_\pi - \chi_\delta \neq 0$

Kanazawa & Yamamoto, JHEP 01(2016)141

Lattice setup

4 PFlops local GPU machine

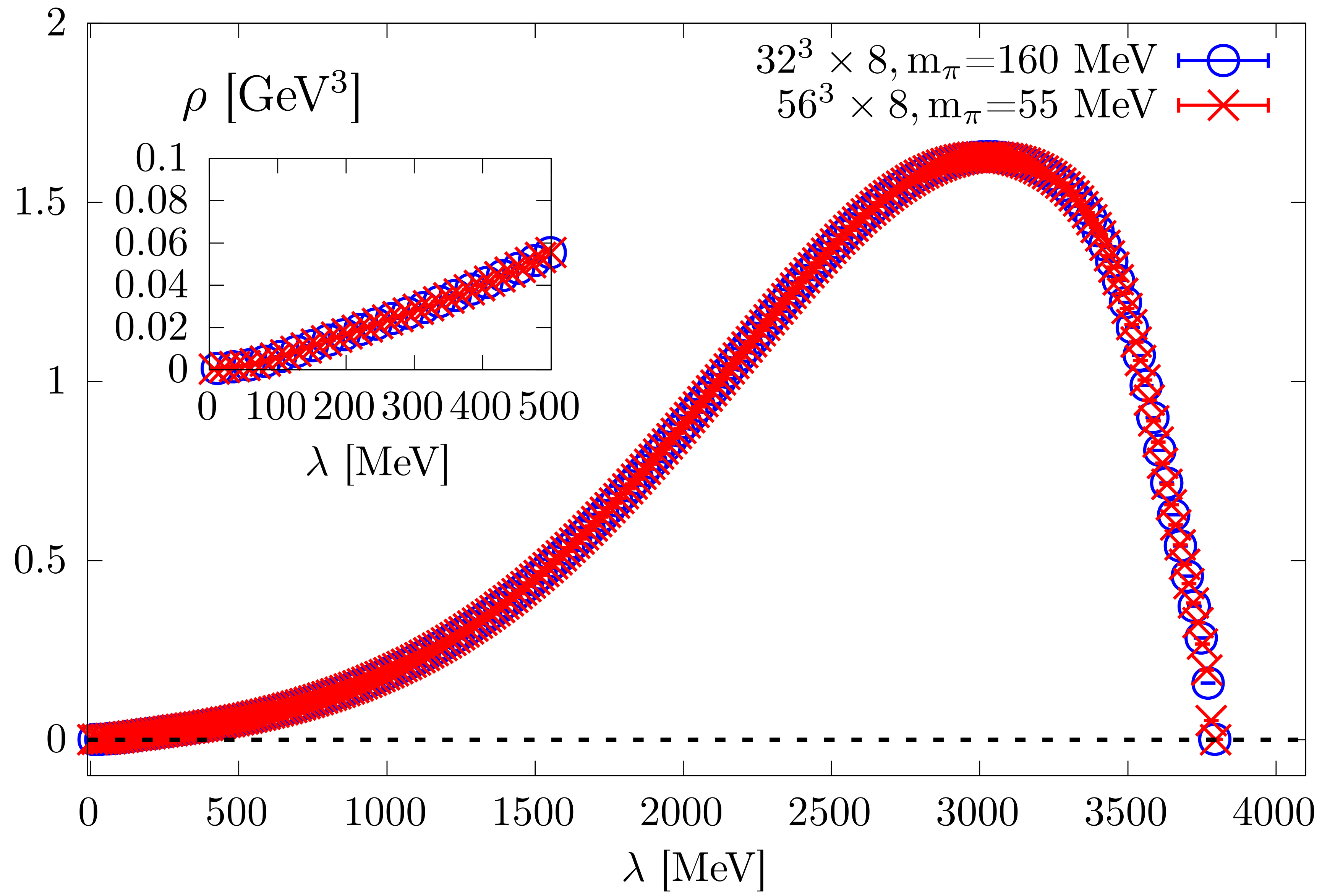


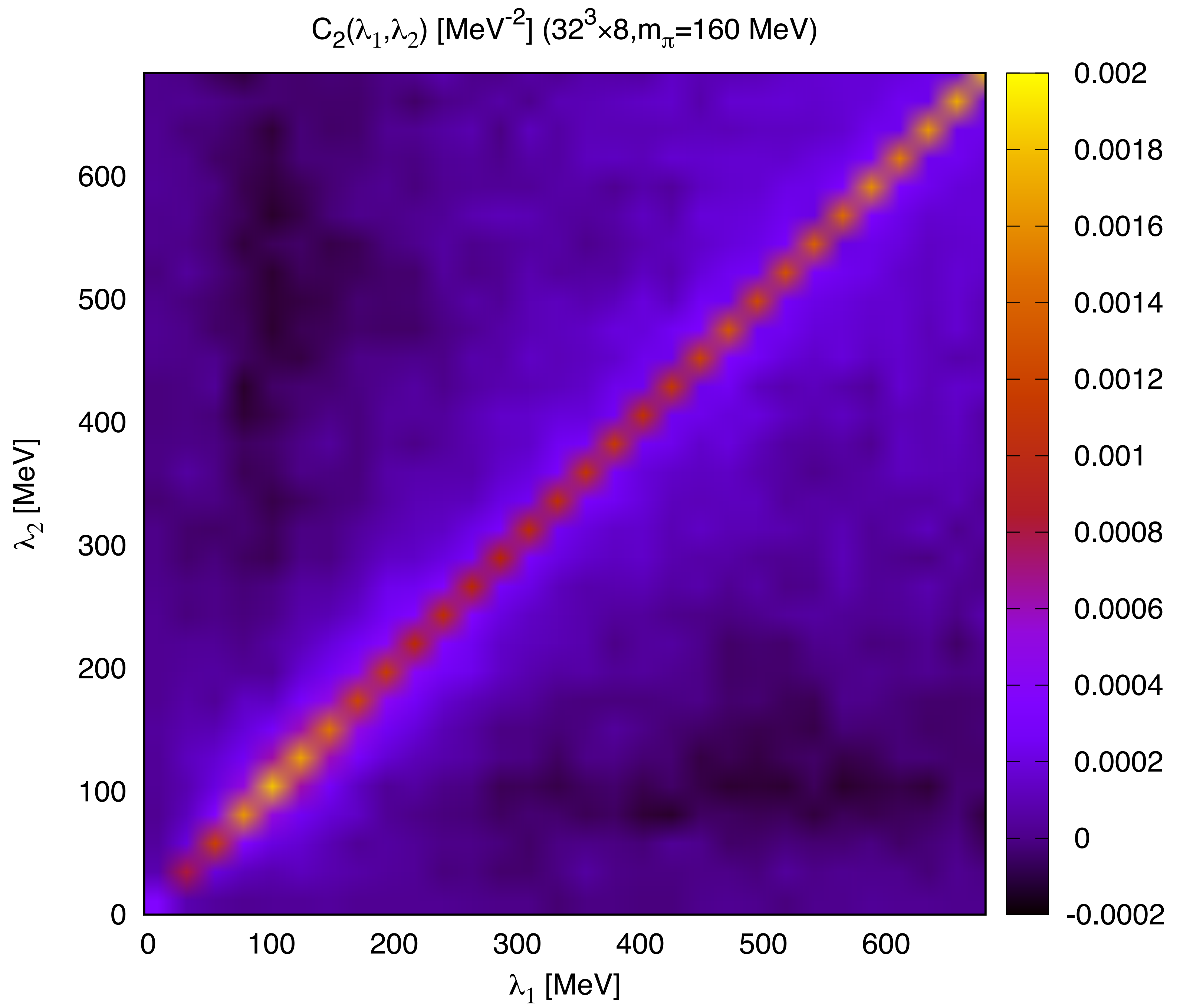
HTD, S.-T. Li, A. Tomiya, S. Mukherjee, X.-D. Wang, Y. Zhang
Phys. Rev. Lett. 126 (2021) 082001

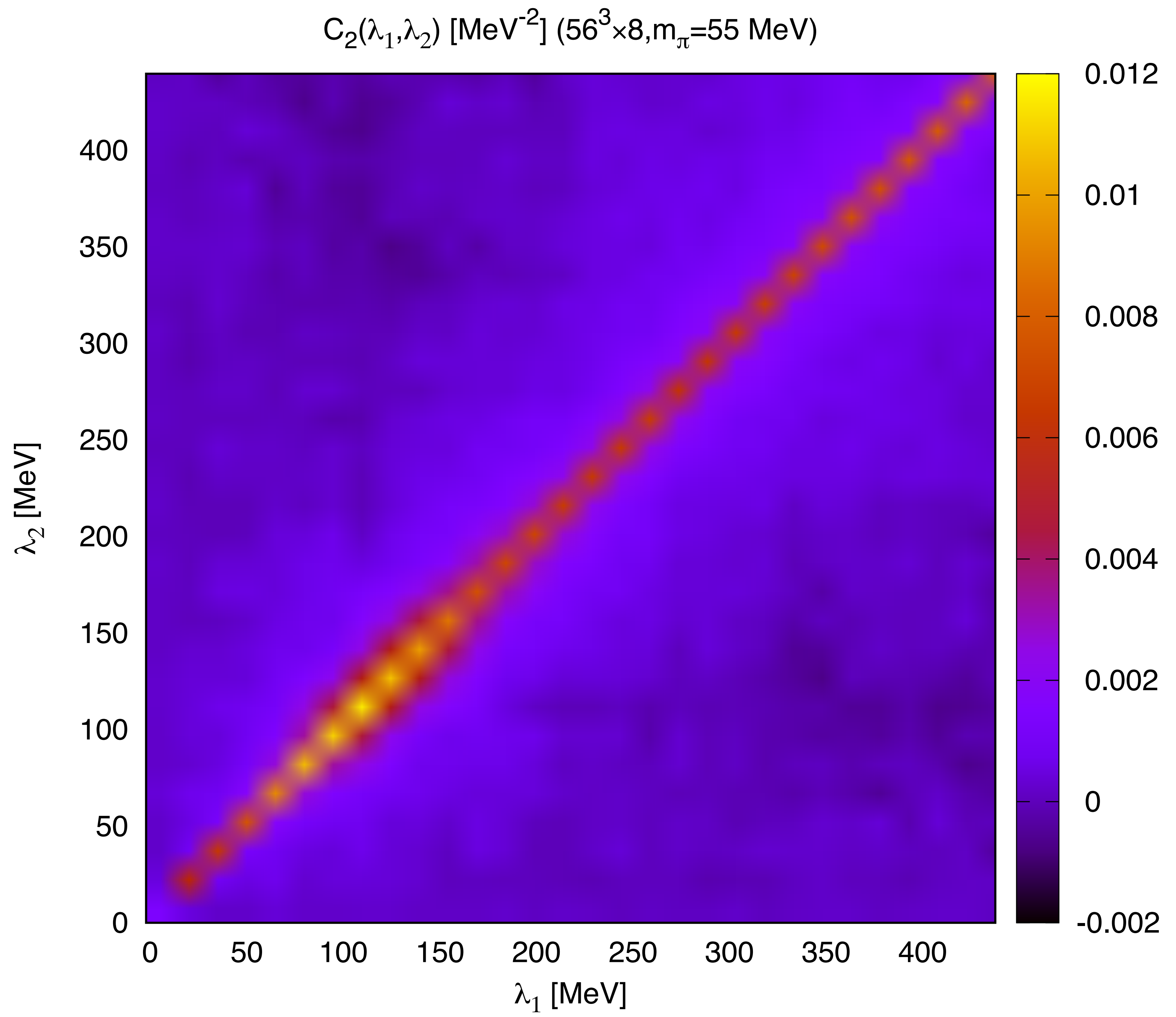
- At a single T~205 MeV
- HISQ/tree action
- N_f=2+1:
  - ☑ Nt=8,12,16 (a=0.12,0.08,0.06 fm)
  - ☑ $m_s^{phy}/\mathbf{m_l}$ = 20, 27, 40, 80, 160

    $m_\pi \approx$160,140,110, 80, 55 MeV
  - ☑ $9 \geq N_s/N_t \geq 4$

Complete eigenvalue spectrum ρ and C_2 at T=205 MeV





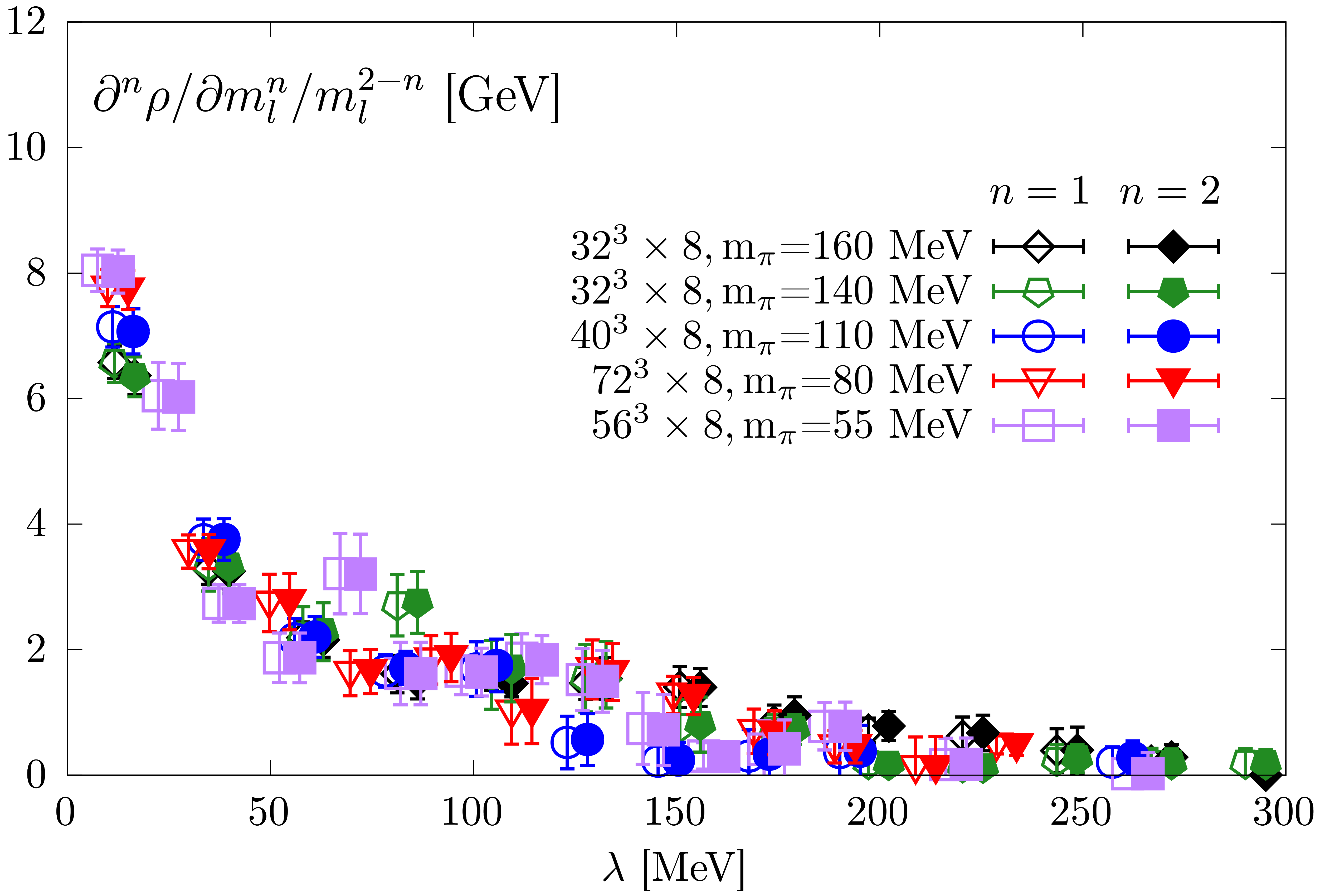


HTD, S.-T. Li, S. Mukherjee, A. Tomiya, X.-D. Wang, Y. Zhang, Phys. Rev. Lett. 126 (2021) 082001

via Chebyshev Polynomial filtering technique

Giusti and Luscher, JHEP03(2009)013, Patella PRD86(2012)025006, Cossu et al., PTEP 2016(2016)093B06
Itou & Tomiya, arXiv:1411.1155, Fodor et al., arXiv:1605.08091, de Forcrand & Jäger, arXiv: 1710.07305, HTD et al., arXiv:2001.05217,2008.00493

