LOCALIZATION OF THE DIRAC MODES IN THE IR PHASE

Andrei Alexandru & Ivan Horvath

New Trends in Thermal Phases of QCD
Nuclear Physics Institute, CAS
Prague, Apr 2023

THE GEORGE WASHINGTON UNIVERSITY
WASHINGTON, DC

OVERVIEW

- Dirac spectrum as a glue probe
- IR phase
- IR dimension for low-lying Dirac modes
- Localization for low-lying Dirac modes
- Summary and outlook

"QCD-LIKE" THEORIES

QCD with SU(3) color and various numbers of quark flavors

$$S = \int d^4x \left[-\frac{1}{2g^2} \mathrm{tr}\, F_{\mu\nu} F_{\mu\nu} + \sum_{i=1}^{N_f} \bar{\psi}_f (D(A) + m_f) \psi_f \right]$$

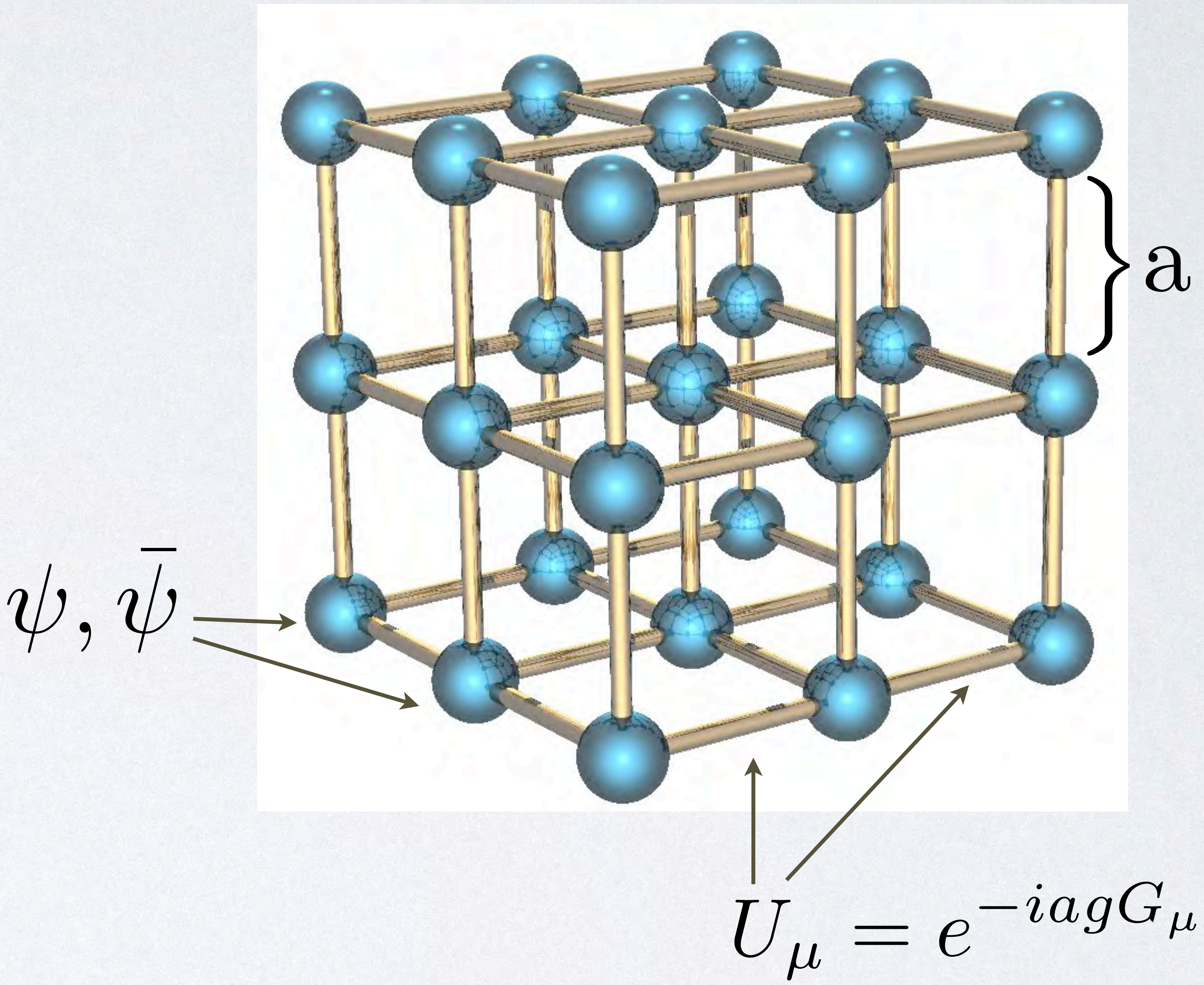
$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \qquad A_\nu \in su(3)$$

The spectrum of the covariant derivative D(A) operator will be used as a probe for the glue field A

$$D(A)\psi \equiv \gamma_\mu (\partial_\mu + A_\mu)\psi \qquad D(A)\psi_\lambda = \lambda \psi_\lambda$$

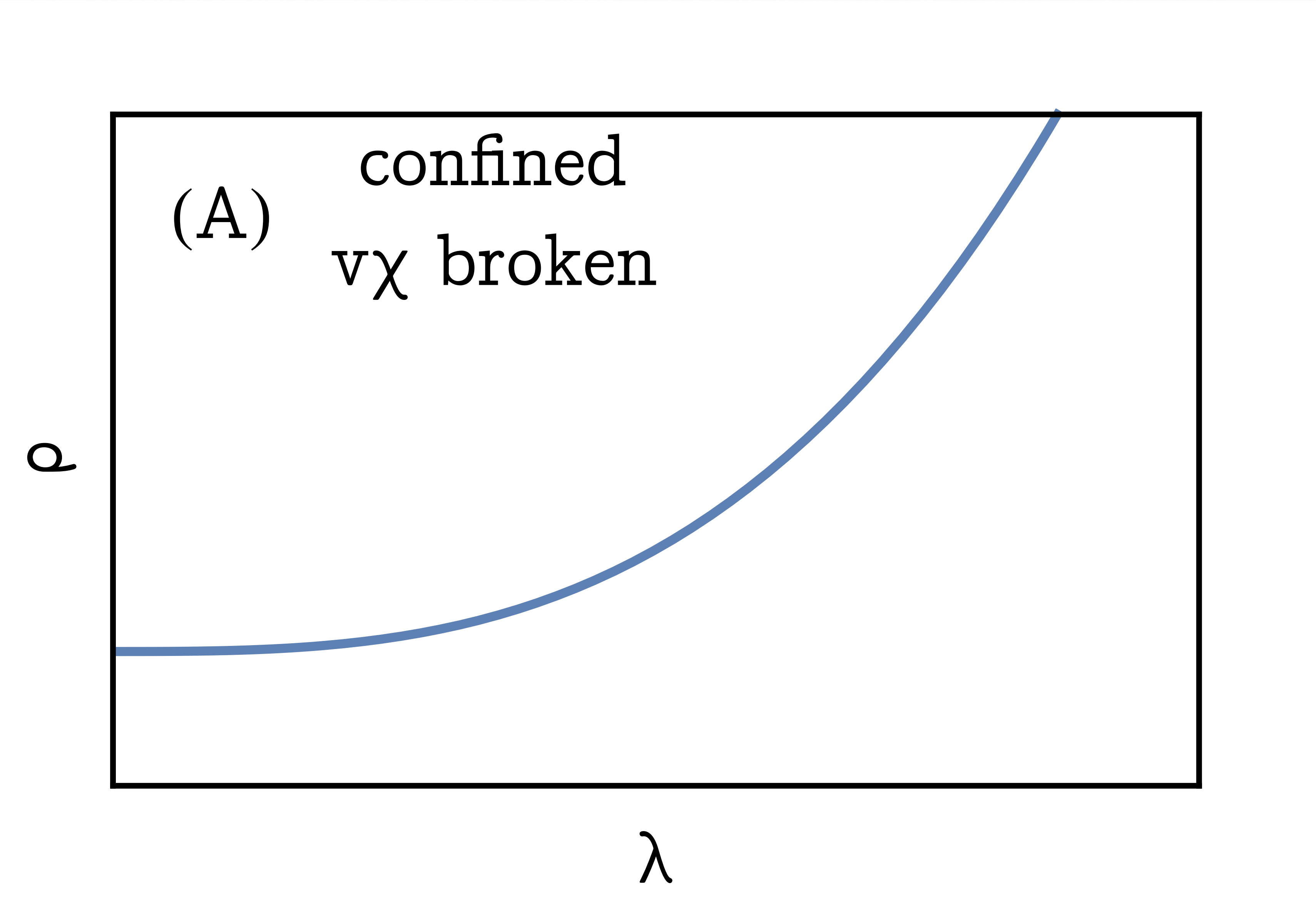
LATTICE QCD

- Non-perturbative formulation of QCD.
- Quark and gluon fields are sampled on a discrete lattice: quarks at sites and glue on links.
- Discretization of the quark covariant derivative is done using overlap formulation.
- This preserves chiral symmetry exactly even at finite lattice spacing and can be used to differentiate precisely zero-modes from near zero modes.