1st & 2nd mass derivative of ρ on N_τ=8 lattices



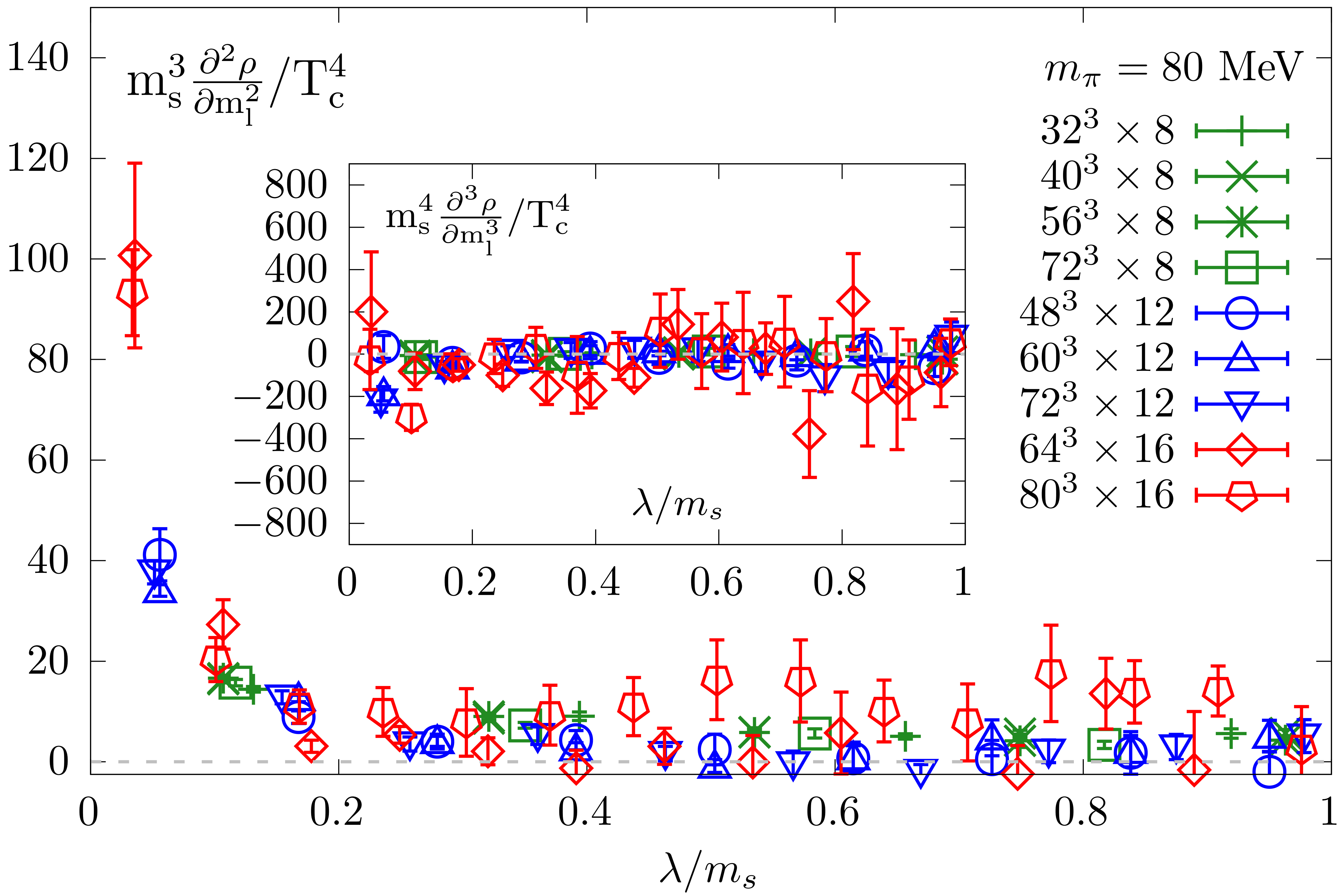
$$m_l^{-1}\partial\rho/\partial m_l \approx \partial^2\rho/\partial m_l^2$$

Quark mass independent

Peaked structure developed in the small λ region

Drops rapidly towards zero for λ/T >1

2nd & 3rd mass derivative of ρ: volume and a dependences



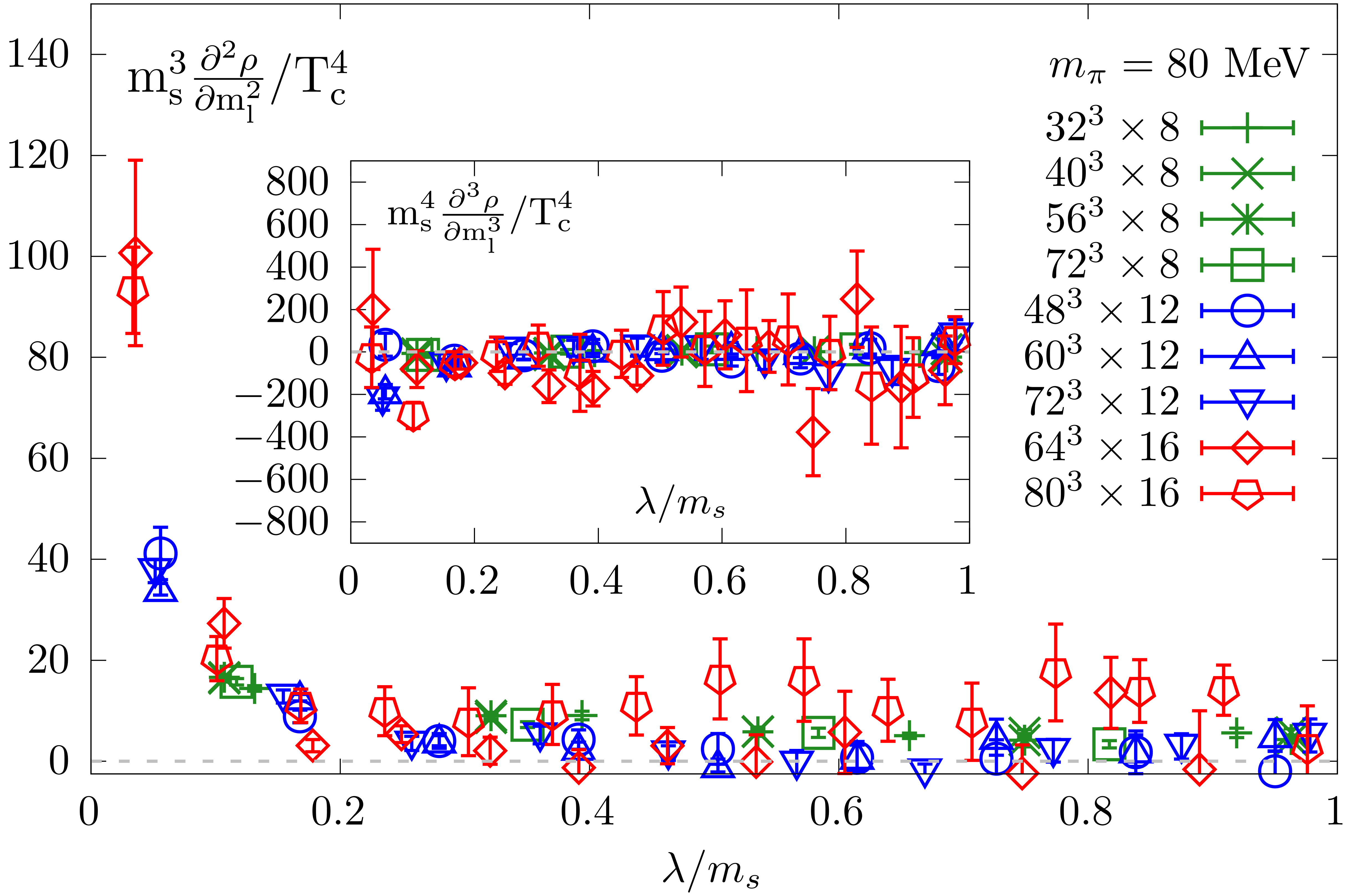
T_c=132 MeV is used from
HTD et al, [HotQCD] PRL 19'

Peaked structure becomes sharper towards continuum limit

Mild volume dependence

$$\partial^3\rho/\partial m_l^3 \approx 0$$

2nd & 3rd mass derivative of ρ: volume and a dependences



T_c=132 MeV is used from
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Peaked structure becomes sharper towards continuum limit

Mild volume dependence

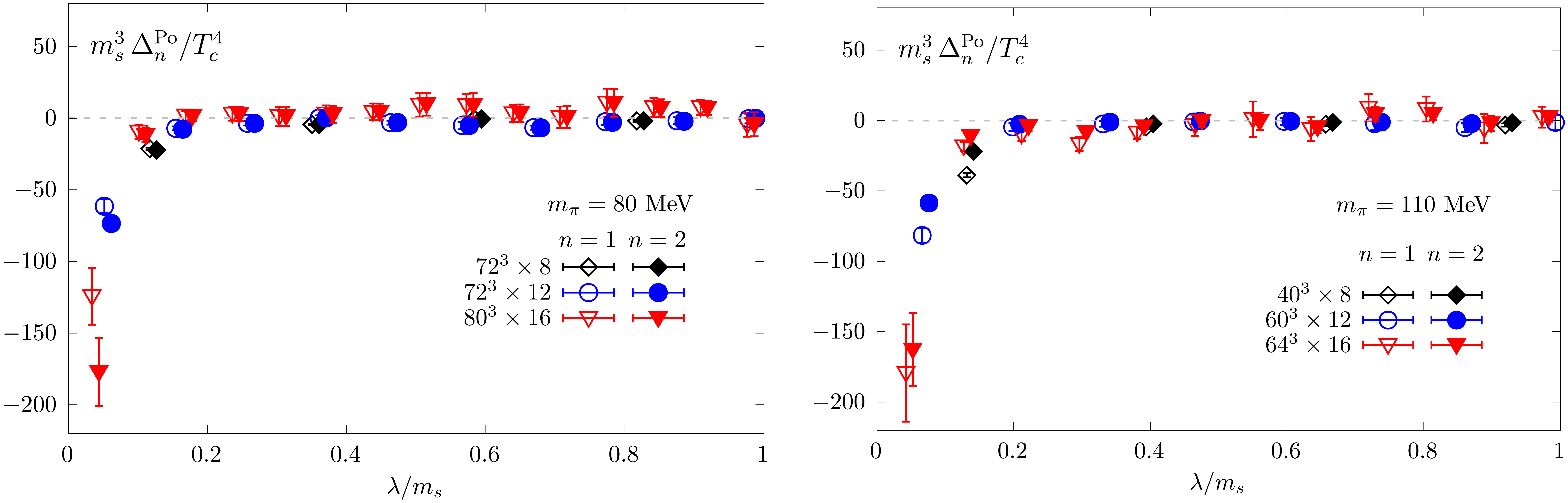
$$\partial^3\rho/\partial m_l^3 \approx 0$$

$$m_l^{-1}\partial\rho/\partial m_l \approx \partial^2\rho/\partial m_l^2$$

$$\rho(\lambda \to 0, m_l \to 0) \propto m_l^2$$

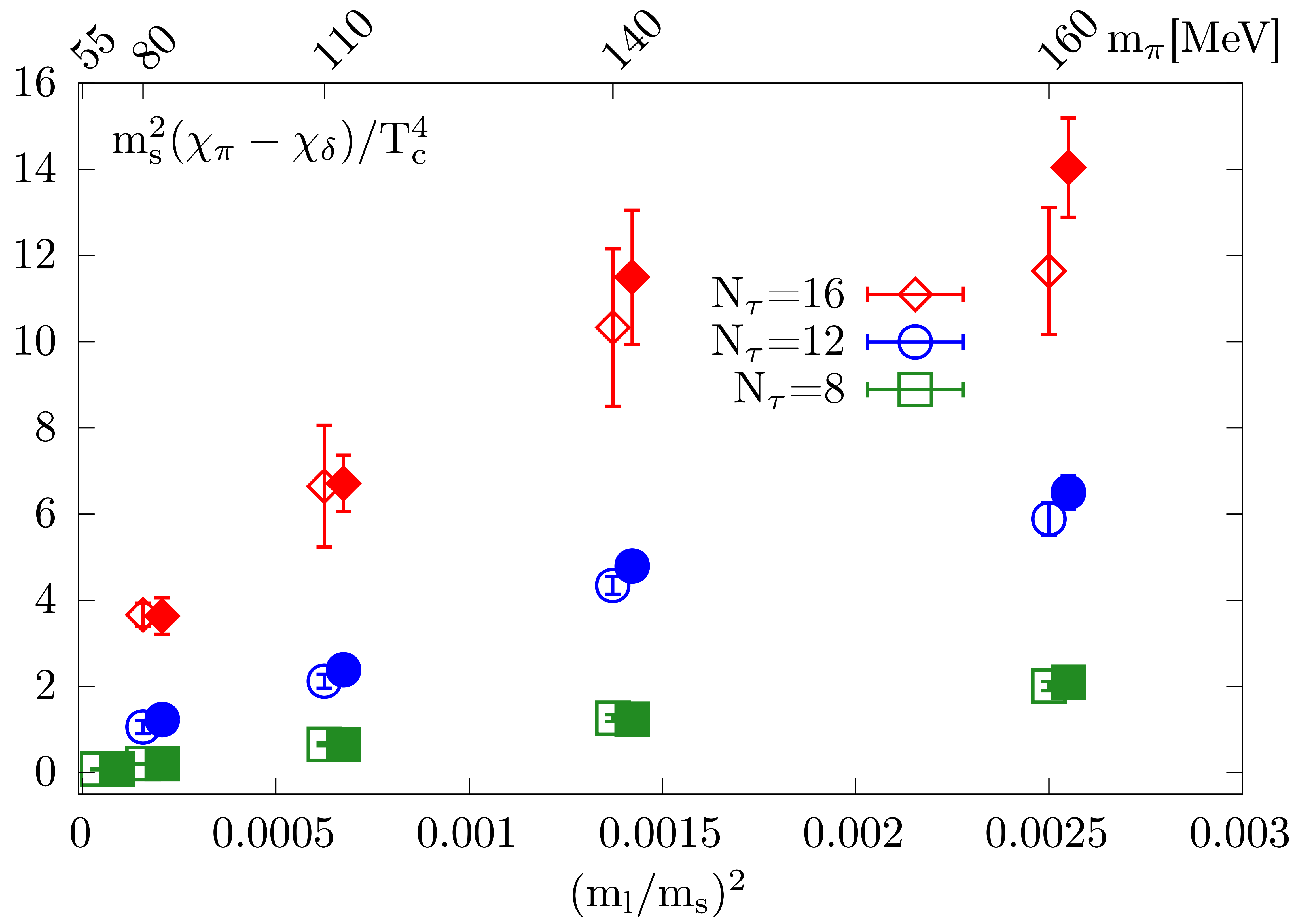
Non-Poisson correlations

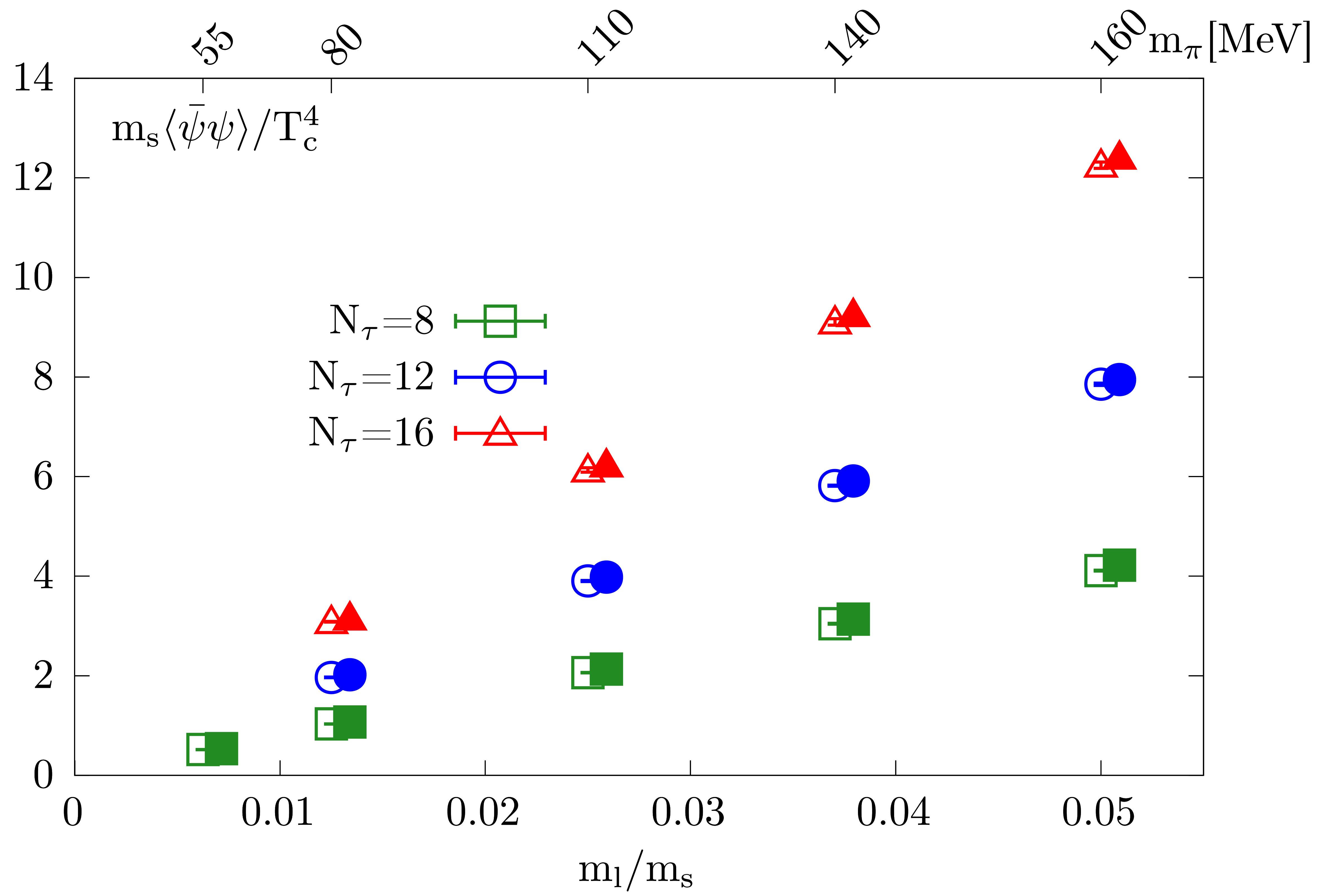
$$\Delta_n^{\mathrm{Po}} = m_l^{n-2}\left[\partial^n \rho/\partial m_l^n - (\partial^n \rho/\partial m_l^n)^{\mathrm{Po}}\right]$$



Repulsive non-Poisson correlation gives rise to the ϱ(λ→0) peak

Quantities related to $\boldsymbol{\rho}$

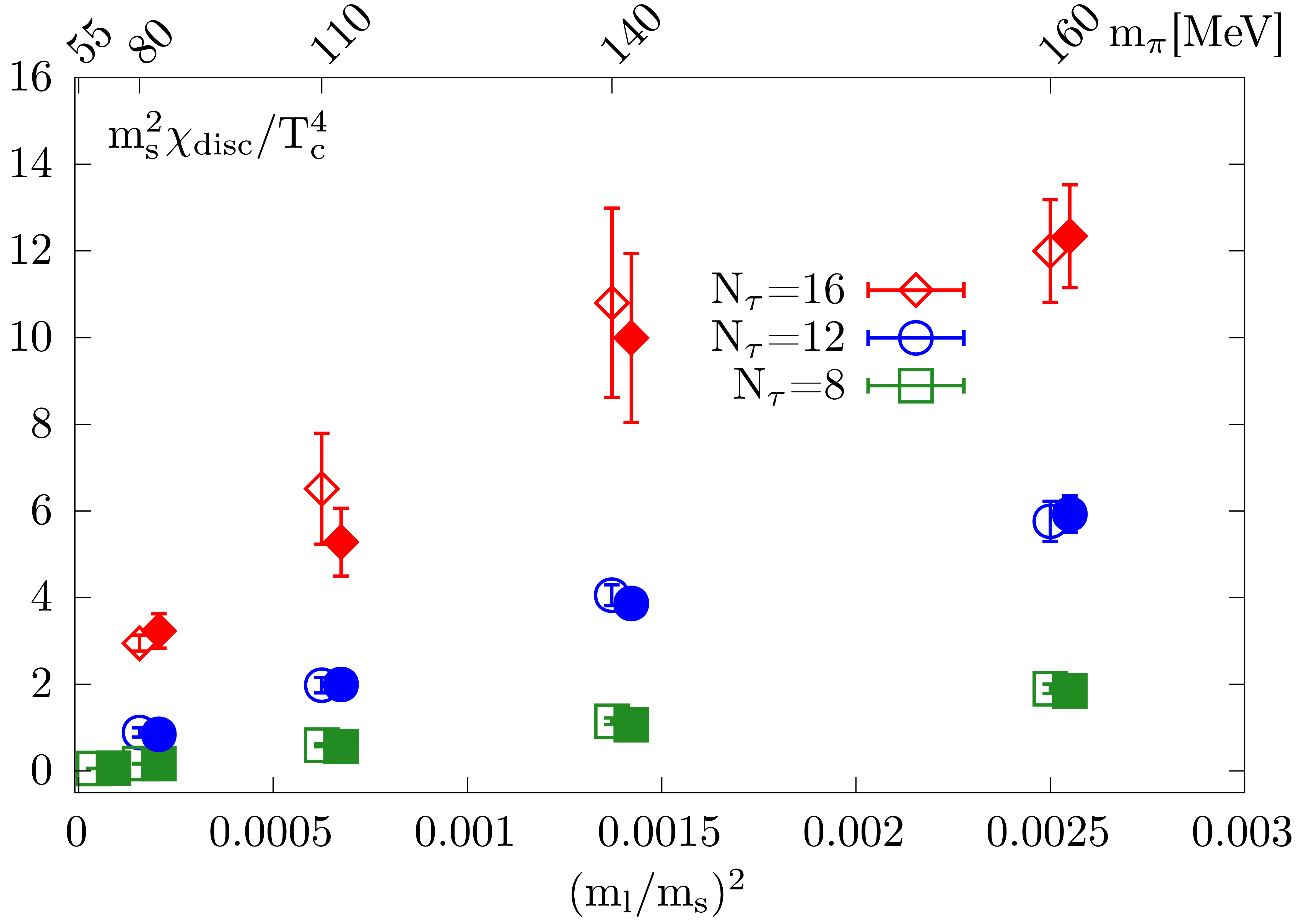


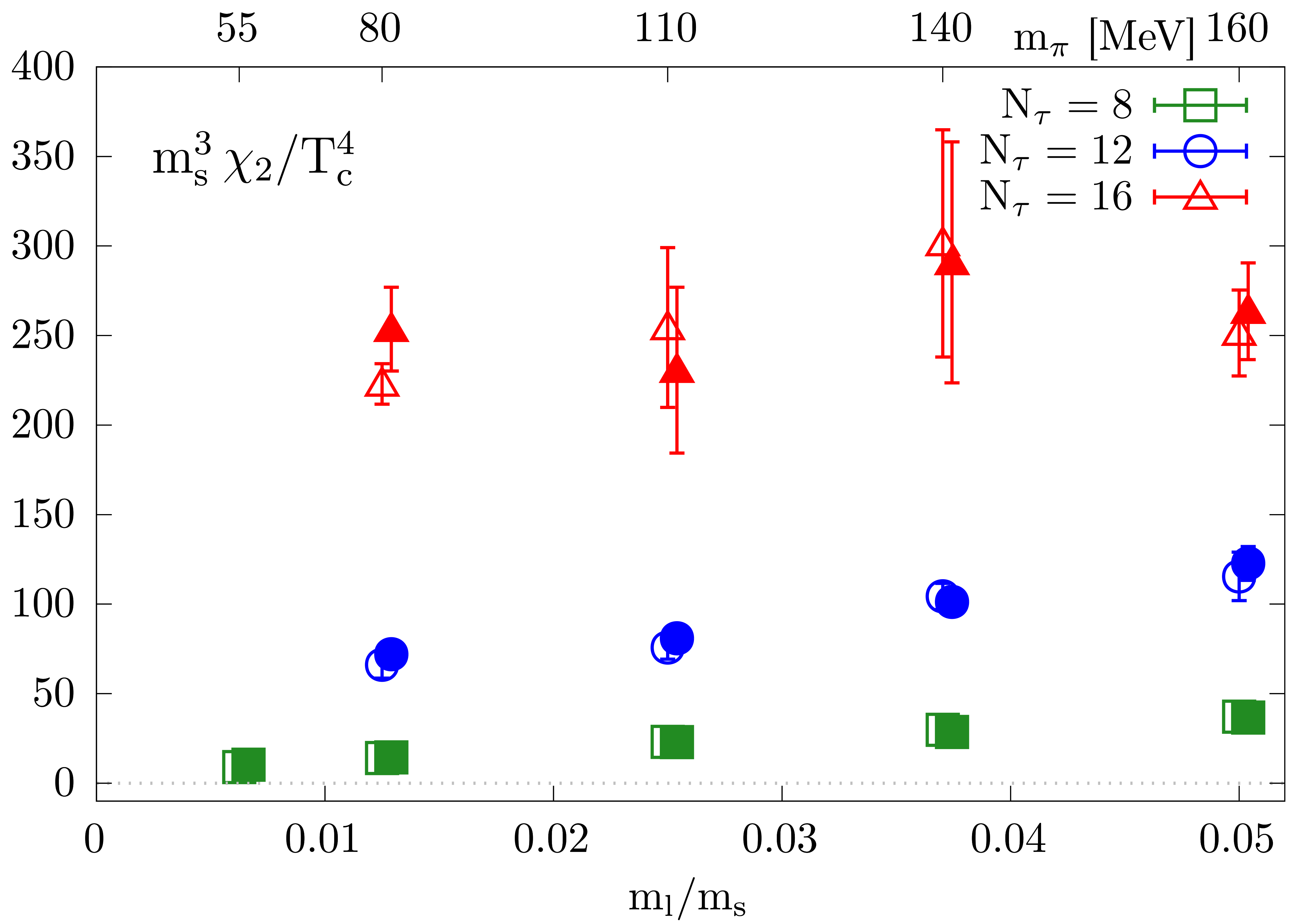


$$\chi_\pi - \chi_\delta = \int_0^\infty d\lambda \, \frac{8m_l^2 \rho}{(\lambda^2 + m_l^2)^2}$$

$$\langle\bar\psi\psi\rangle = \int_0^\infty \frac{4m_l \rho}{\lambda^2 + m_l^2} \, d\lambda$$

Quantities related to 1st & 2nd derivatives of ρ

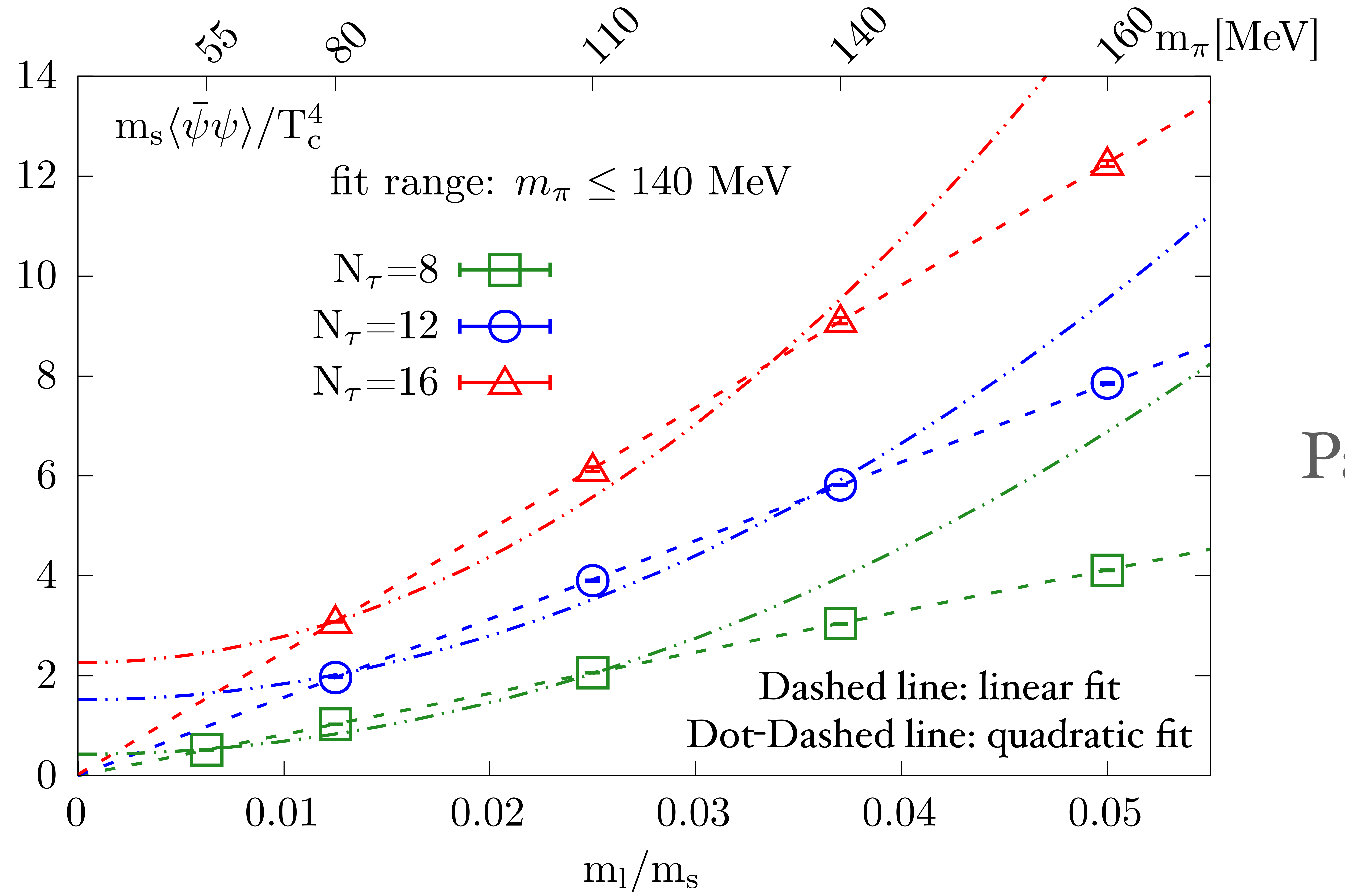




$$\chi_{\rm disc} = \int_0^\infty \mathrm{d}\lambda \, \frac{4m_l \, \partial\rho/\partial m_l}{\lambda^2 + m_l^2}$$

$$\chi_2 = \int_0^\infty \mathrm{d}\lambda \, \frac{4m_l \, \partial^2\rho/\partial m_l^2}{\lambda^2 + m_l^2}$$

SU(2)xSU(2) symmetry restoration at T=205 MeV



In the chiral symmetric phase

Z(2) subgroup of SU(2)xSU(2) sym.