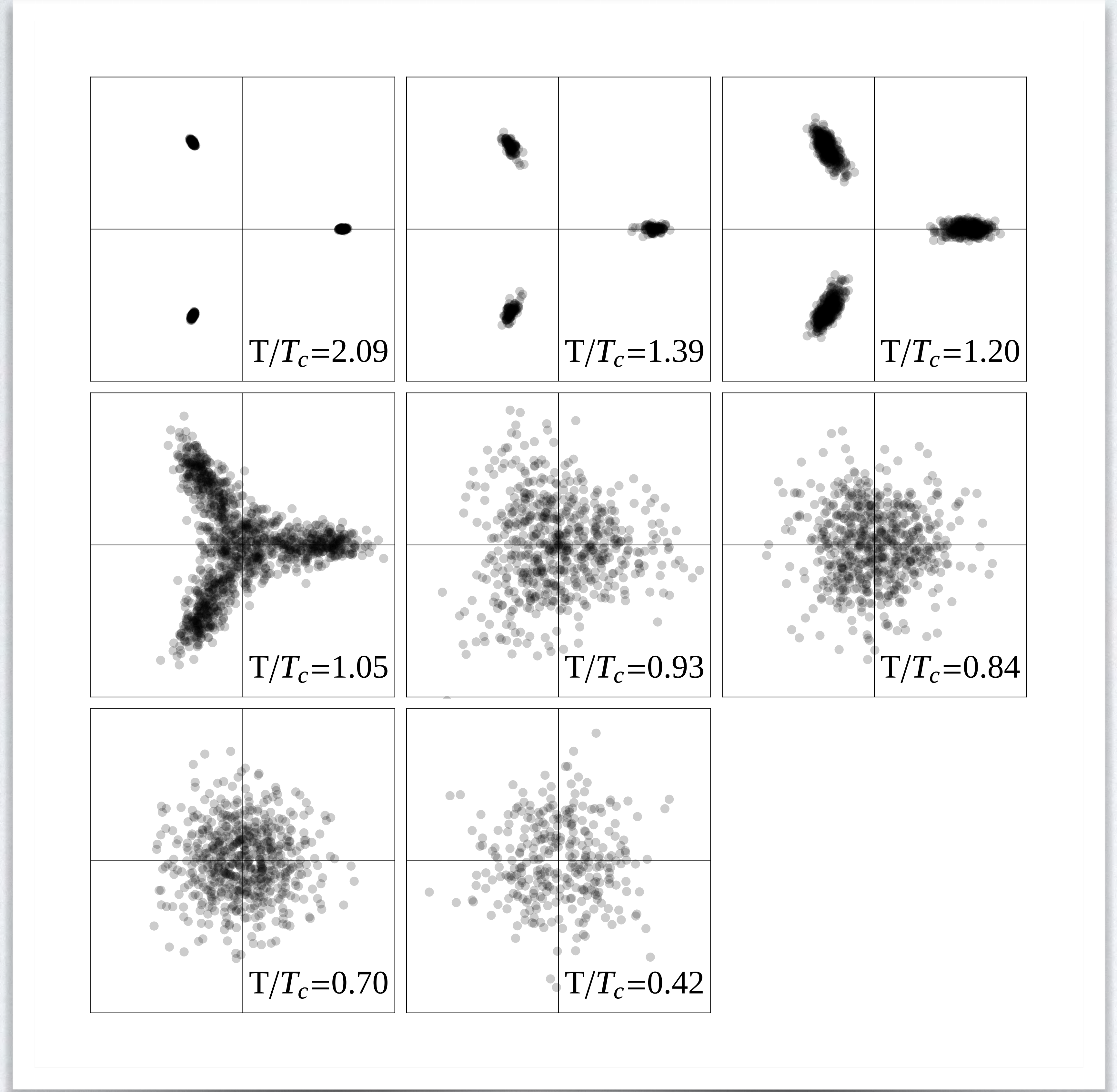
DIRAC SPECTRUM AT T=0

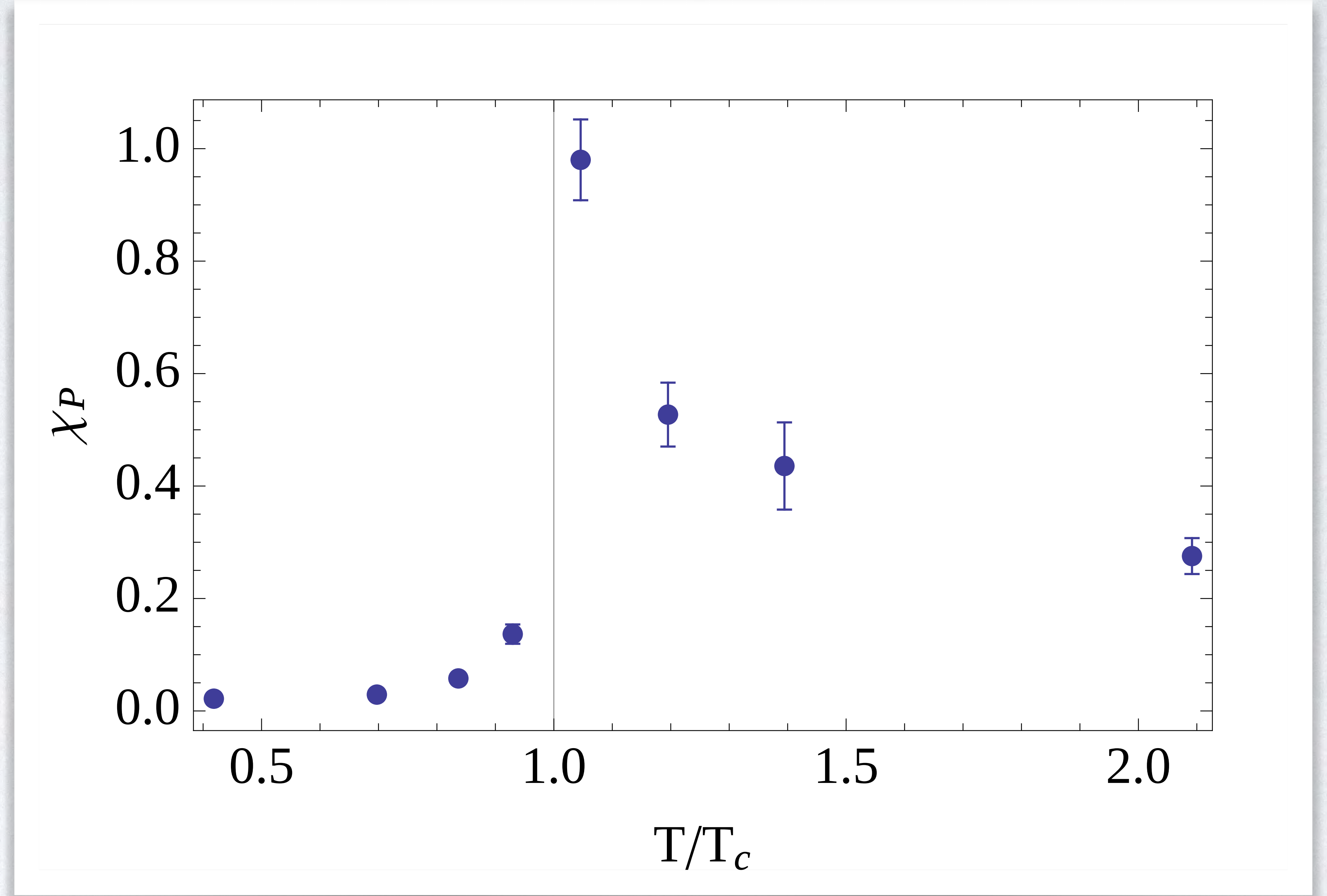
- At zero temperature the spectrum is monotonic with a non-zero value in the infrared
- All Dirac eigenmodes are delocalized, including the deep infrared modes
- Banks-Casher relation connects the density of infrared modes to the chiral condensate



$$\lim_{\lambda\to 0}\lim_{m\to 0}\lim_{V\to\infty}\rho(\lambda,m) = -\frac{1}{\pi}\langle\bar{\psi}\psi\rangle$$

THERMAL PHASE TRANSITION



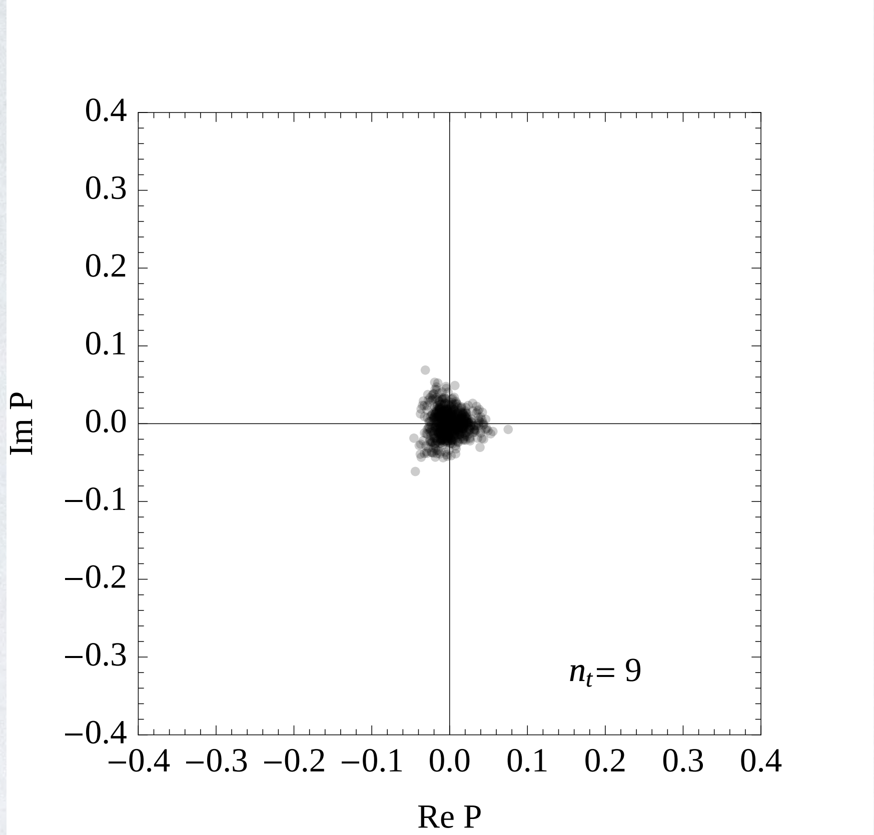


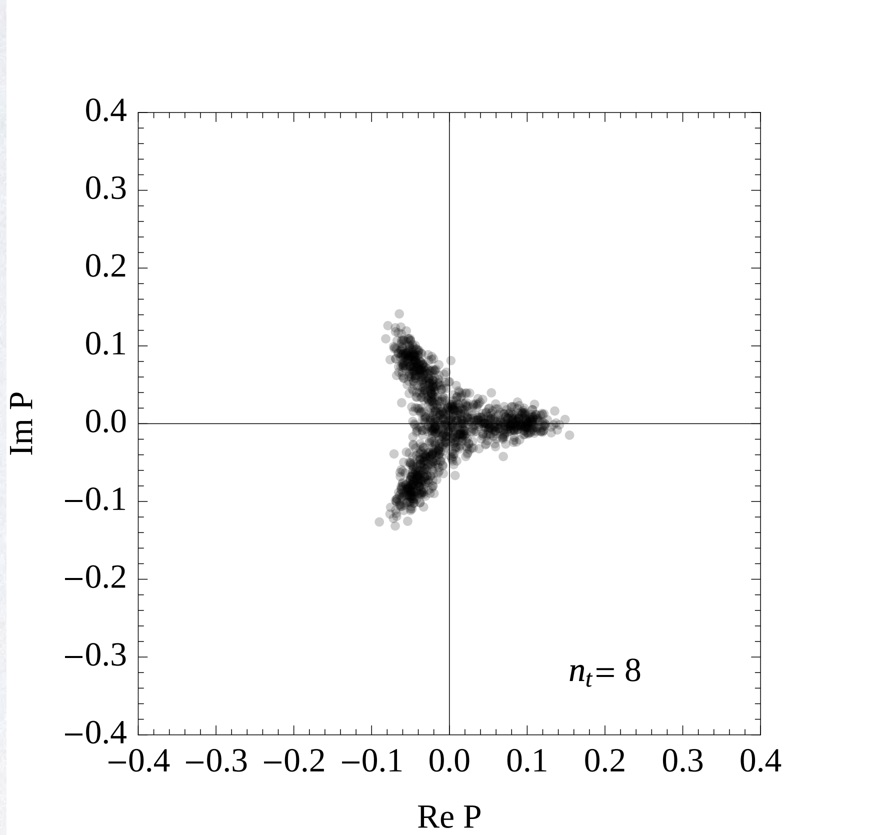
$$P = \frac{1}{3V_s} \sum_{\vec{x}} \mathrm{Tr} \left(\prod_{t=0}^{N_t - 1} U_4(\vec{x}, t) \right)$$

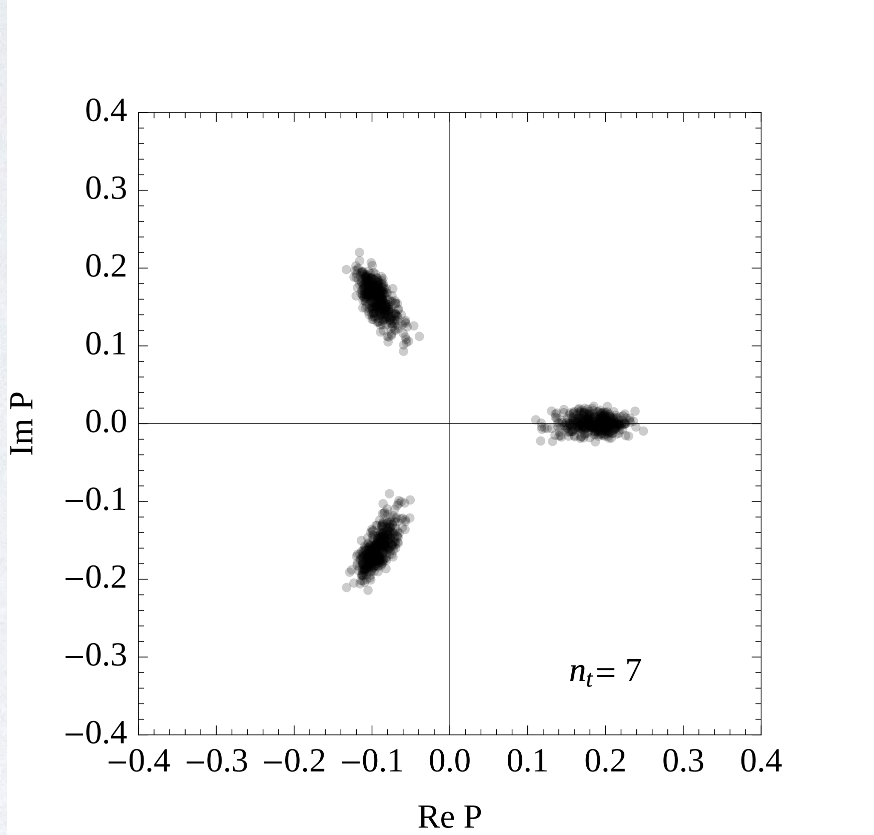
$$\chi_P = V_s \left(\langle |P|^2 \rangle - \langle |P| \rangle^2 \right)$$