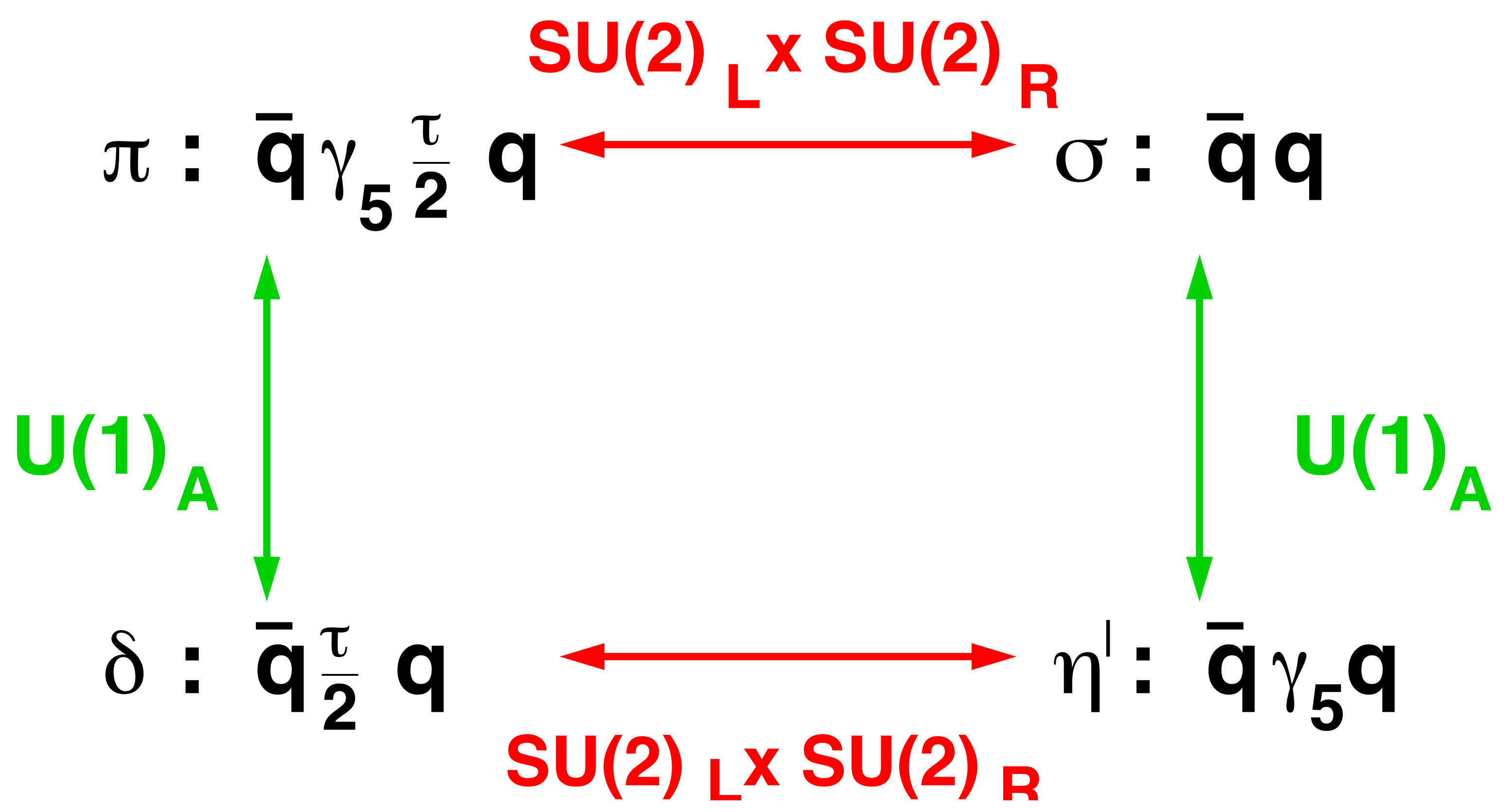
Partition function: even function of m

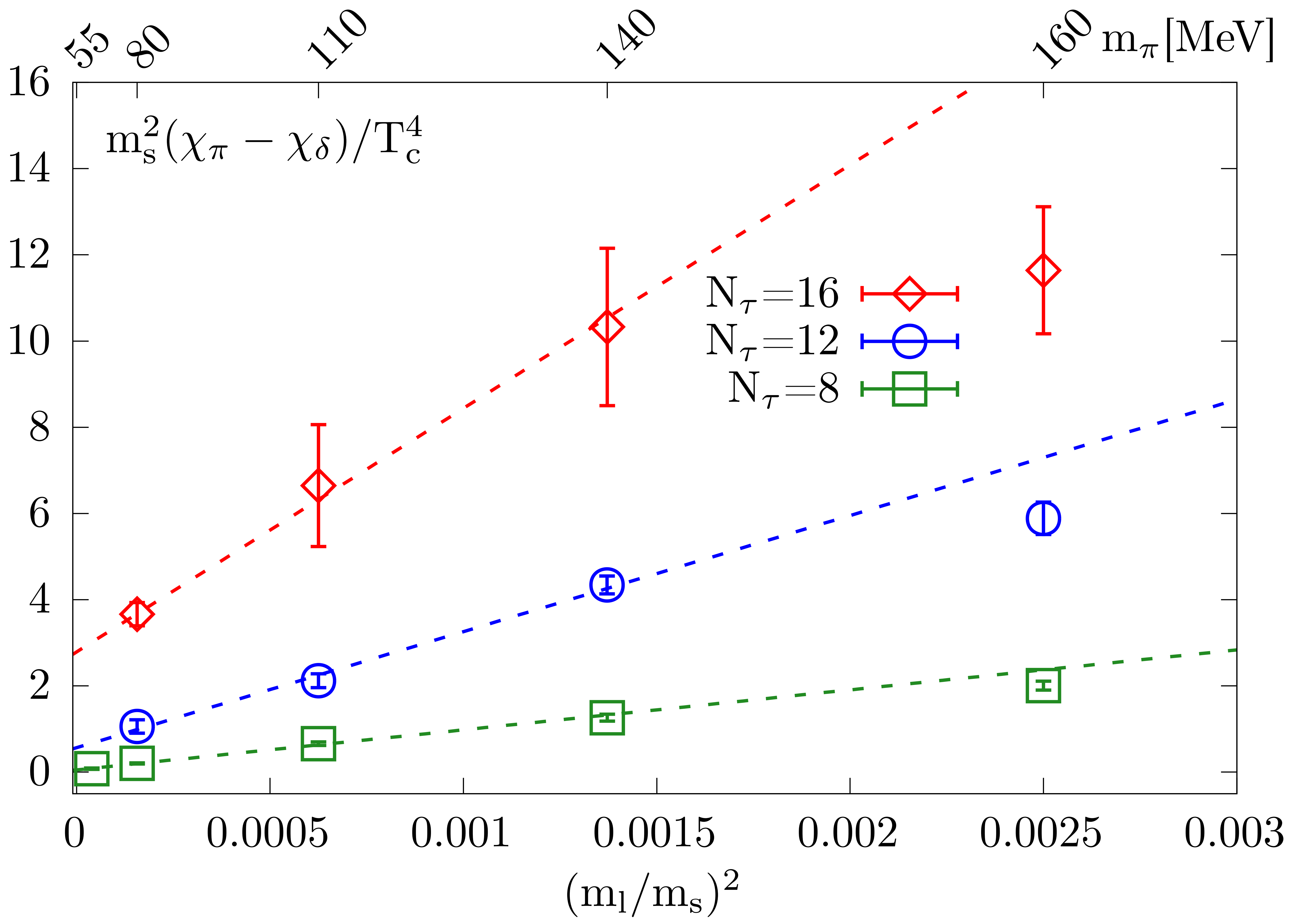
$$\langle\bar\psi\psi\rangle \propto m \text{ as } m \to 0$$

$$\chi_{disc} \propto m^2 \text{ as } m \to 0$$

χ^2/dof	Linear fits	Quadratic fits
N_τ=8	0.43	13972.7
N_τ=12	4.4	1504.0
N_τ=16	0.1	198.5

Difference between π and δ susceptibilities





- Linear behavior in quark mass squared at $m_\pi \leq 140$ MeV
- Linear fits wo m_π =160 MeV data at each Nt yield values at m=0:

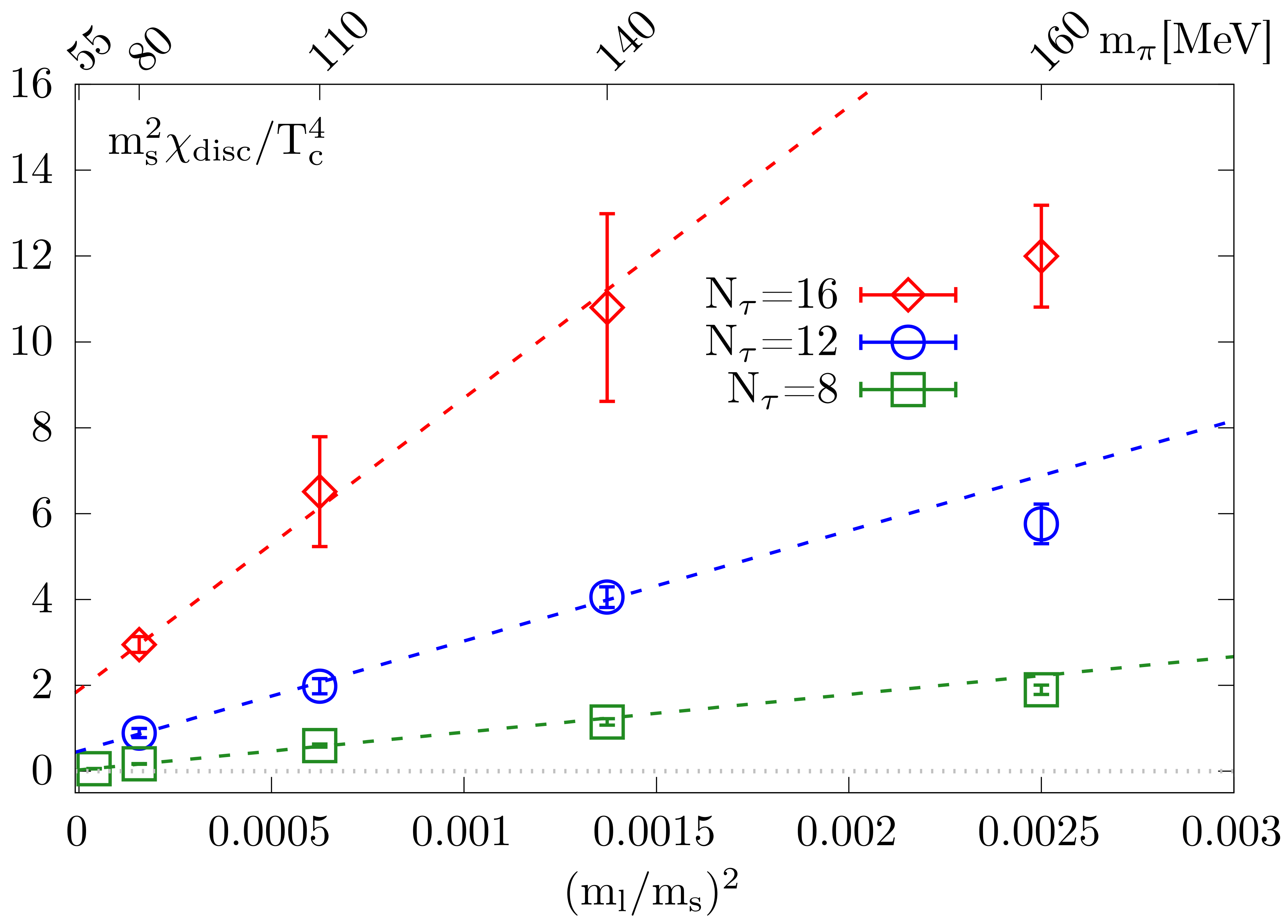
N_τ=8: 0.05(1)

N_τ=12: 0.6(2)

N_τ=16: 2.8(1)

Two $U(1)_A$ measures

$\chi_\pi - \chi_\delta$ should equal to χ_{disc} in chiral symmetric QCD



N_τ=8: 0.0030(7)

N_τ=12: 0.47(8)

N_τ=16: 1.9(1)

Values in the chiral limit at each N_τ

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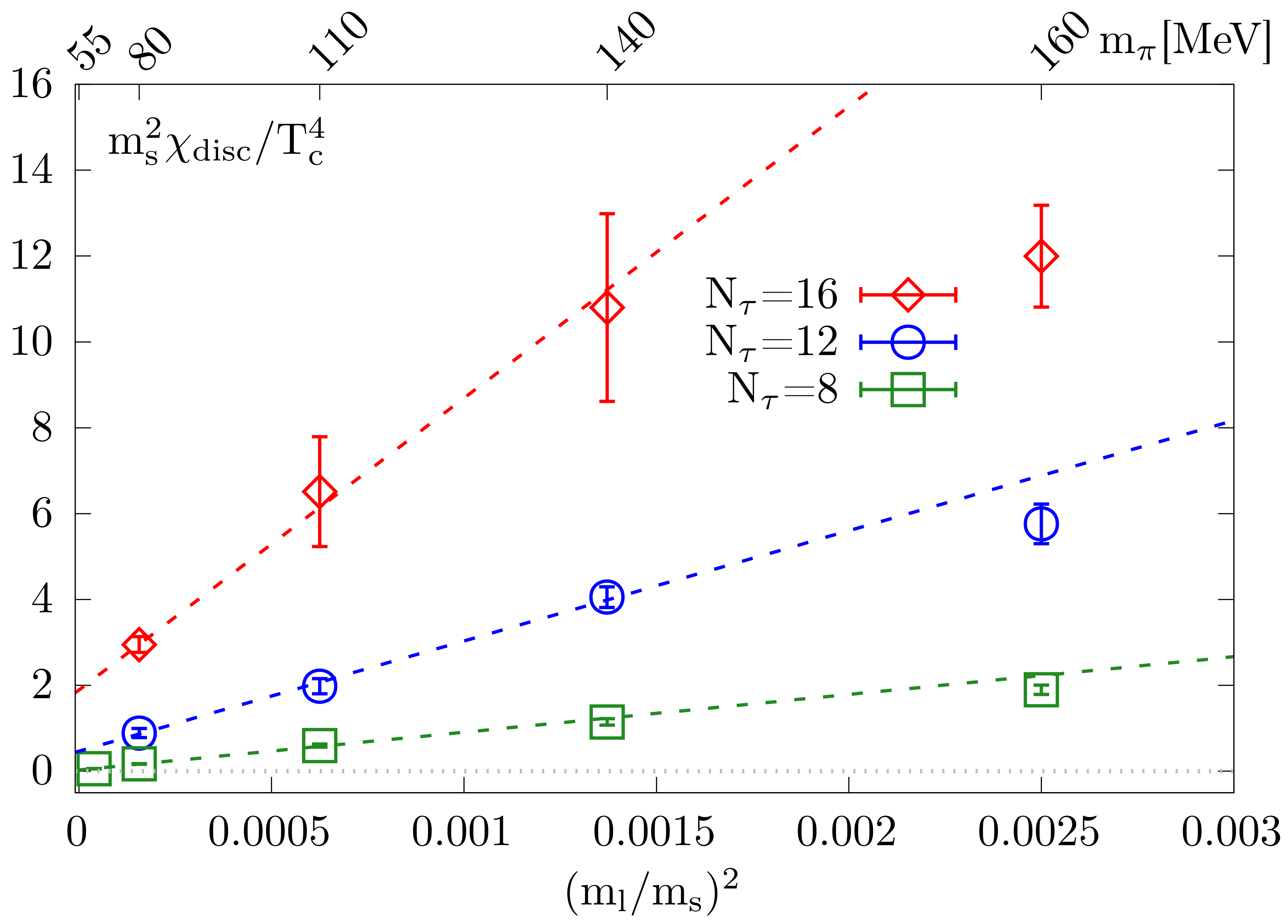
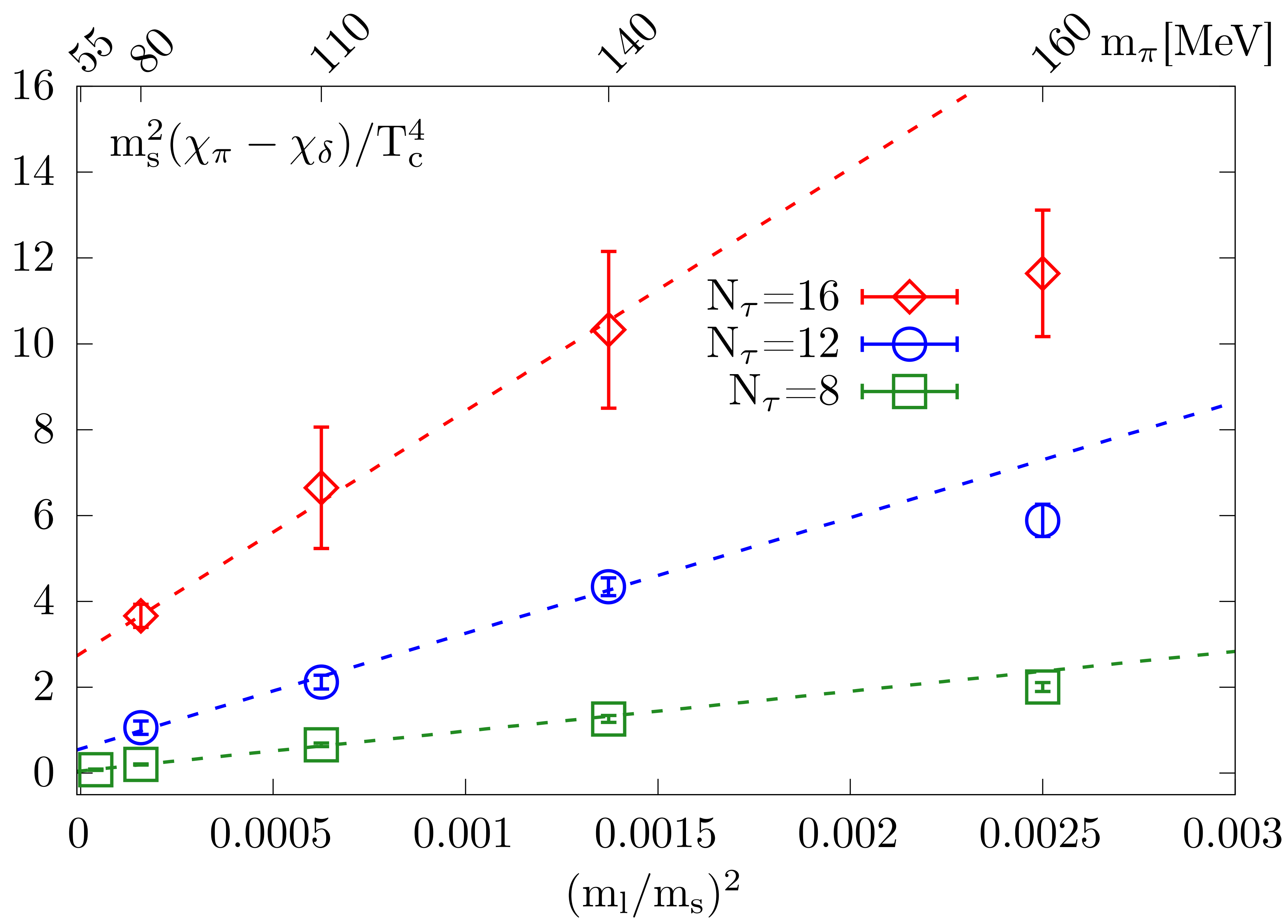




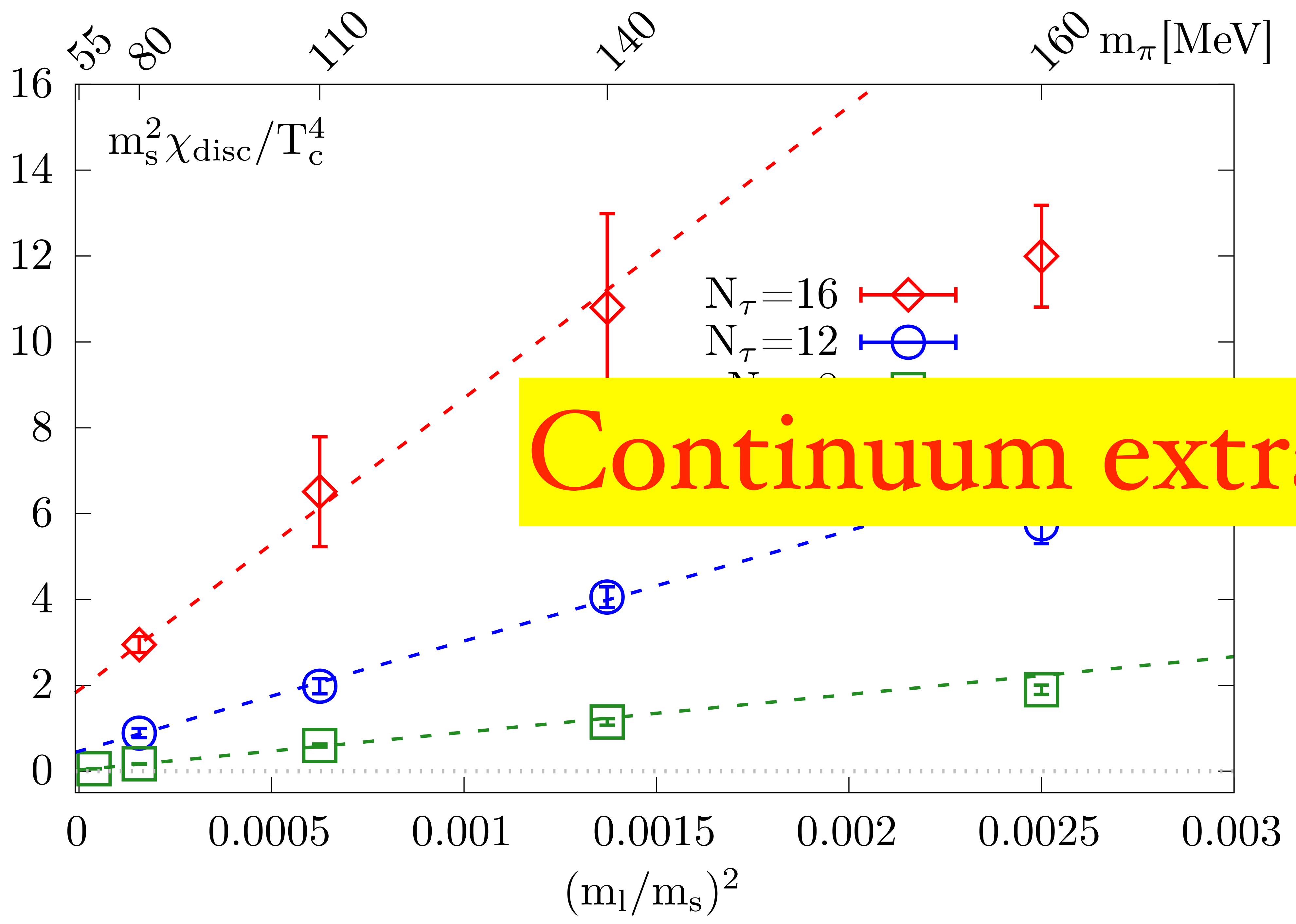
N_τ=8: 0.0030(7)
N_τ=12: 0.47(8)
N_τ=16: 1.9(1)

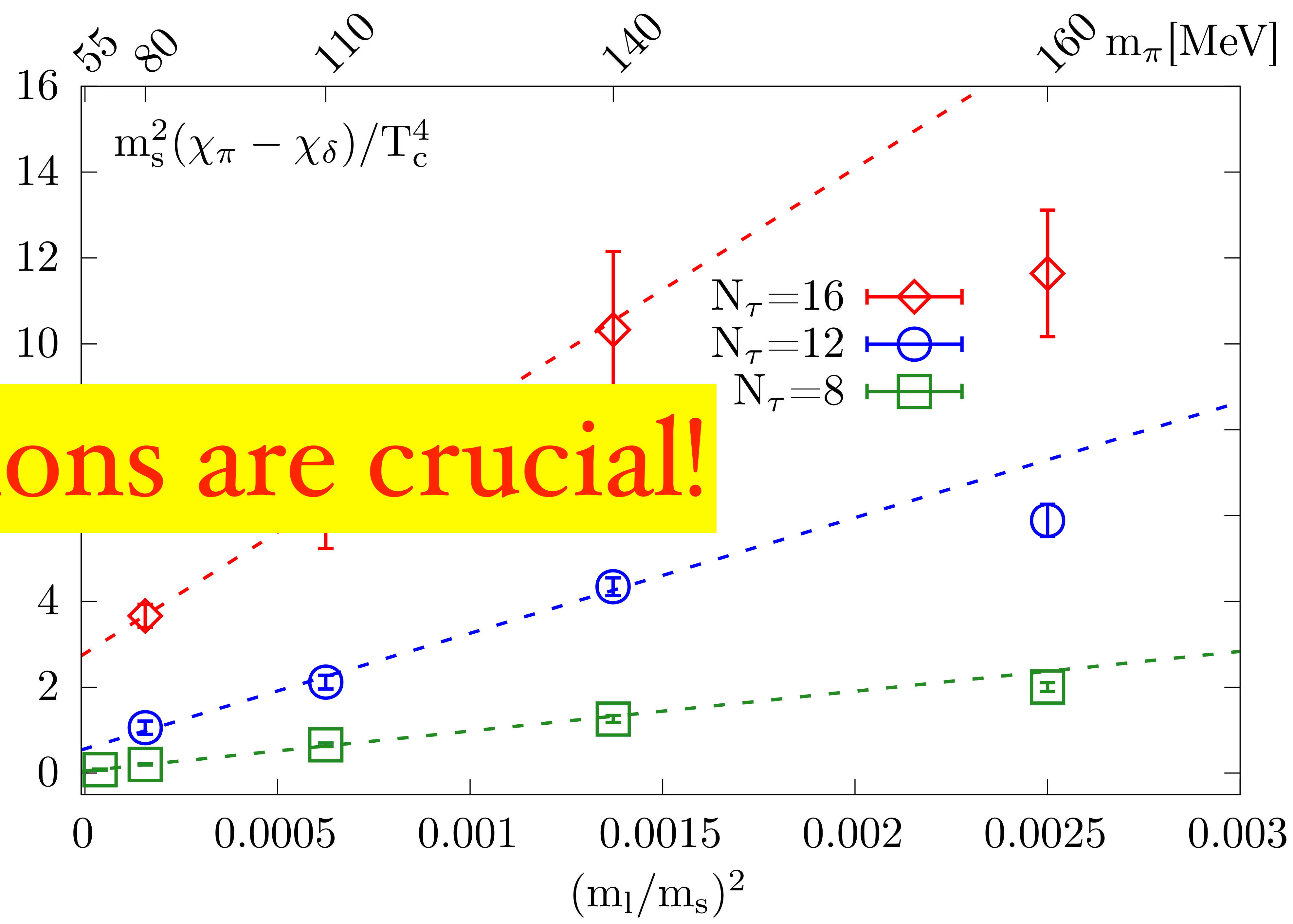
Values in the chiral limit
at each N_τ

N_τ=8: 0.05(1)
N_τ=12: 0.6(2)
N_τ=16: 2.8(1)

Two U(1)$_A$ measures

$\chi_\pi - \chi_\delta$ should equal to χ_{disc} in chiral symmetric QCD





Continuum extrapolations are crucial!

N_τ=8: 0.0030(7)

N_τ=12: 0.47(8)

N_τ=16: 1.9(1)

Values in the chiral limit at each N_τ

N_τ=8: 0.05(1)

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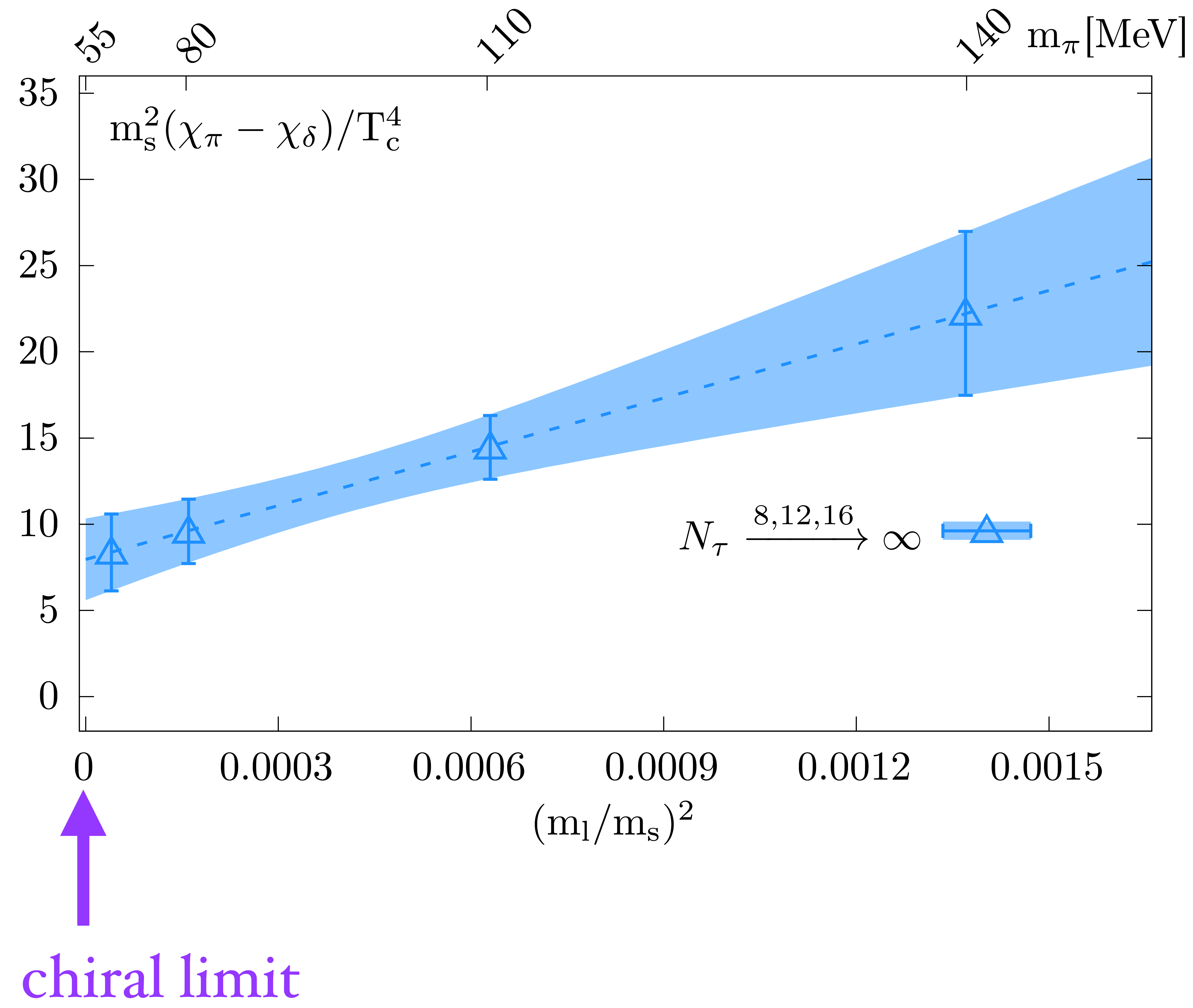
Continuum and chiral extrapolations with $m_\pi \leq 140$ MeV data



Joint fit: simultaneous fits

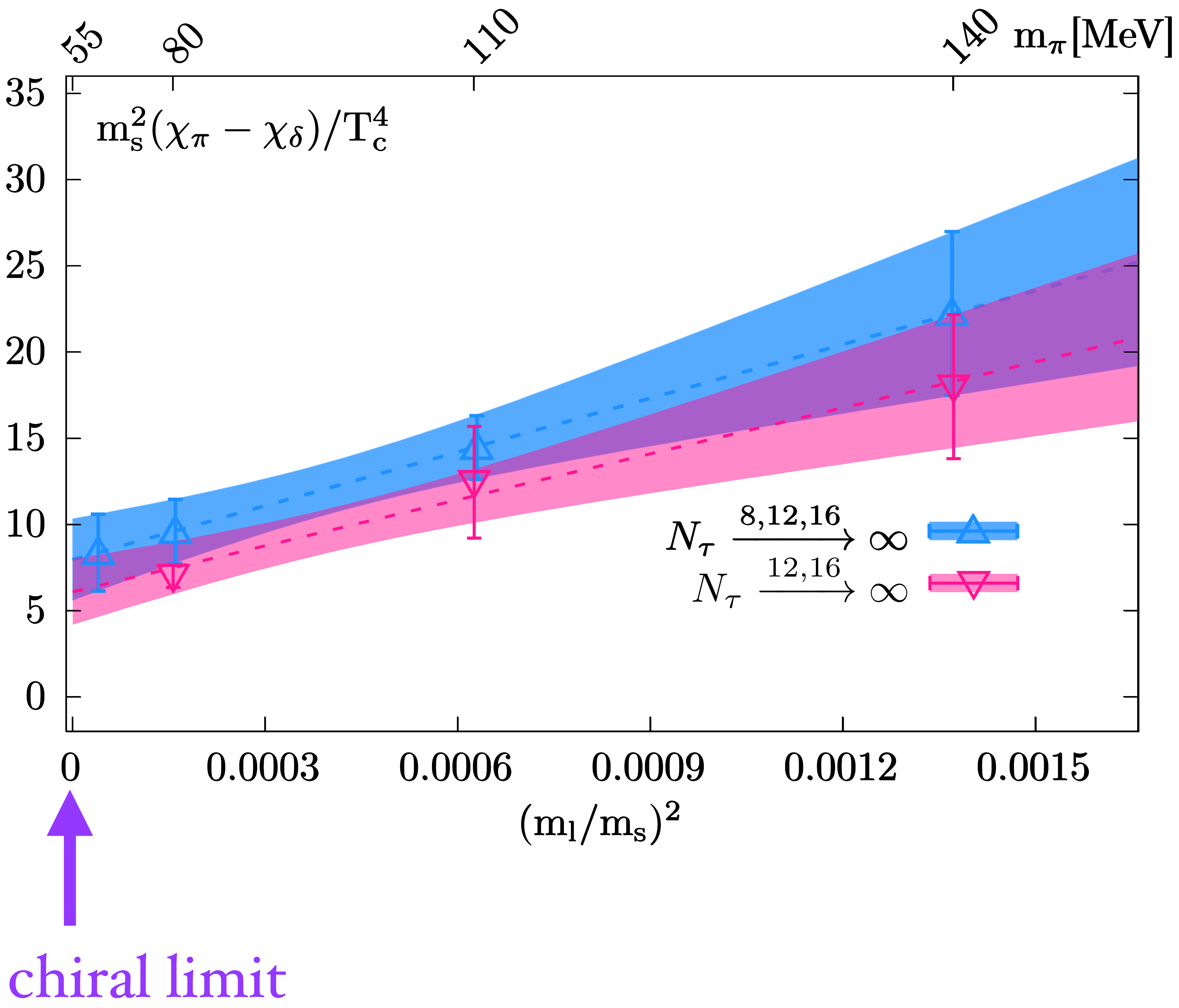
Continuum: $c_0 + c_1/N_\tau^2 + c_2/N_\tau^4$