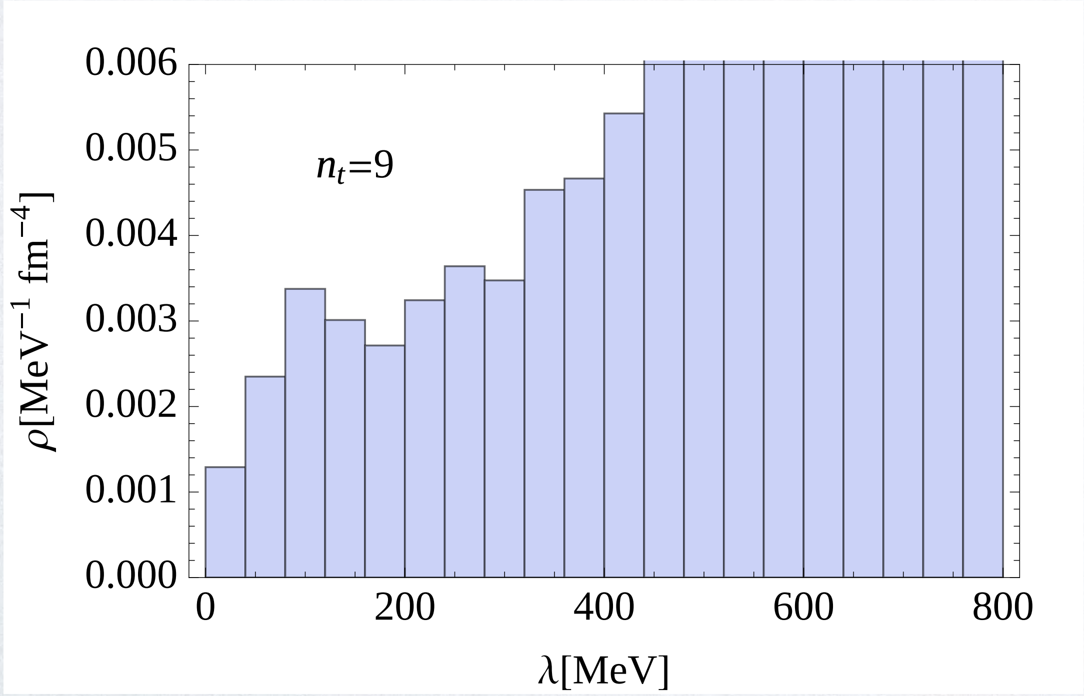
NEAR-ZERO PEAK MODES

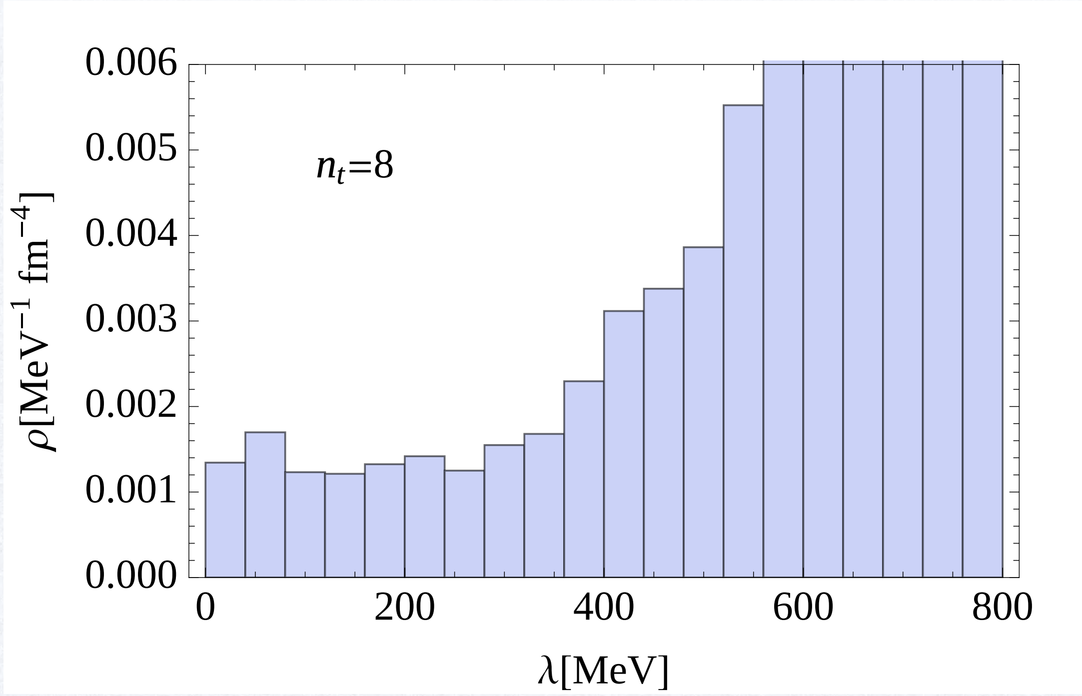
T/T_c=0.87 | T/T_c=0.98 | T/T_c=1.12

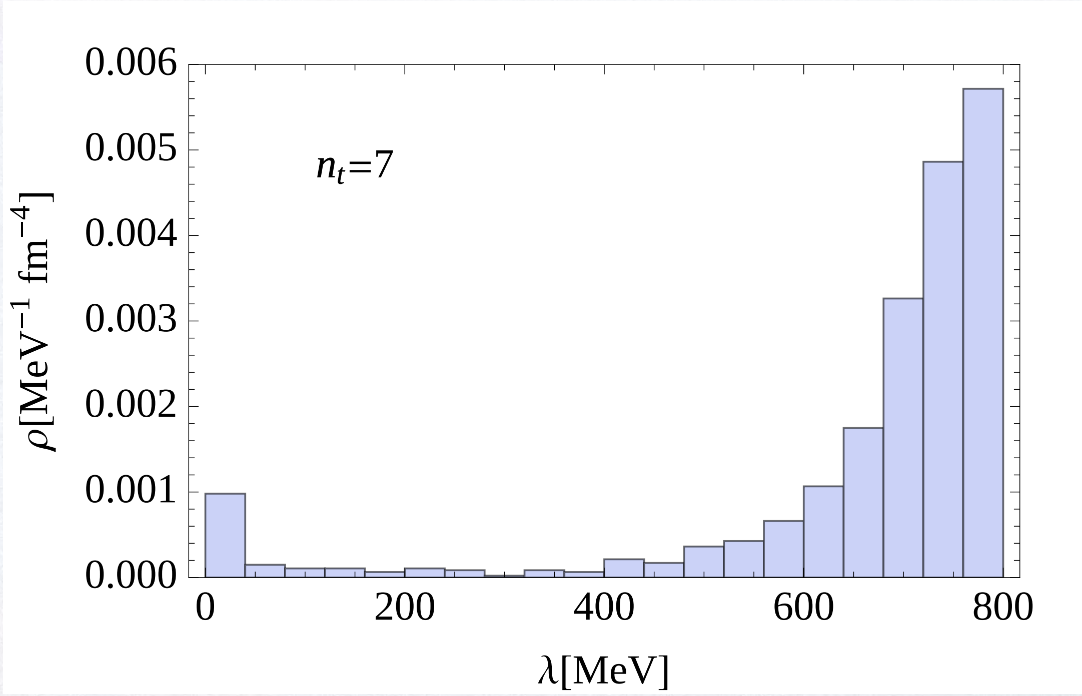










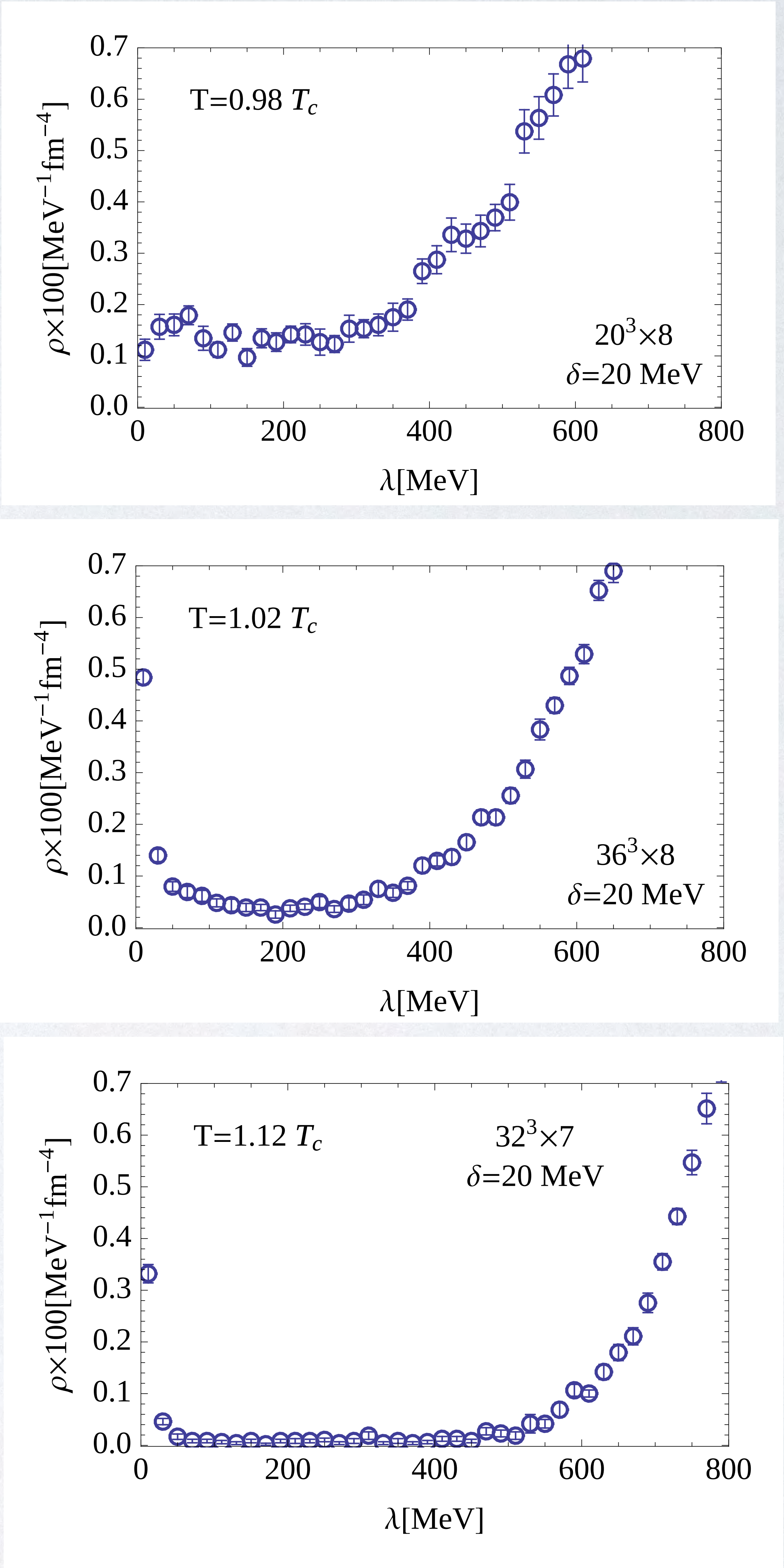


R. G. Edwards, U. M. Heller, J. E. Kiskis, and R. Narayanan, Phys.Rev. D61 (2000) 074504

NEAR-ZERO PEAK MODES

Dirac spectra around deconfinment

- We verified that the peak survives the thermodynamic limit and continuum limit
- We found that the peak appears above the deconfinement transition
- For pure glue theory the transition is sharp and coincides with Tc



AA and I. Horvath, Phys. Rev. D92 (2015), no. 4 045038

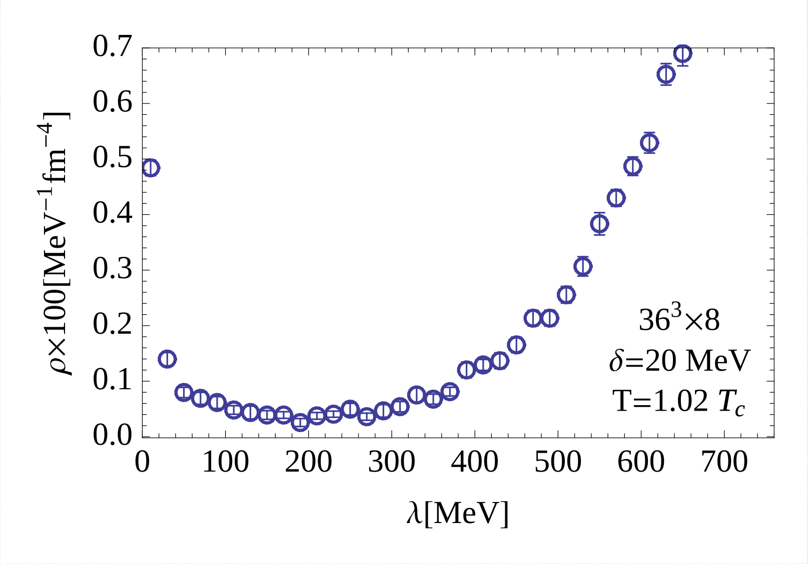
IR PHASE

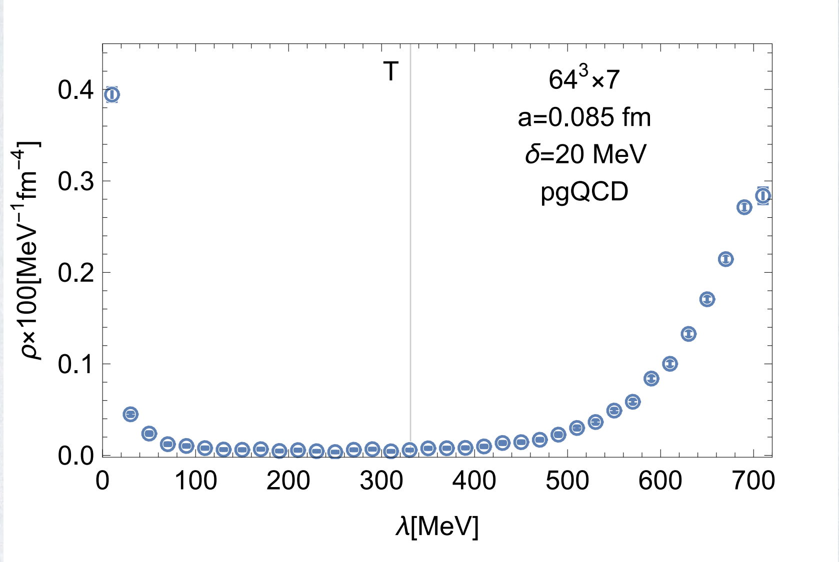
Dirac spectra for QCD like theories

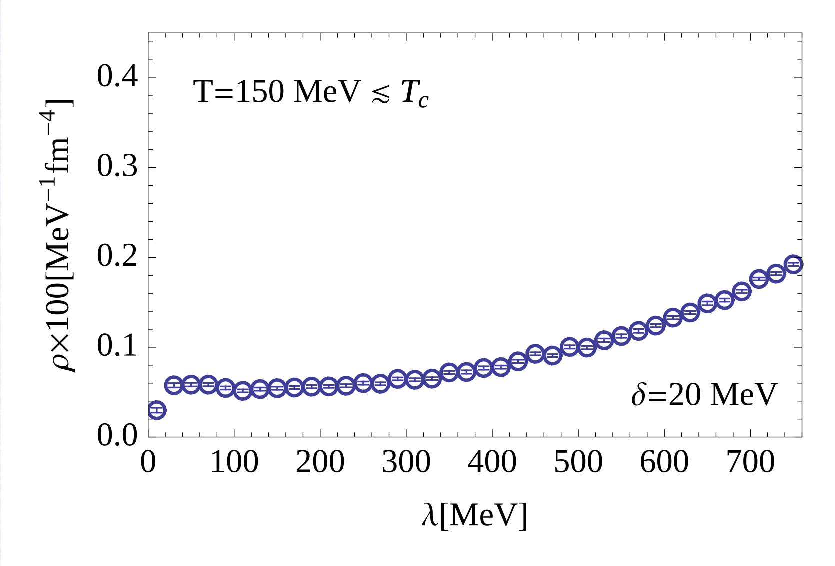
- Pure gauge theories at temperature above Tc have unusual behavior

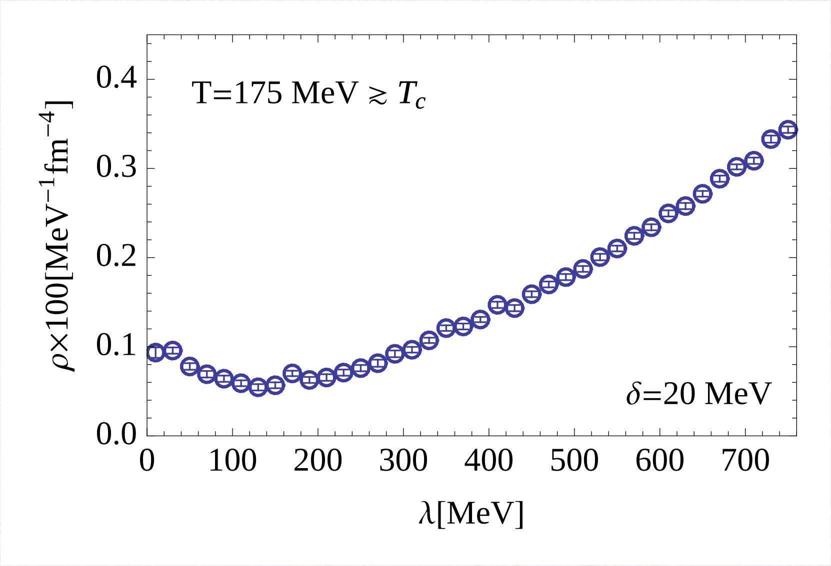
- The same qualitative behavior is present with dynamical quarks

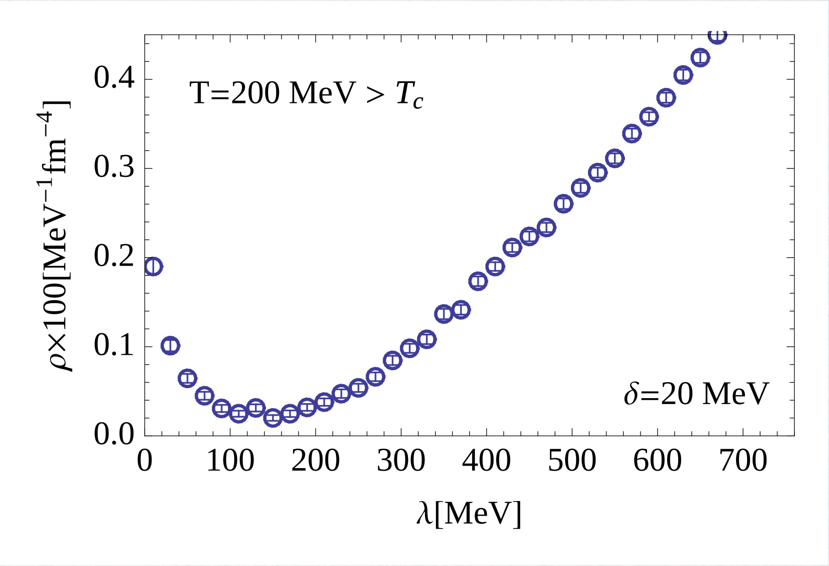
- Similar behavior is visible in theories with Nf=12 light quarks at T=0