Chiral: quadratic in quark mass

Value at $N_\tau \to \infty$ and $m \to 0$:

8.0 ± 2.4

Continuum and chiral extrapolations with $m_\pi \leq 140$ MeV data



- **Joint fit**: simultaneous fits

  Continuum: $c_0 + c_1/N_\tau^2 + c_2/N_\tau^4$

  Chiral: quadratic in quark mass

  Value at $N_\tau \to \infty$ and $m \to 0$:

  8.0 ± 2.4

- **Sequential fit:** first continuum and then chiral extrapo.

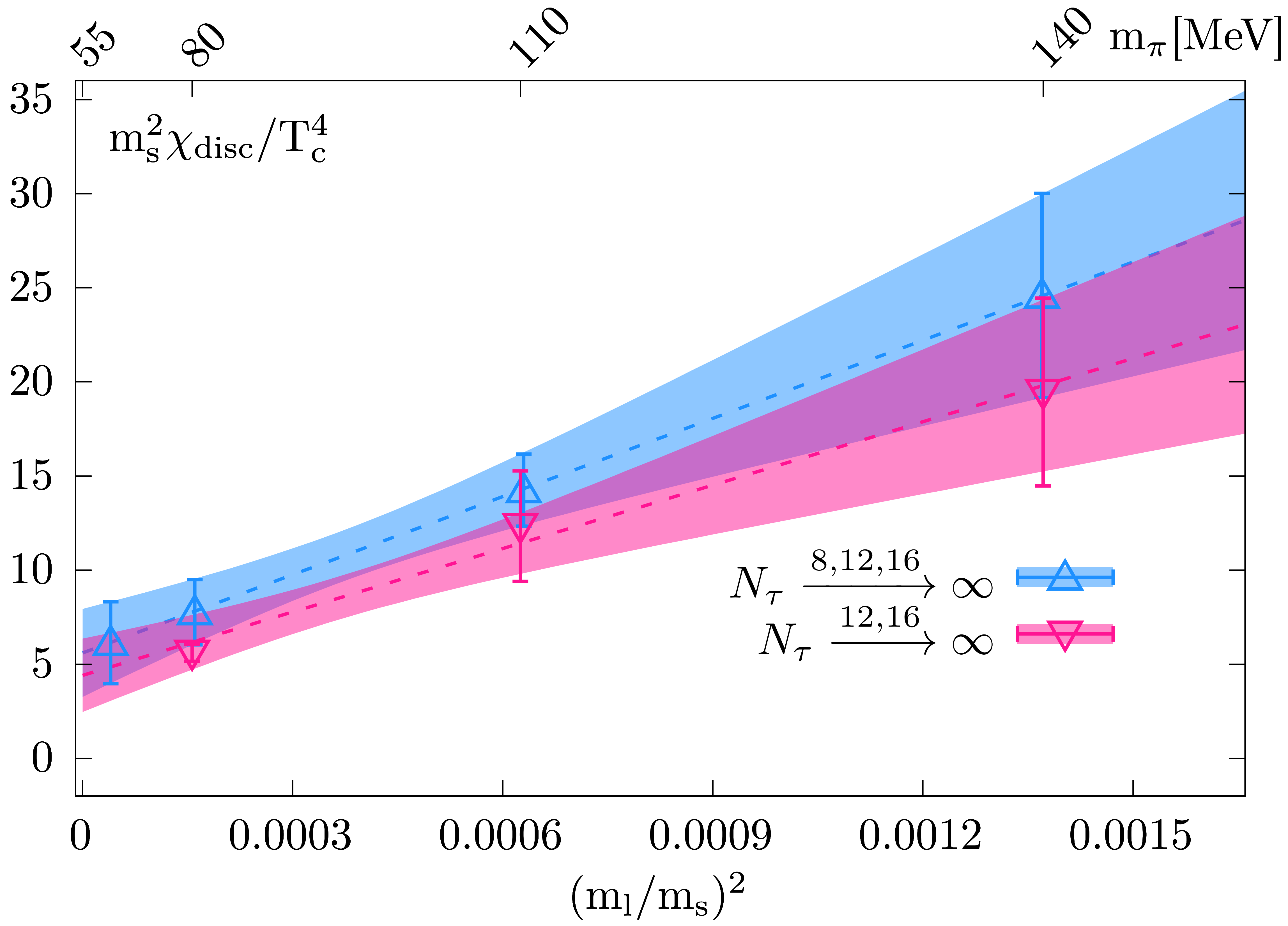
  Continuum: quadratic in $1/N_\tau$ with N_τ=12 & 16 data

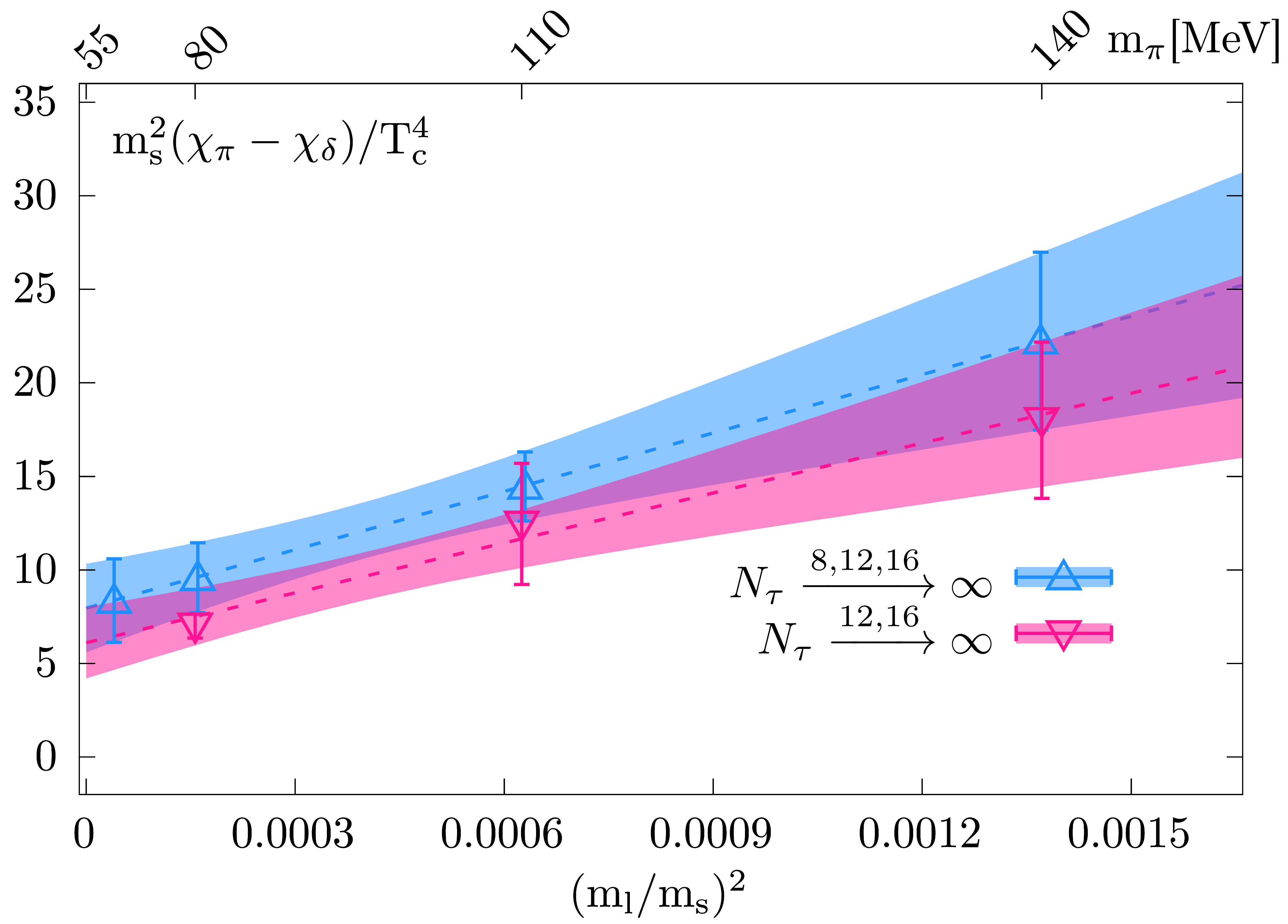
  Chiral: quadratic in quark mass

  Value at $N_\tau \to \infty$ and $m \to 0$:

  6.1 ± 1.9

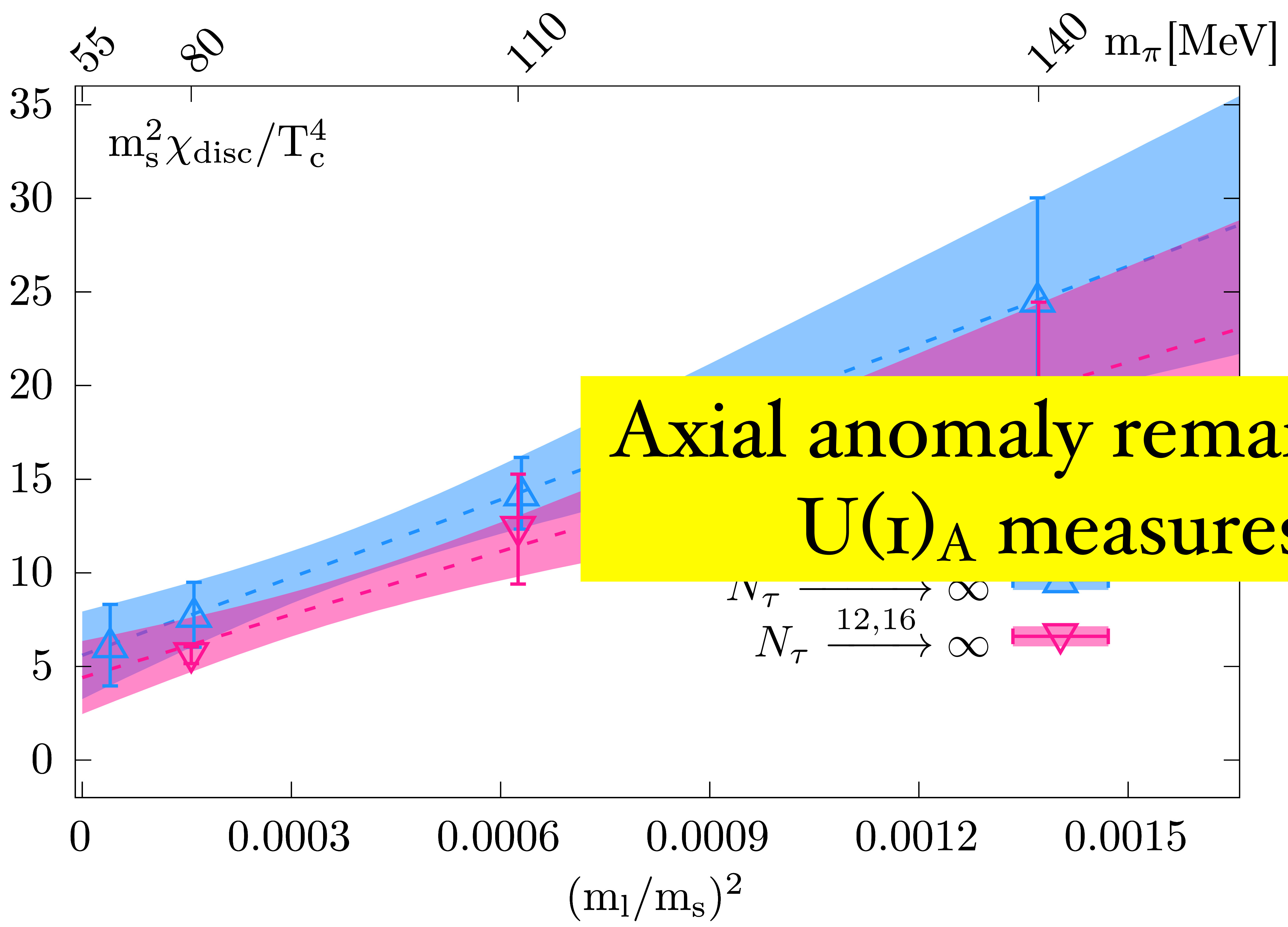
Continuum and chiral extrapolations with $m_\pi \leq 140$ MeV data

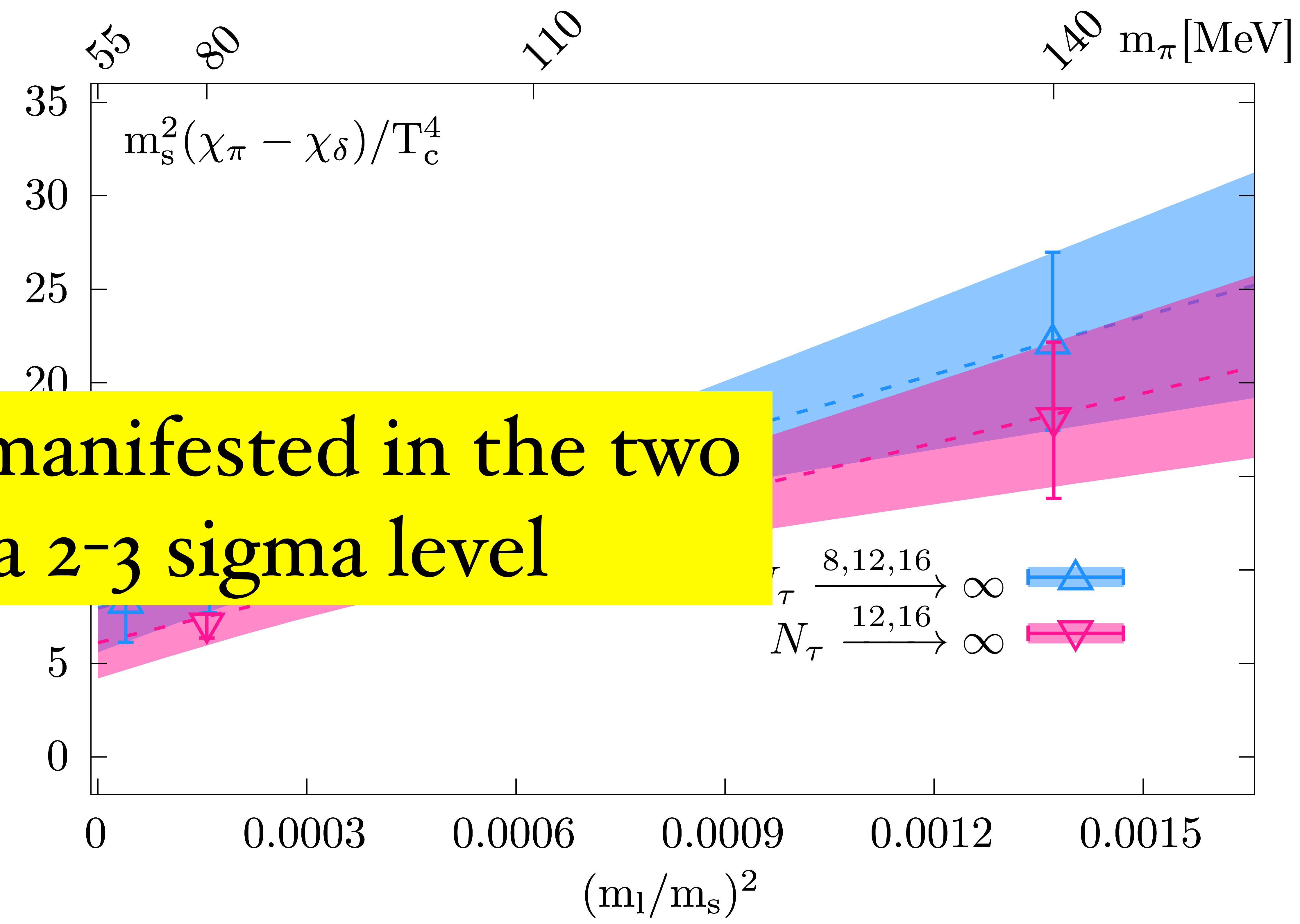




$N_\tau \to \infty$ and $m \to 0$	$m_s^2 \chi_{disc}/T_c^4$	$m_s^2(\chi_\pi - \chi_\delta)/T_c^4$
Joint fit	5.6 ± 2.3	8.0 ± 2.4
Sequential fit	4.4 ± 1.9	6.1 ± 1.9

Continuum and chiral extrapolations with $m_\pi \leq 140$ MeV data





Axial anomaly remains manifested in the two $U(1)_A$ measures at a 2-3 sigma level

$N_\tau \to \infty$ and $m \to 0$	$m_s^2 \chi_{disc}/T_c^4$	$m_s^2(\chi_\pi - \chi_\delta)/T_c^4$
Joint fit	**5.6 ± 2.3**	**8.0 ± 2.4**
Sequential fit	**4.4 ± 1.9**	**6.1 ± 1.9**

Recap

Our study suggests:

- At $T \gtrsim 1.6\ T_c$ the microscopic origin of axial anomaly is driven by the weakly interacting (quasi-) instanton gas motivated $\rho(\lambda\rightarrow 0, m\rightarrow 0) \propto m^2\, \delta(\lambda)$

- N_f=2+1 QCD: 2nd order chiral phase transition belonging to 3-d O(4)

Outlook:

- ☐ the methodology would be useful for other discretization schemes

How about the case in the proximity of Tc?

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Cumulant of chiral order parameter & criticality

n-th order Cumulant of chiral condensate:

$$K_n \equiv \frac{T}{V}\,\kappa_n(\bar{\psi}\psi) = \int_0^\infty C_n(\lambda_1,\cdots,\lambda_n)\prod_{i=1}^{n}\left(\frac{4m_l\,\mathrm{d}\lambda_i}{\lambda_i^2+m_l^2}\right) = \int_0^\infty D_n(\lambda)\frac{4m_l}{\lambda^2+m_l^2}\mathrm{d}\lambda\,.$$

$$C_n \equiv \frac{T}{V}\,\kappa_1\Big(\rho_U(\lambda_1),\cdots,\rho_U(\lambda_n)\Big) \qquad D_n(\lambda) \equiv \int_0^\infty C_n(\lambda,\lambda_2,\cdots,\lambda_n)\prod_{i=2}^{n}\left(\frac{4m_l\,\mathrm{d}\lambda_i}{\lambda_i^2+m_l^2}\right), n \geq 2$$

Criticality manifested in order parameter: $\langle\bar{\psi}\psi\rangle = h_0^{-1/\delta}H^{1/\delta}f_G(z)$

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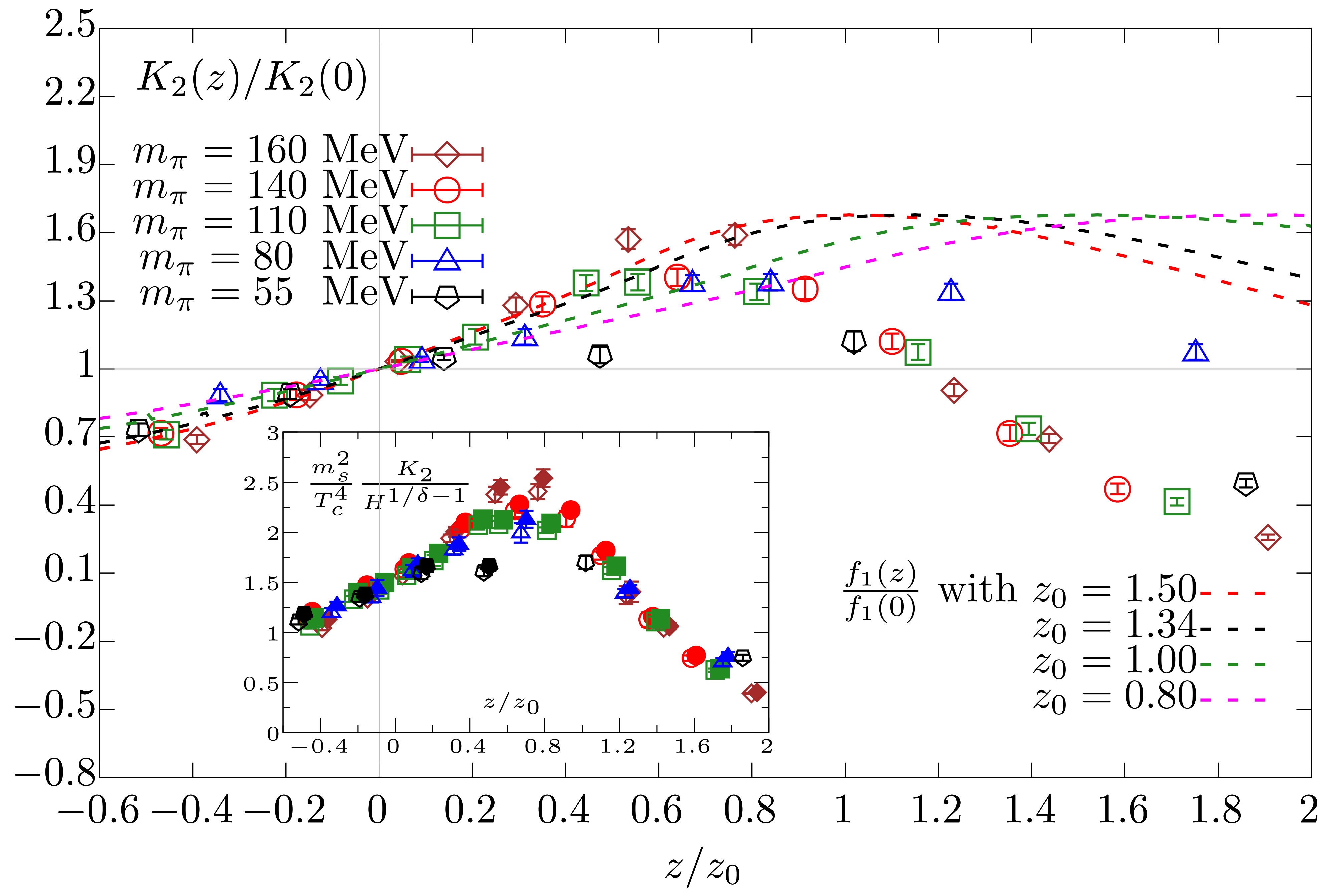
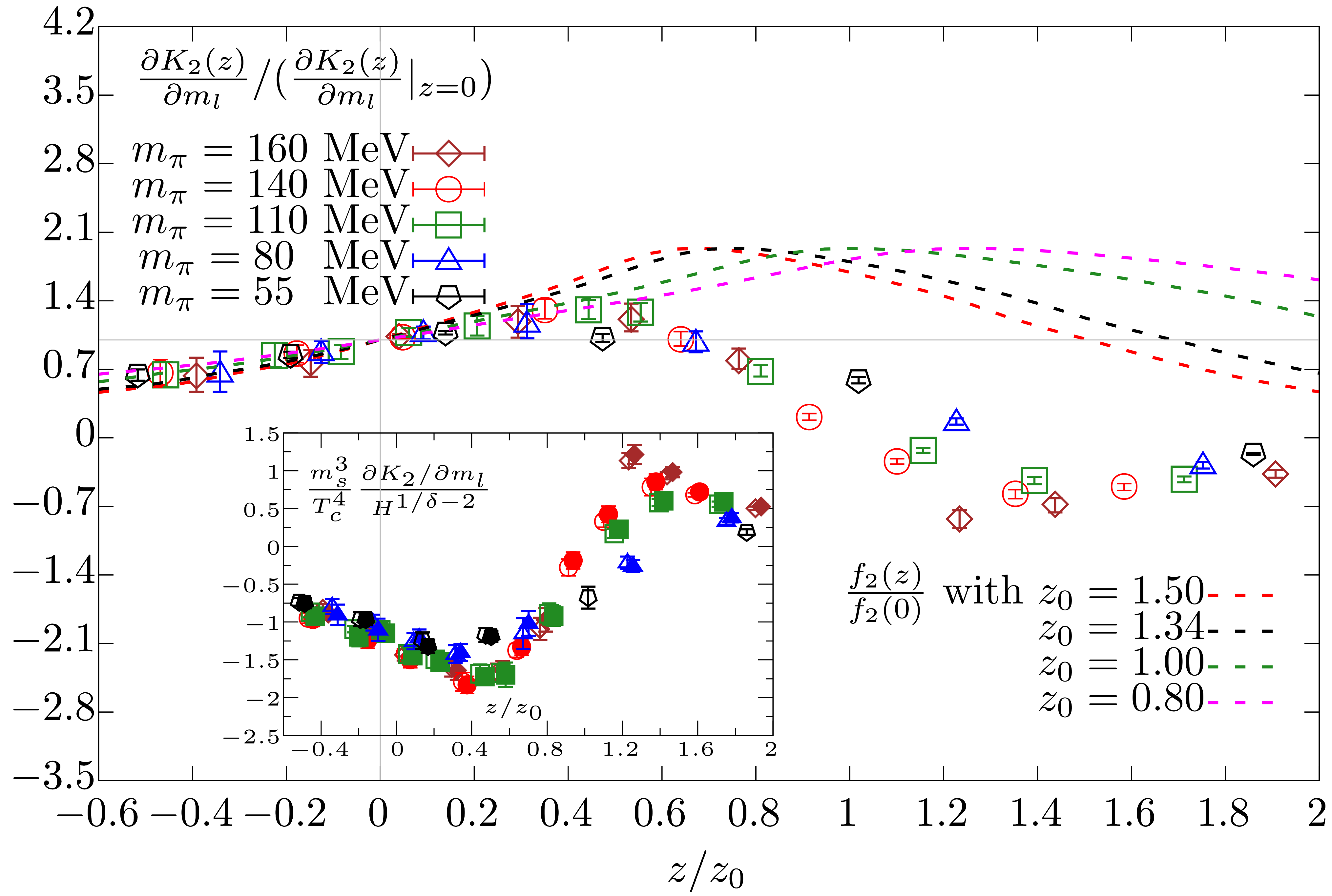
Criticality manifested in the 2nd order cumulant of order parameter

Nt=8 lattices with HISQ fermions





$$z = z_0 H^{-1/\beta\delta}(T - T_c)/T_c$$

with O(2) critical exponents

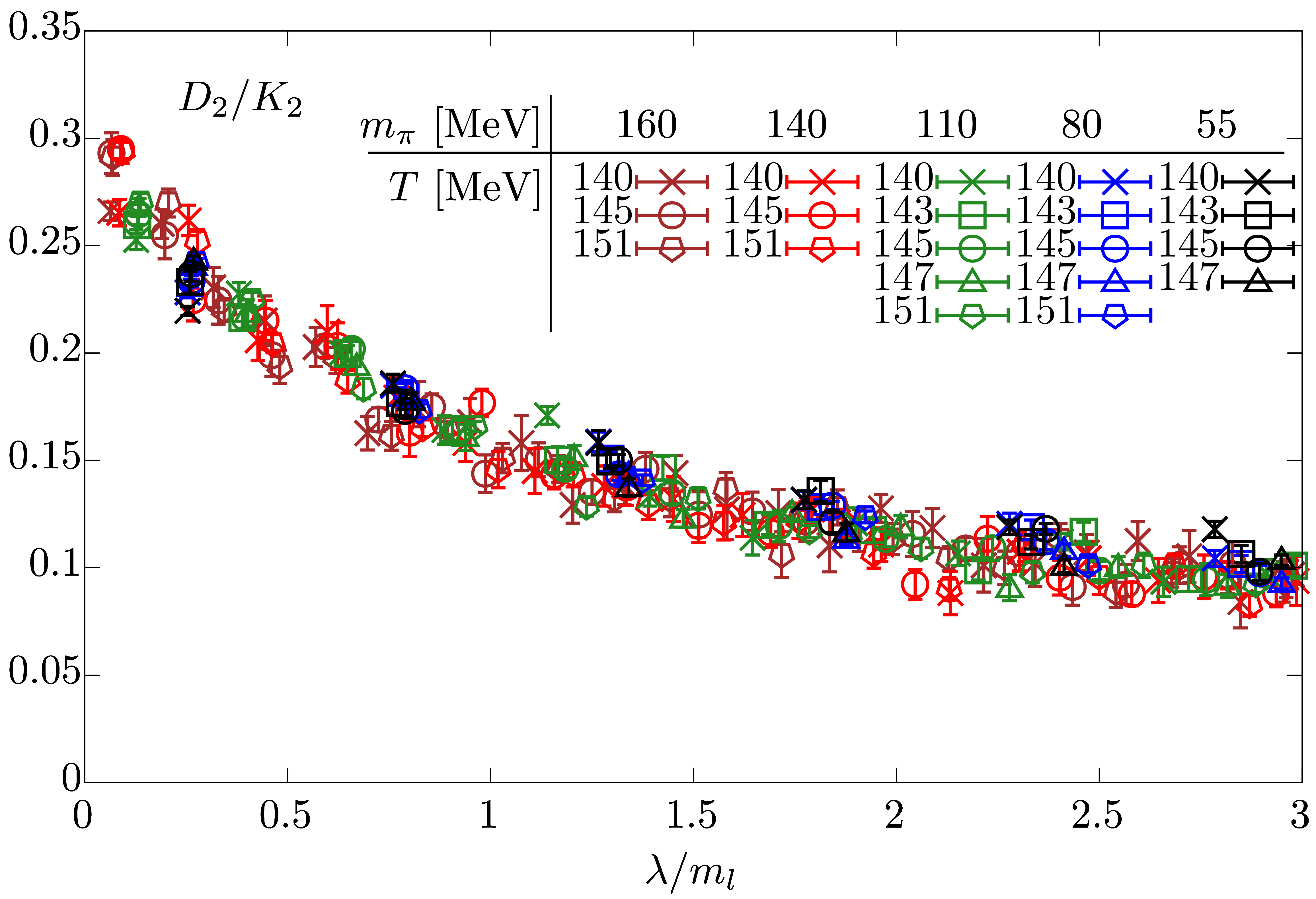
$$K_2(z) = K_2(0)\frac{f_1(z)}{f_1(0)}$$