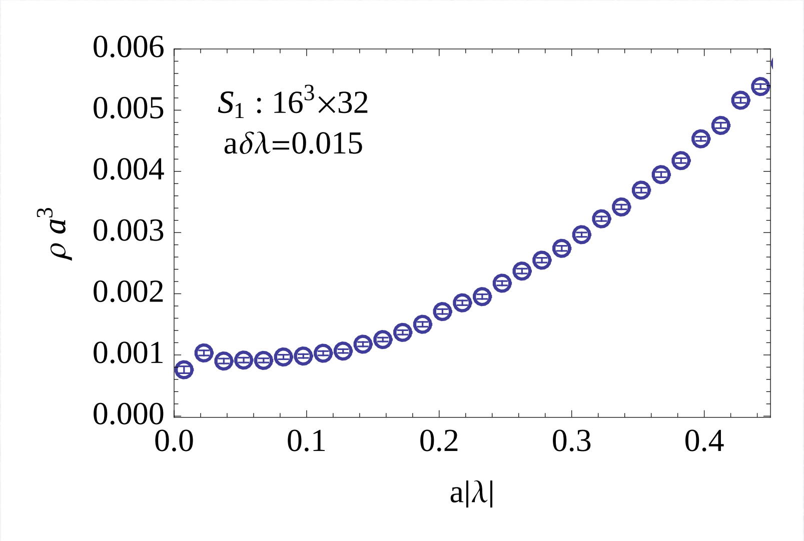


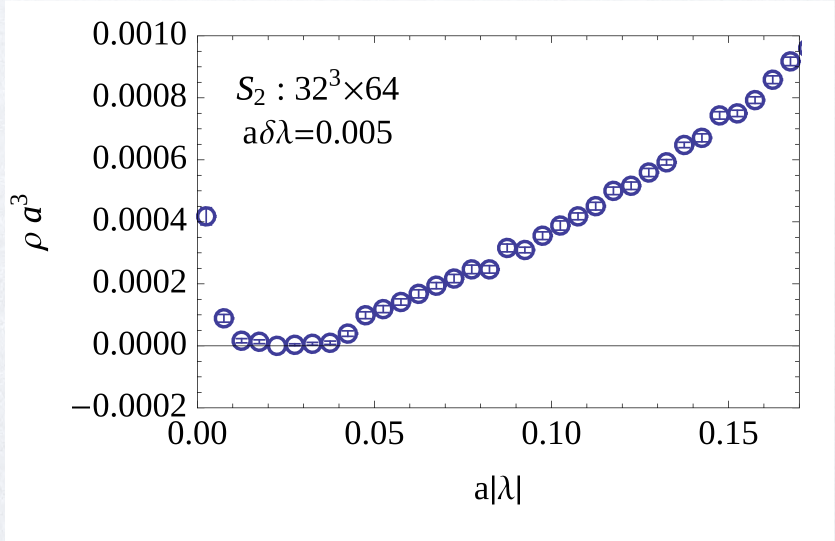










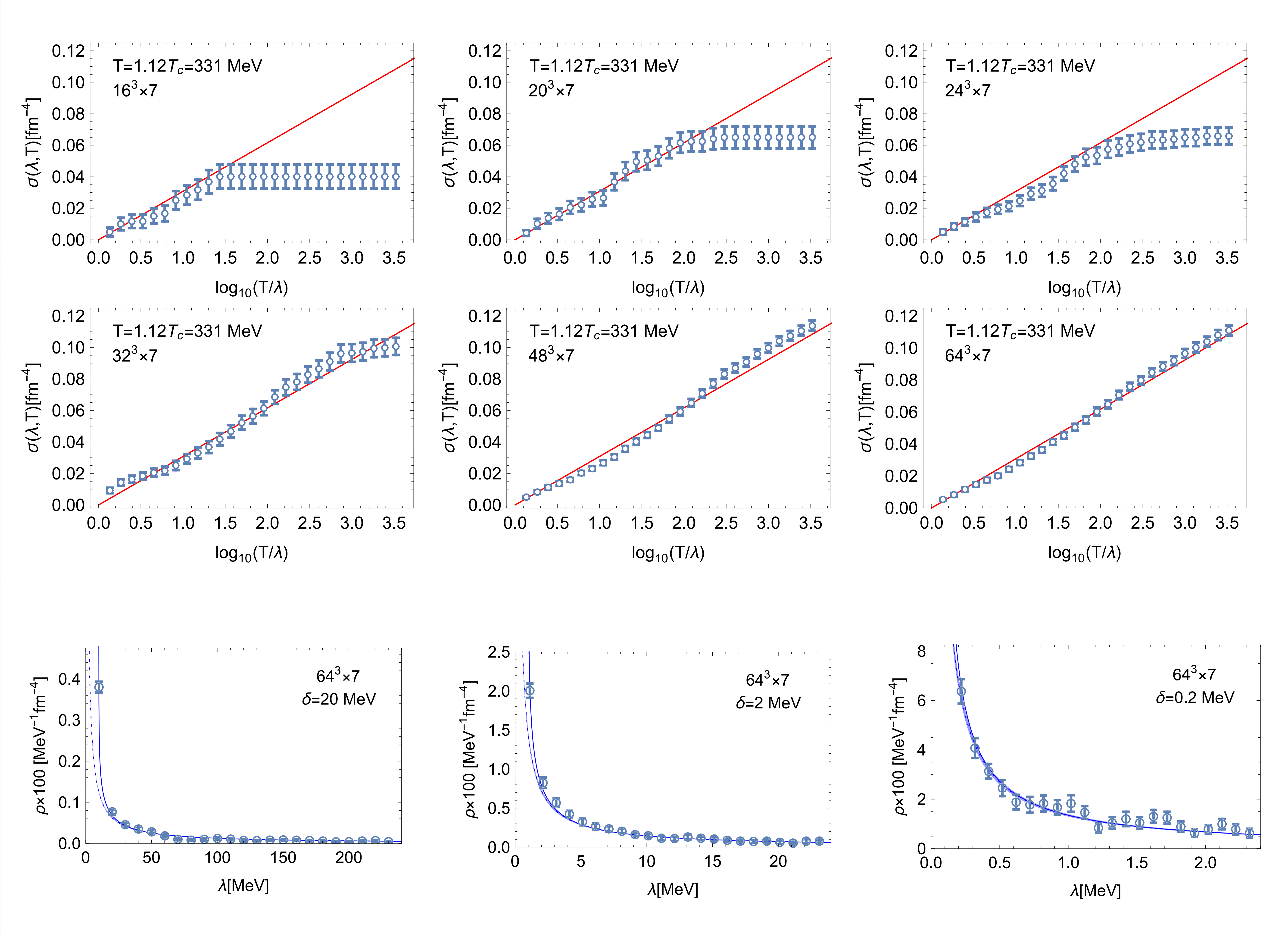


AA and I. Horvath, Nucl.Phys. B891 (2015) 1–41

IR PHASE

Low-lying Dirac spectrum properties

- The spectrum separates in two modes: the "bulk" and an IR peak
- As we increase the volume the peak becomes more pronounced
- The density in the IR peak seems to be to a very good approximation $\rho(\lambda) \propto 1/\lambda$

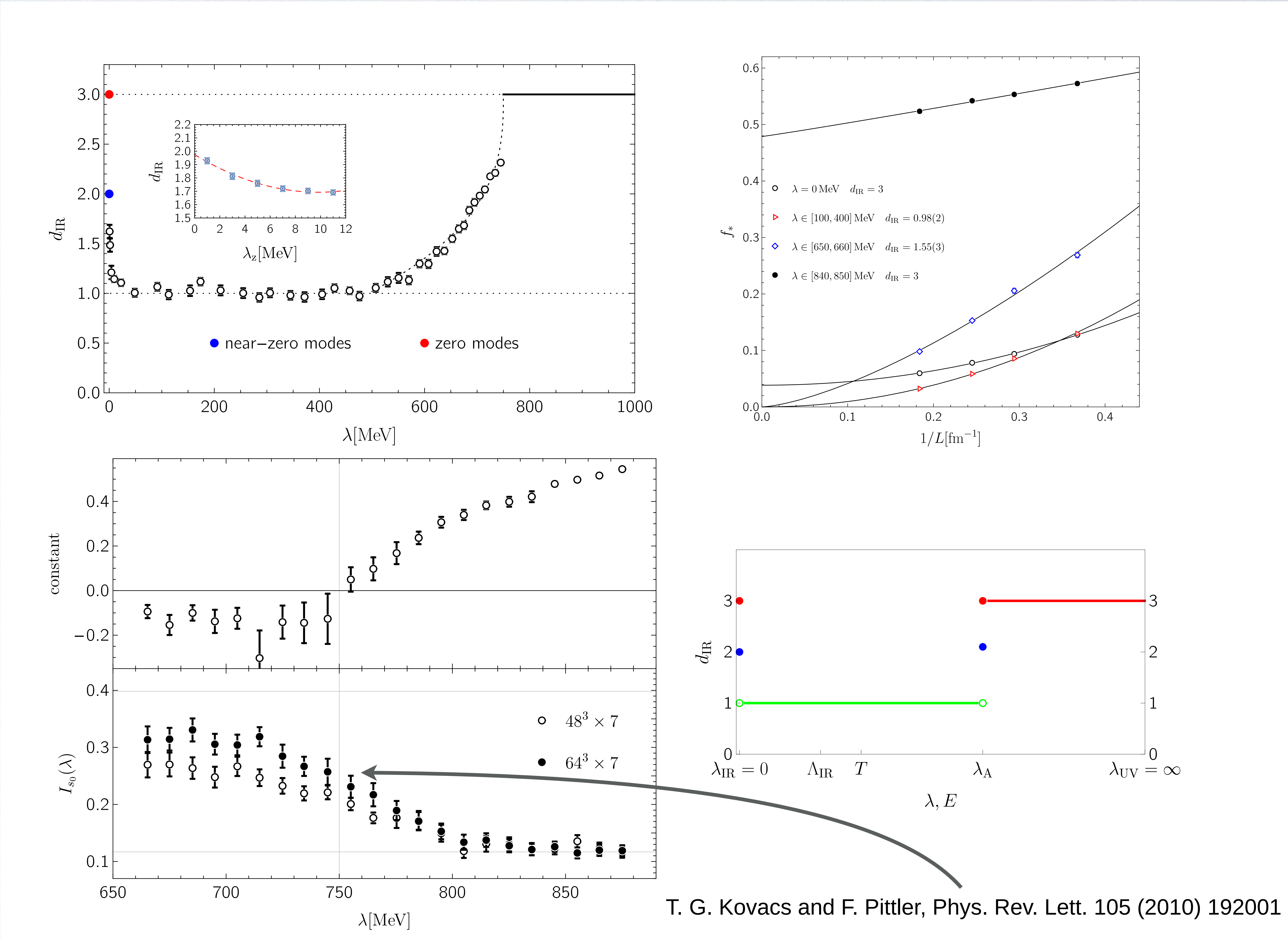


AA and I. Horvath, Phys. Rev. D100 (2019), no. 9 09450

IR DIMENSION

Eigenmode support scaling

- The "support" for each eigenvector, is roughly the number of points N where $|\psi(x)|^2$ is above average
- The IR dimension is defined by the scaling with volume (at fixed UV cutoff): $N \propto L^{d_{IR}}$
- We find that the dimension depends on the spectral band: bulk (~3), gap (~1), IR peak (2), zero modes (~3)
- The transition between bulk and gap is close to the mobility edge and we conjecture that they coincide in the infinite volume limit



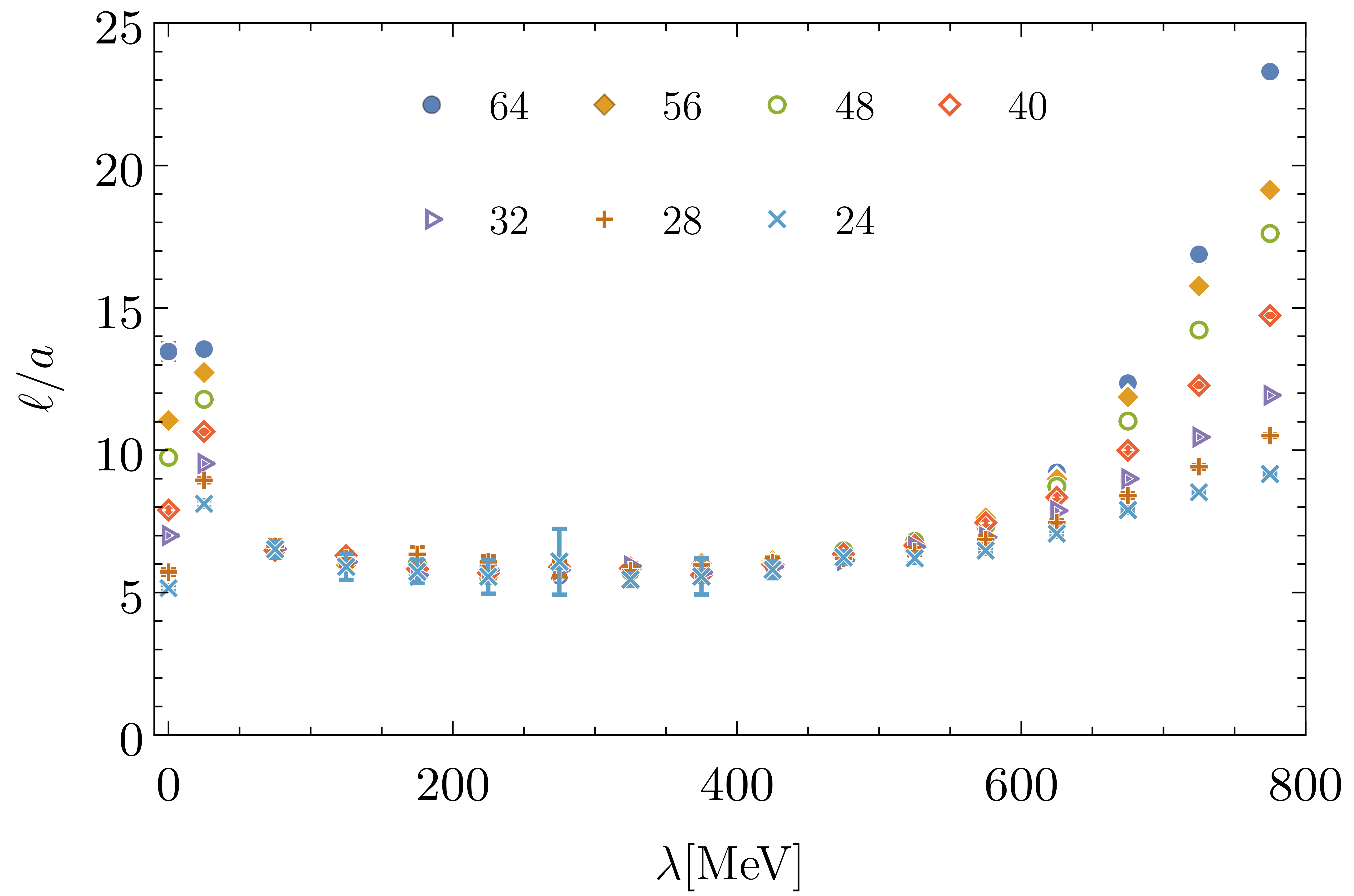
T. G. Kovacs and F. Pittler, Phys. Rev. Lett. 105 (2010) 192001