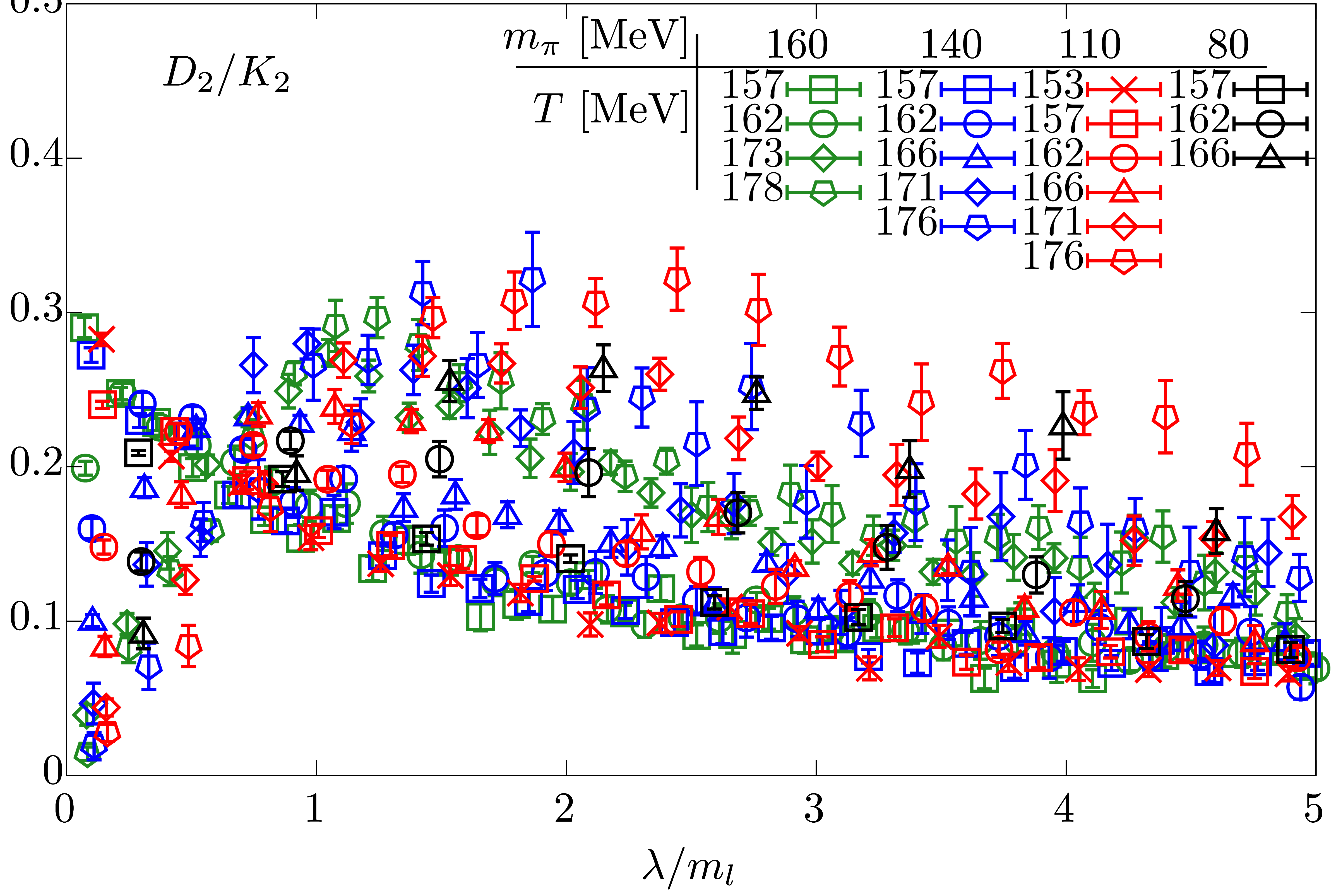
$$K_2(z)_{\text{fixed } z} \propto H^{1/\delta-1}$$

$$\frac{\partial K_2(z)}{\partial m_l} = \left.\frac{\partial K_2(z)}{\partial m_l}\right|_{z=0}\frac{f_2(z)}{f_2(0)}$$

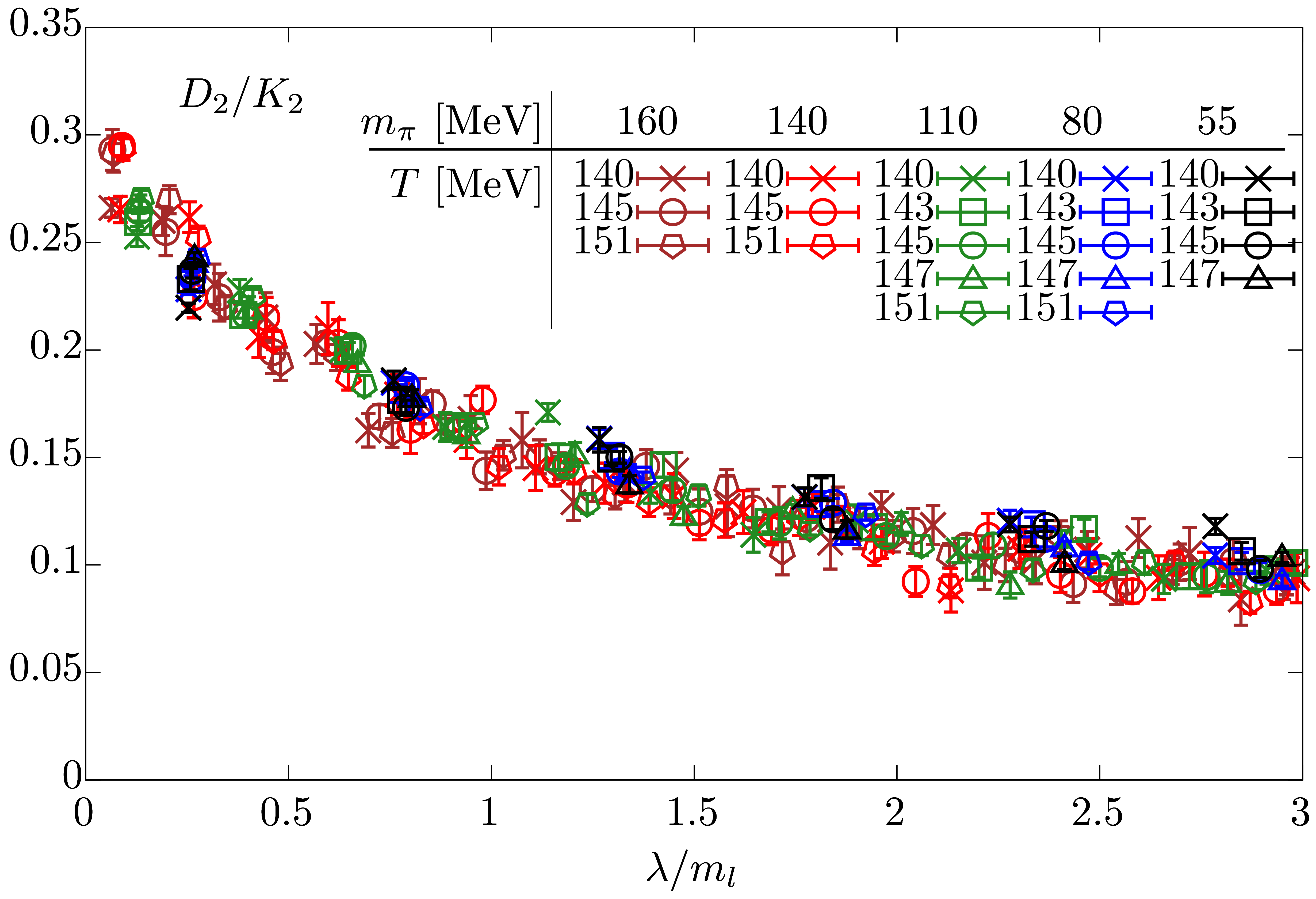
$$\left.\frac{\partial K_2(z)}{\partial m_l}\right|_{\text{fixed } z} \propto H^{1/\delta-2}$$

Scaling window in correlated Dirac eigenvalues





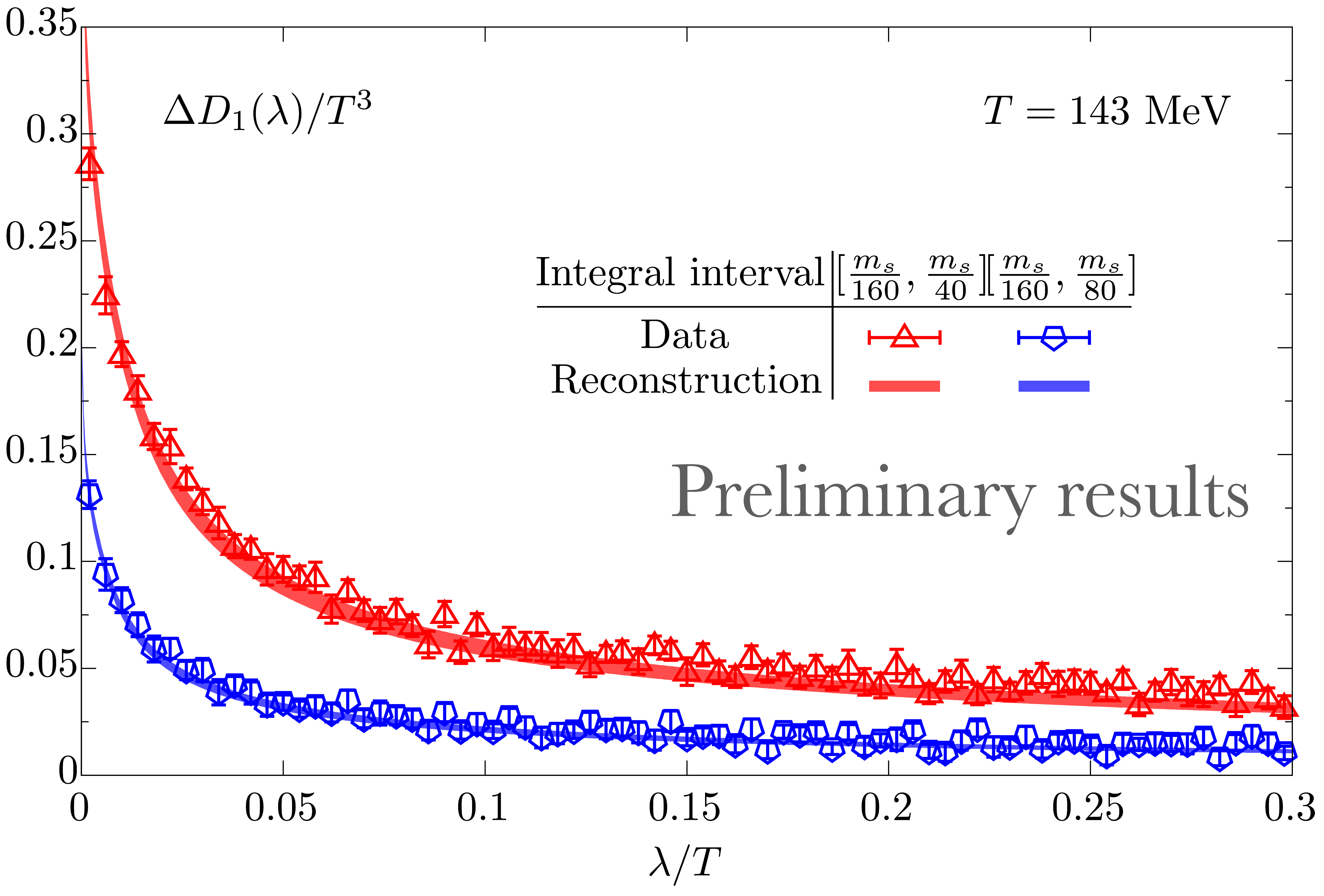
Criticality in correlated Dirac eigenvalues

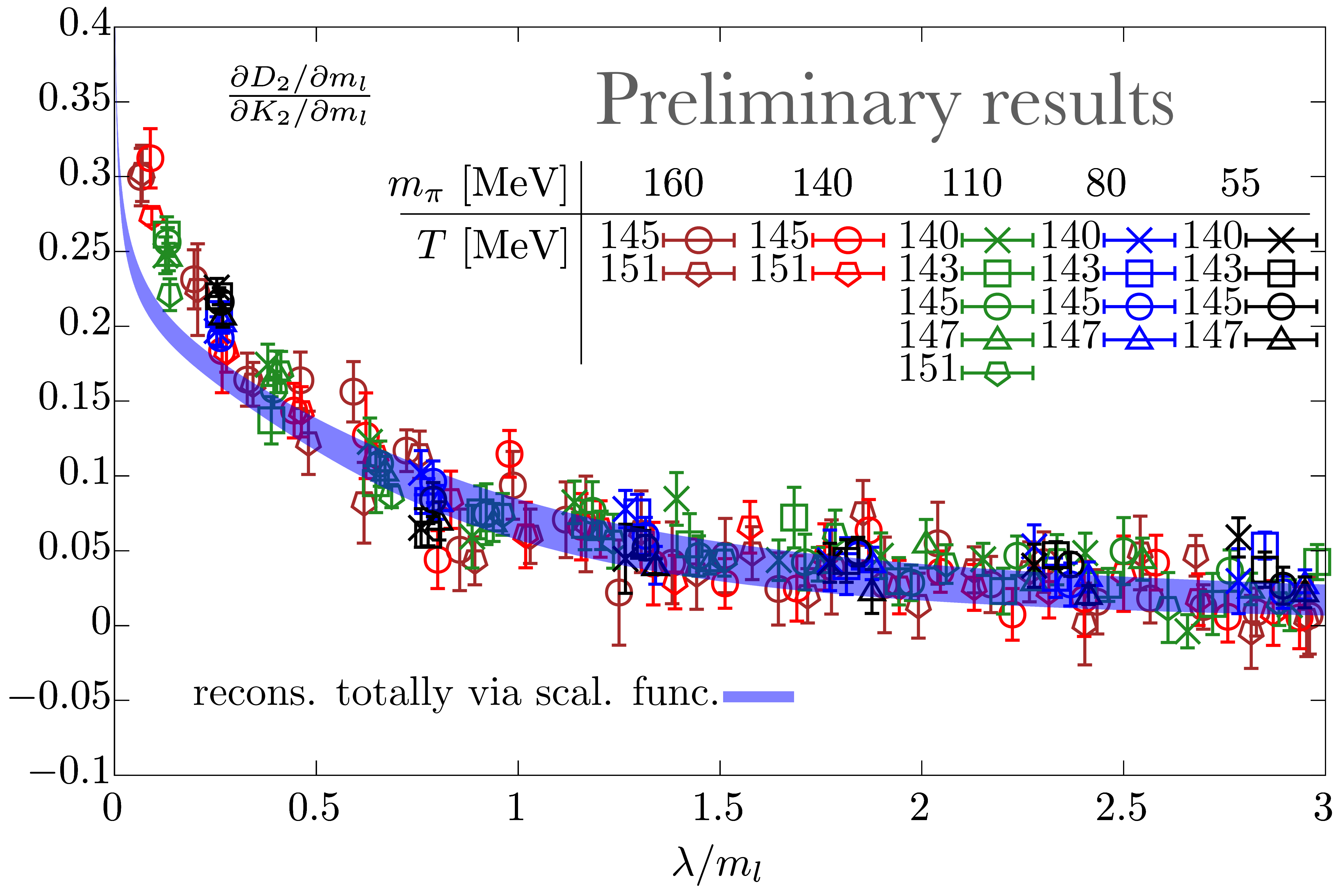


$$D_2 = g(\lambda/m_l) \times K_2(z) = g(\lambda/m_l) \times K_2(0)\frac{f_1(z)}{f_1(0)}$$

$$g(\lambda/m_l) = \frac{A\lambda^k m_l^{2n-k}}{\left(\lambda^2 + m_l^2\right)^n} = \frac{A(\lambda/m_l)^k}{[(\lambda/m_l)^2 + 1]^n}$$

Reconstruction of m_l derivative of D2 and integral of D2





$$D_1(m_{l,2}) - D_1(m_{l,1}) = \int_{m_{l,1}}^{m_{l,2}} g(\lambda/m_l) K_2 \, \mathrm{d}m_l \, .$$

$$\frac{\partial D_2/\partial m_l}{\partial K_2/\partial m_l} = \tilde{g}(\lambda/m_l) \, \frac{f_1(z)}{f_2(z)} \, \frac{f_2(0)}{f_1(0)} \left(\frac{1}{\delta} - 1\right)^{-1} + g(\lambda/m_l)$$

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Summary & Conclusion

- We established novel relations between $\partial^n\rho/\partial m^n$, K_n & C_{n+1}

- At 1.6 Tc: Axial U(1) anomaly remains manifested⇒2nd order O(4) chiral phase transition

- In the vicinity of Tc: microscopic encoding of macroscopic criticalities