AA and I. Horvath, Phys. Rev. Lett. 127 (2021), no. 5 052303

MODE EXTENT

Eigenmode extent

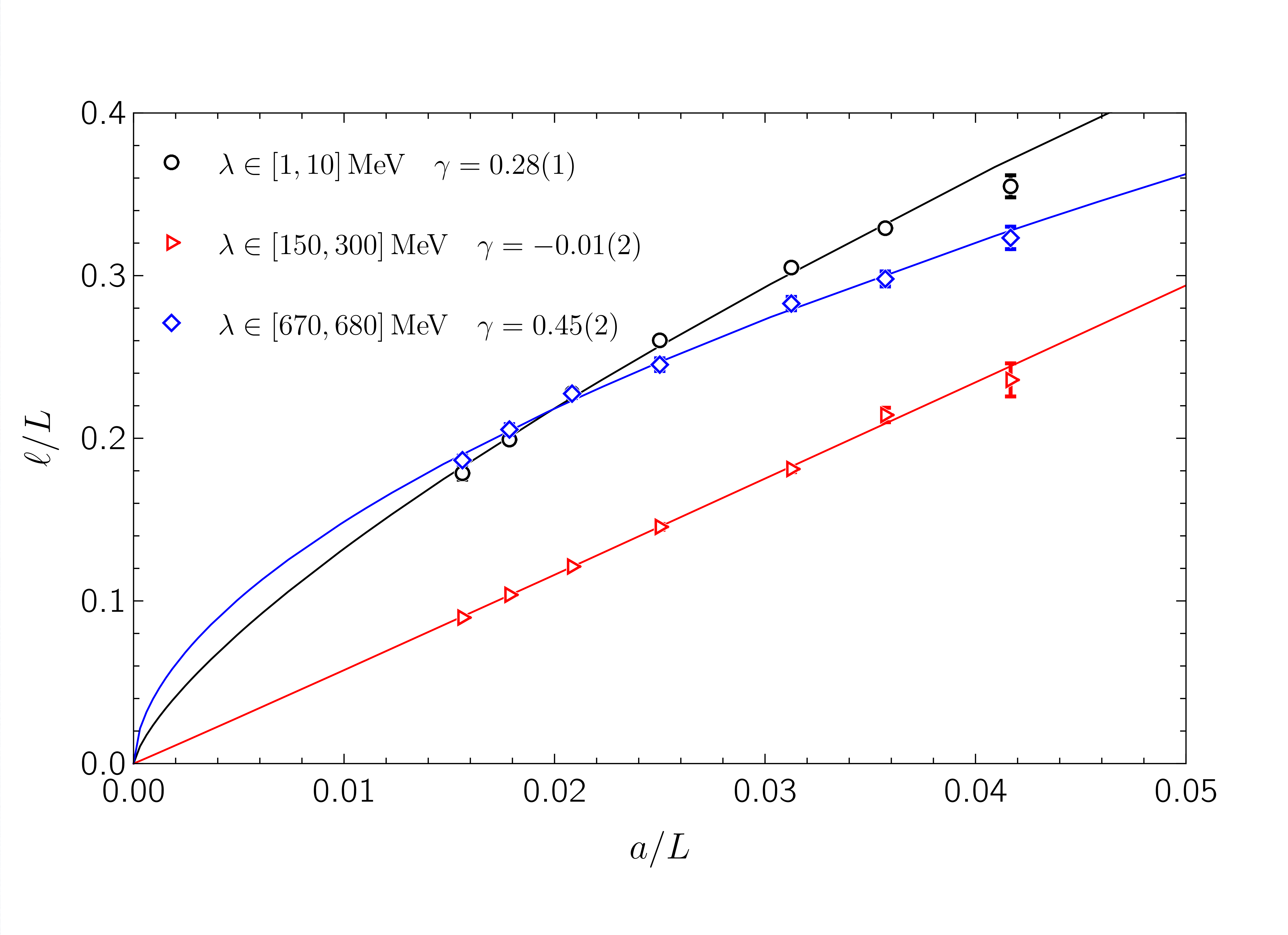
- The "extent" of each eigenmode is given by the weighted average distance from the maximum point
- The weight is controlled by the local magnitude of the eigenvector $p(x) = |\psi(x)|^2$
- The average extent is $\ell = \sum_x p(x)\,|x - x_*|$
- In the "gap" the size of the modes seems volume independent, consistent with localized modes
- For both the "bulk" and "peak" modes the extent varies with the volume



MODE INDEX

Coefficient of scaling

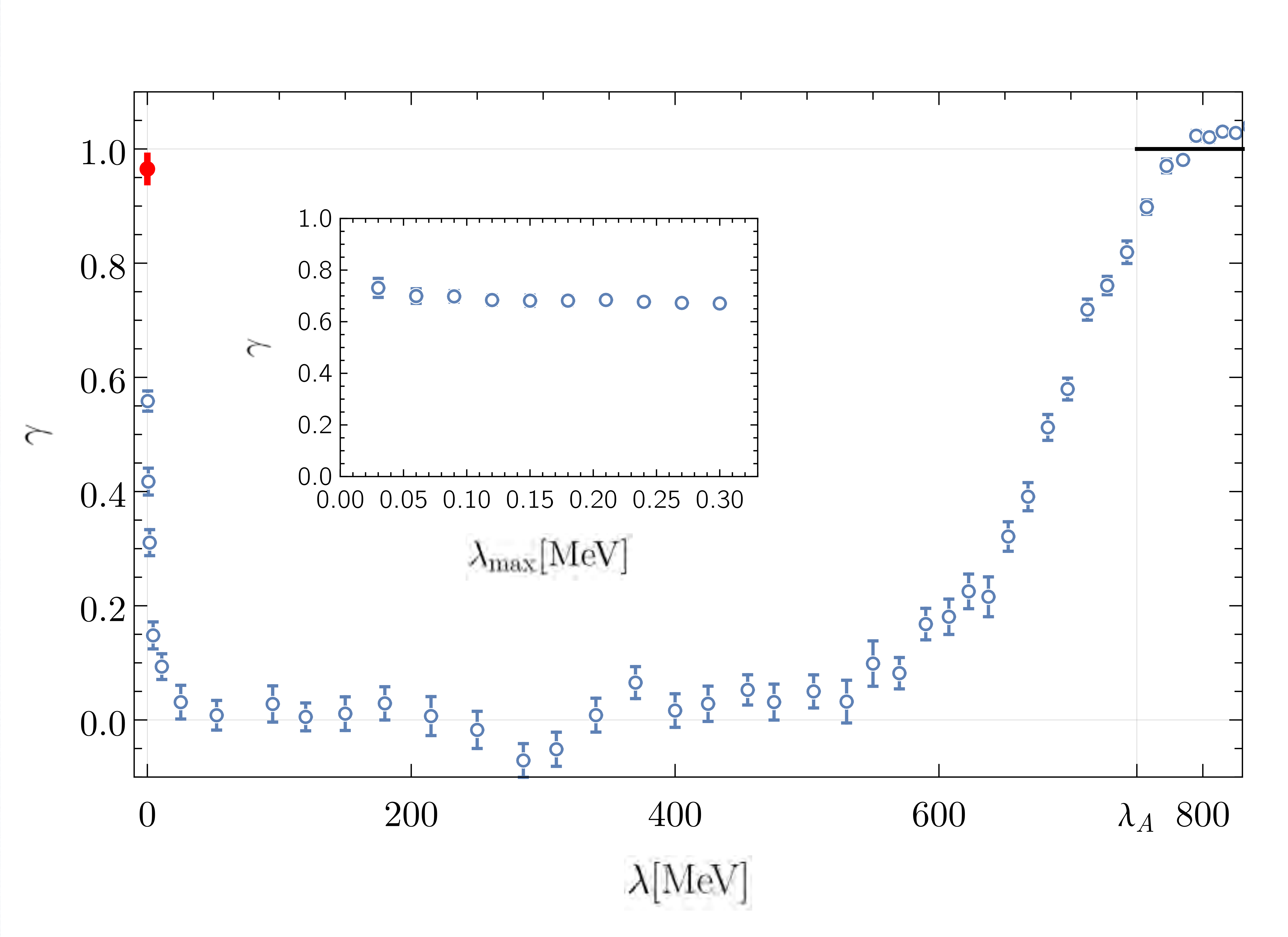
- To characterize the localization properties of the modes we define a "mode index" that quantifies the scaling of the mode size with the size of the box
- The mode index is defined via $\ell \propto L^{\gamma}$ with $0 \leq \gamma \leq 1$
- The index is calculated by fitting the mode extent as a function of the size of the box
- The fits here correspond to typical spectral bands in the "peak", the "gap", and close to the mobility edge



MODE INDEX

Coefficient of scaling as a function of λ

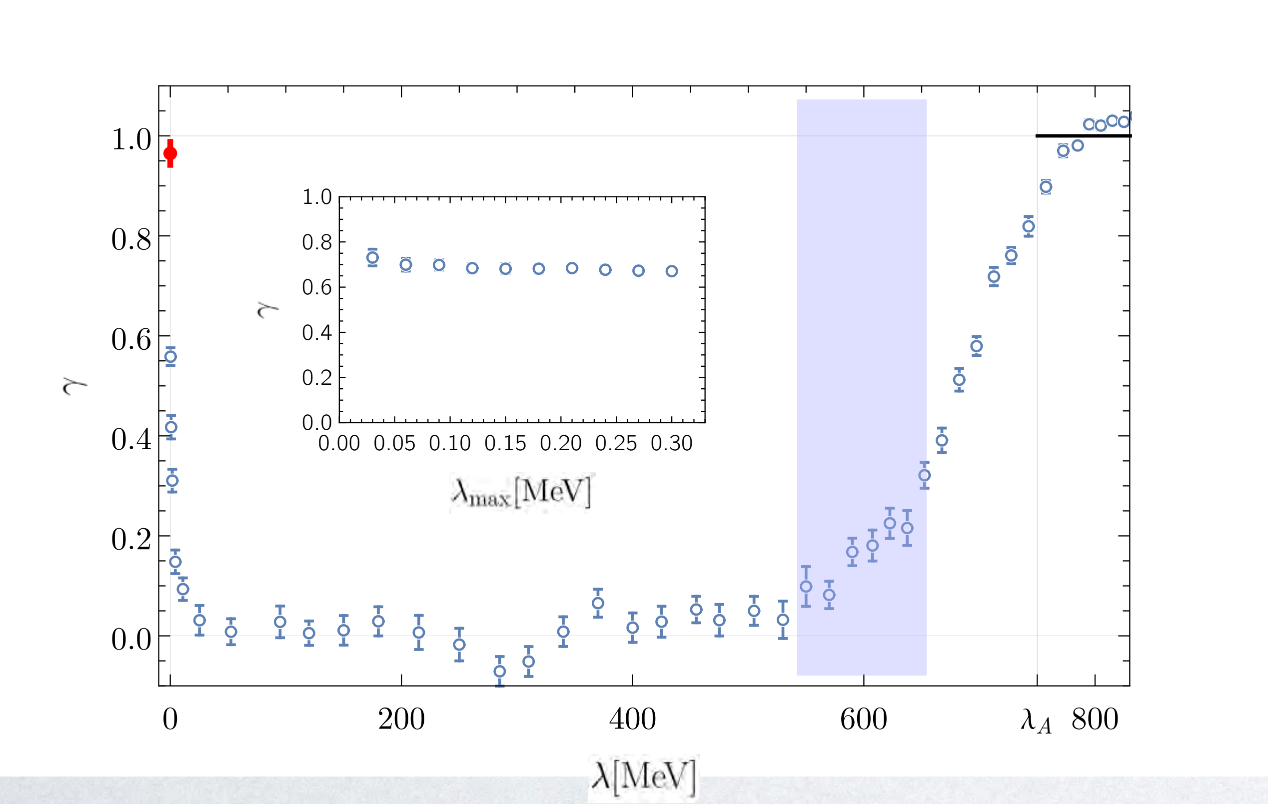
- We calculated the index as a function of the spectral band
- In the "gap" where the modes are localized, the index is 0
- For both the "bulk" modes higher than λ_A, the index is 1, as expected for the "plane-wave" like modes
- Similarly for zero-modes the index is 1, since these modes are delocalized
- We note that for modes around $\lambda = 0^+$ the index we computed is different from 1 (similarly for $\lambda \approx \lambda_A$)

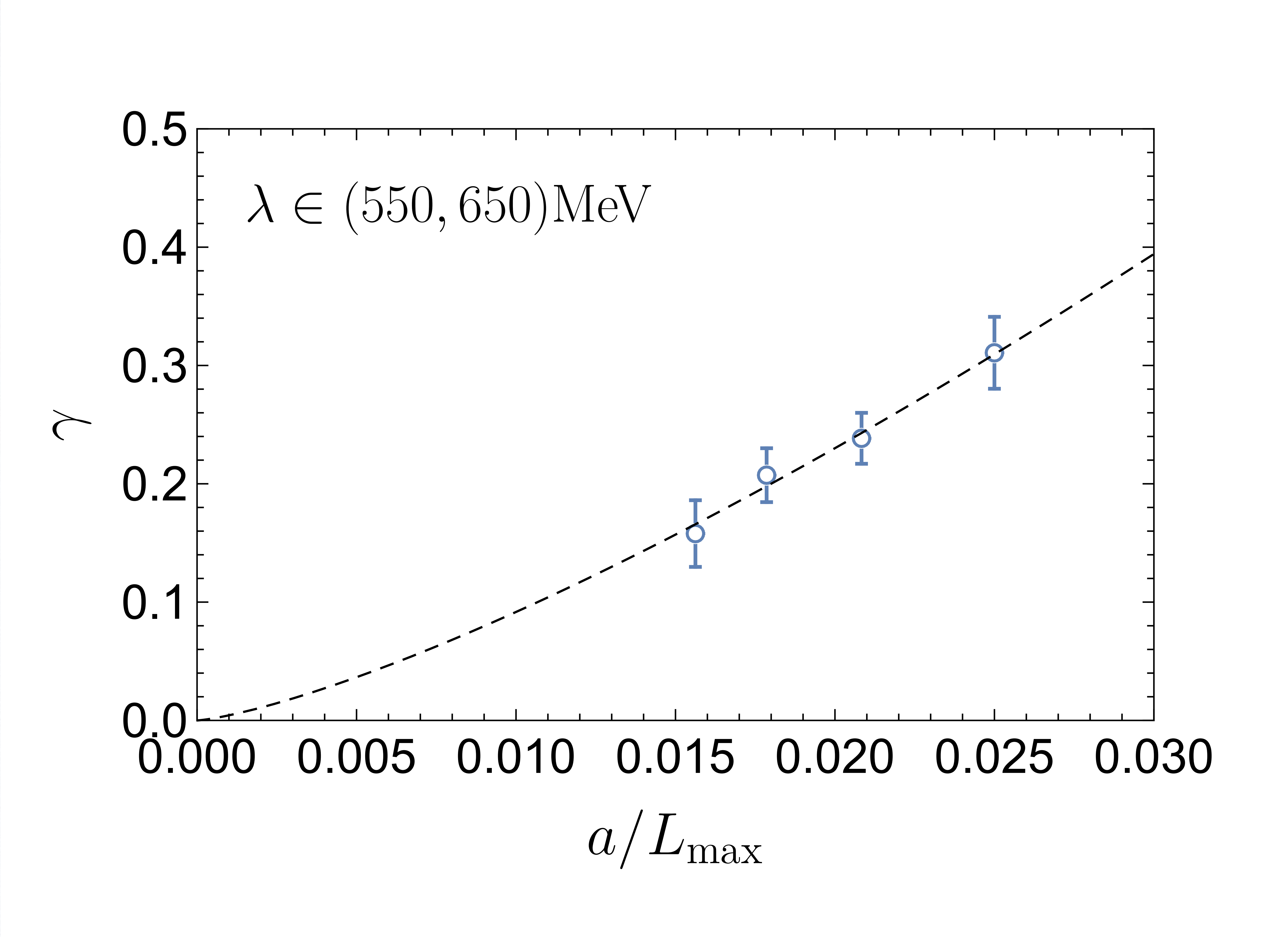


INDEX — THERMODYNAMIC LIMIT

Sliding fit window

- The results for gamma index were computed using fits for all volumes available
- In the transition regions a more detailed view is required to estimate the infinite volume limit
- Here we perform the fits using a sliding window, using 4 consecutive volumes with increasing size

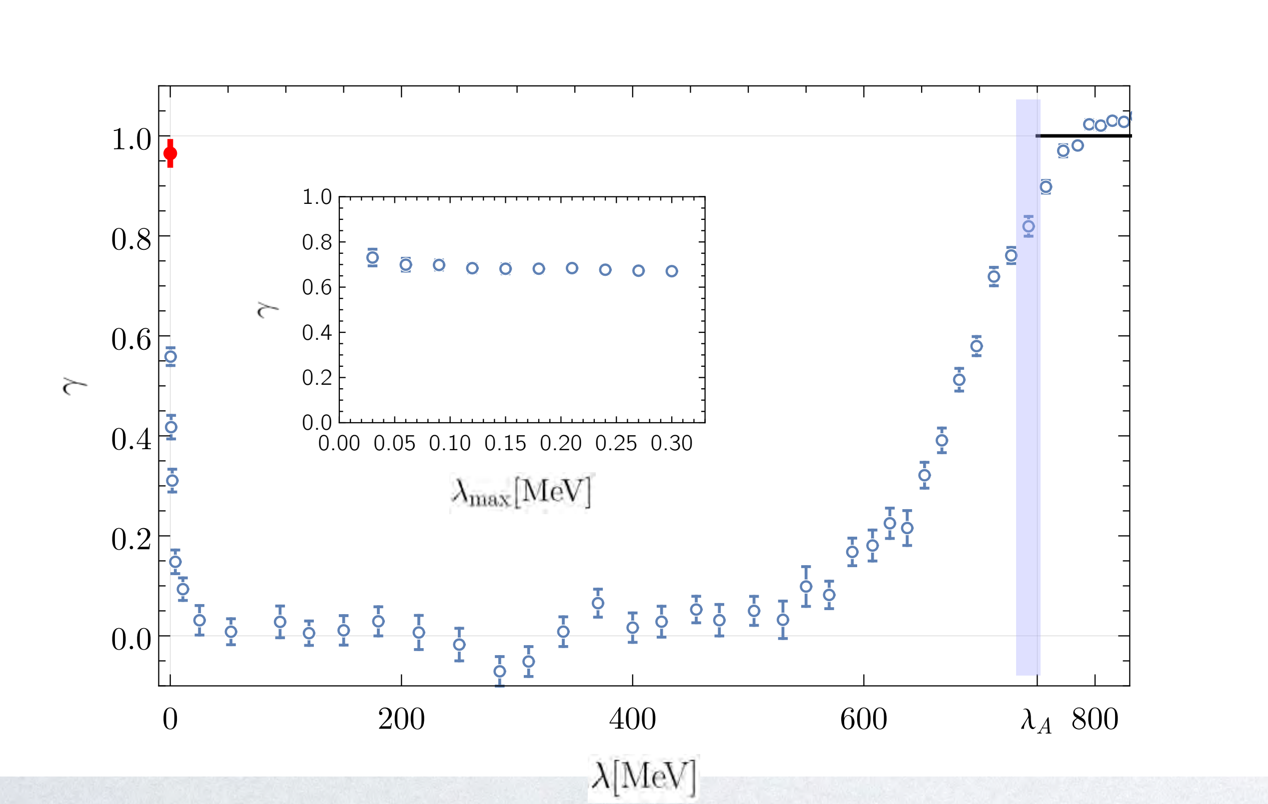


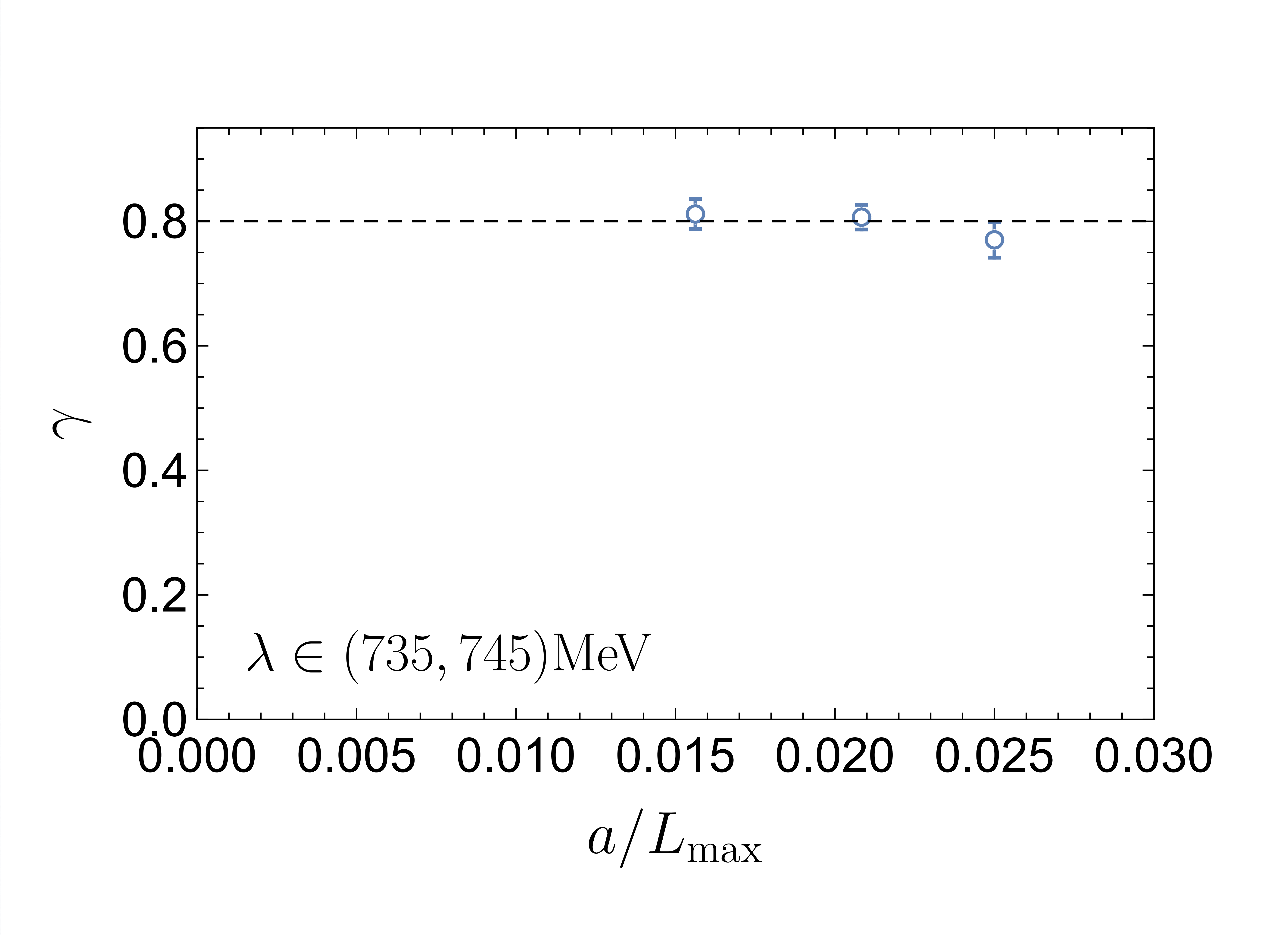


INDEX — THERMODYNAMIC LIMIT

Sliding fit window

- The results for gamma index were computed using fits for all volumes available
- In the transition regions a more detailed view is required to estimate the infinite volume limit
- Here we perform the fits using a sliding window, using 4 consecutive volumes with increasing size

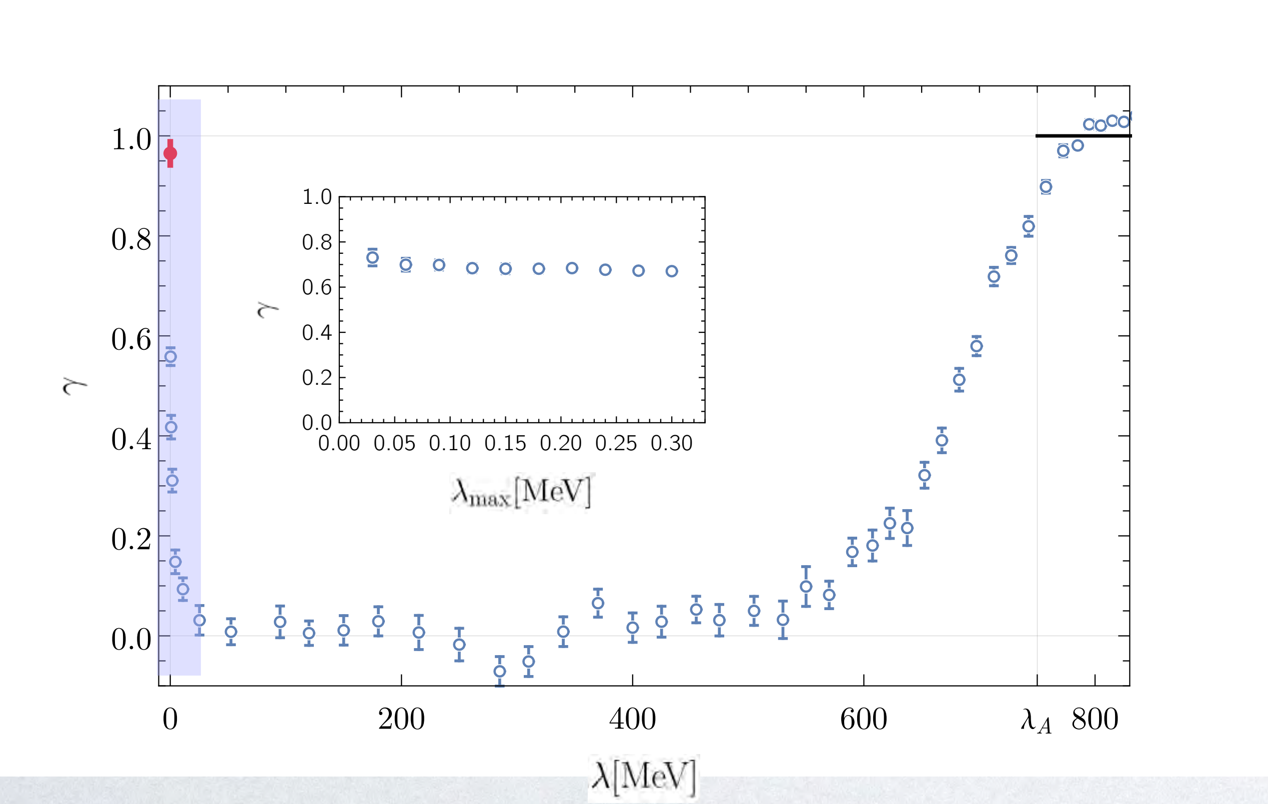


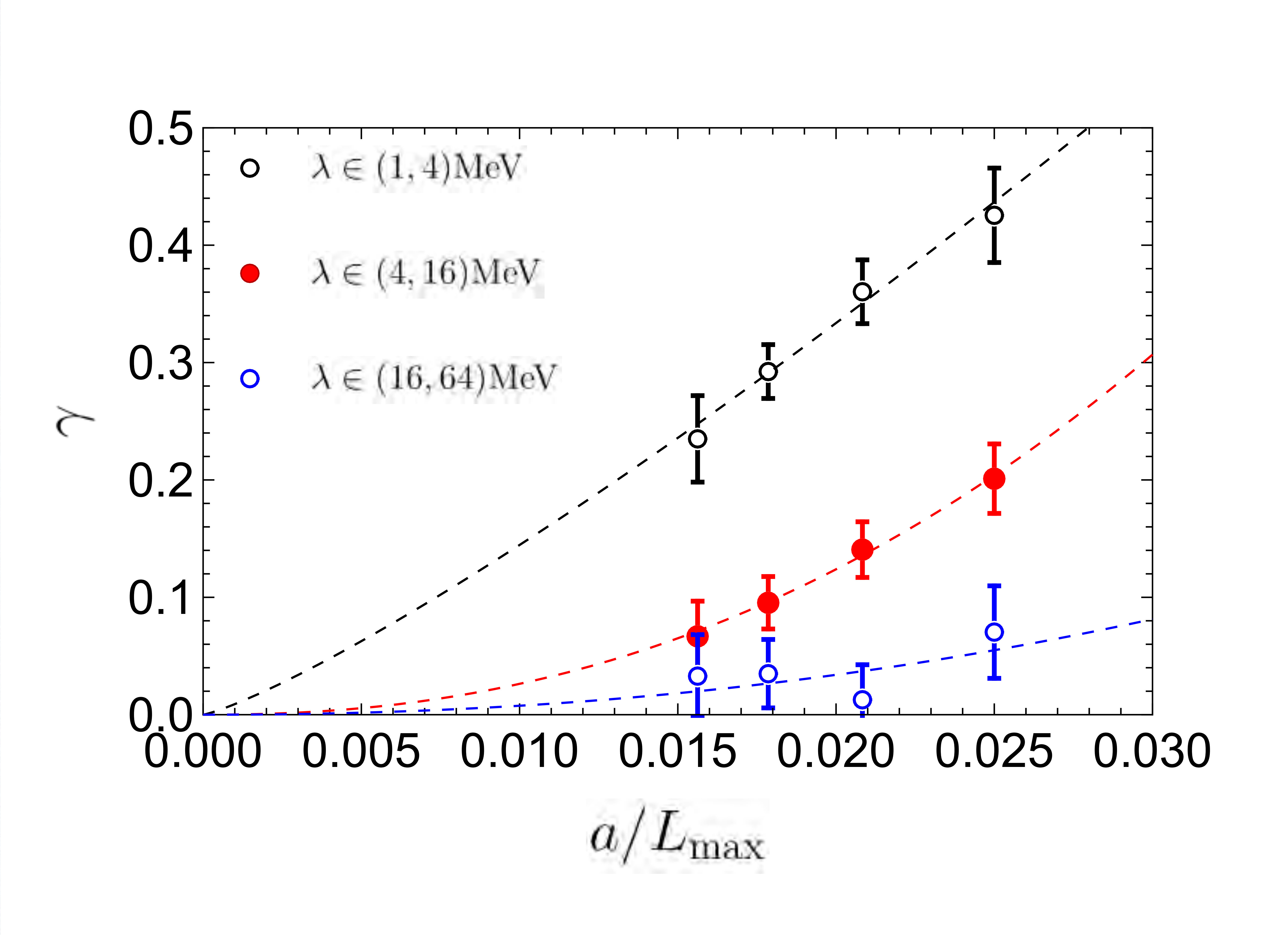


INDEX — THERMODYNAMIC LIMIT

Sliding fit window

- The results for gamma index were computed using fits for all volumes available
- In the transition regions a more detailed view is required to estimate the infinite volume limit
- Here we perform the fits using a sliding window, using 4 consecutive volumes with increasing size

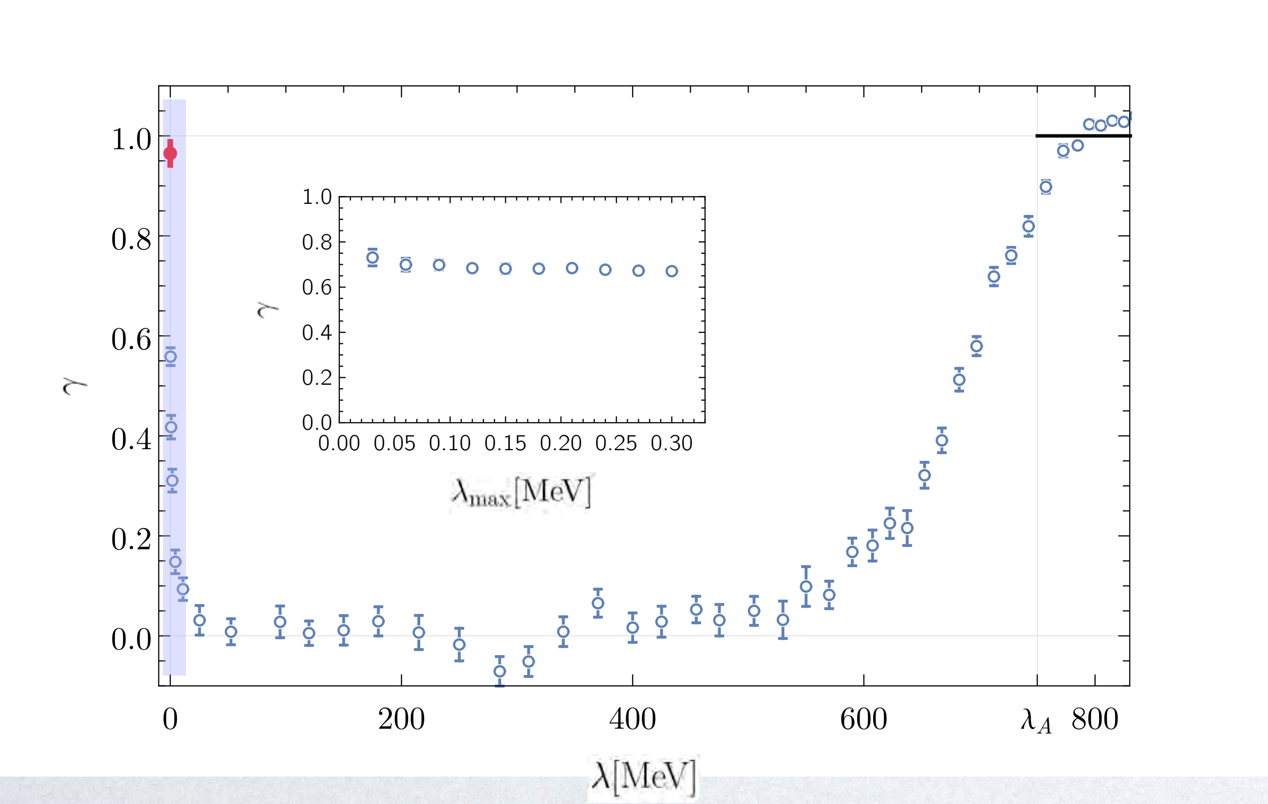


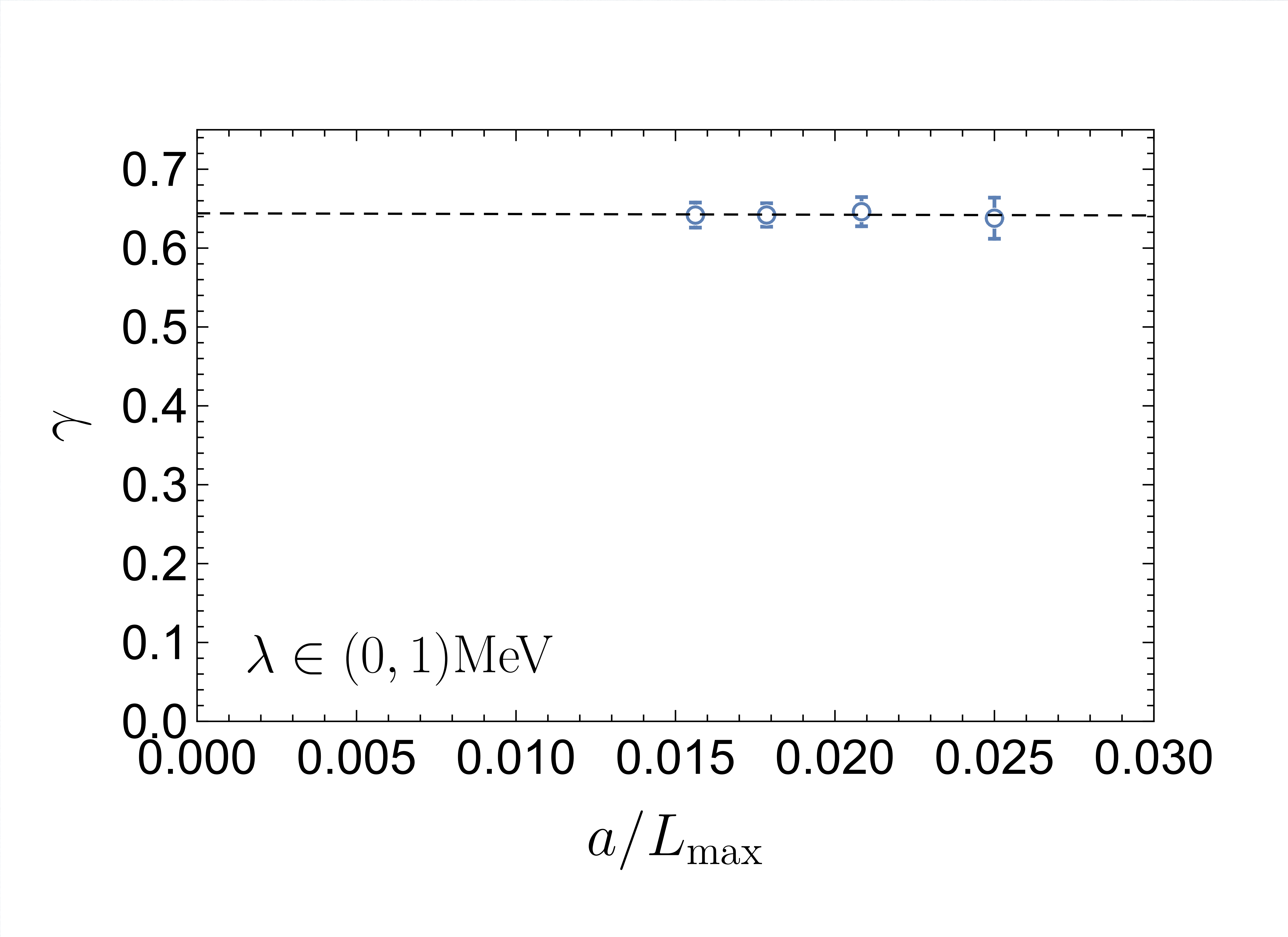


INDEX — THERMODYNAMIC LIMIT

Sliding fit window

- The results for gamma index were computed using fits for all volumes available
- In the transition regions a more detailed view is required to estimate the infinite volume limit
- Here we perform the fits using a sliding window, using 4 consecutive volumes with increasing size





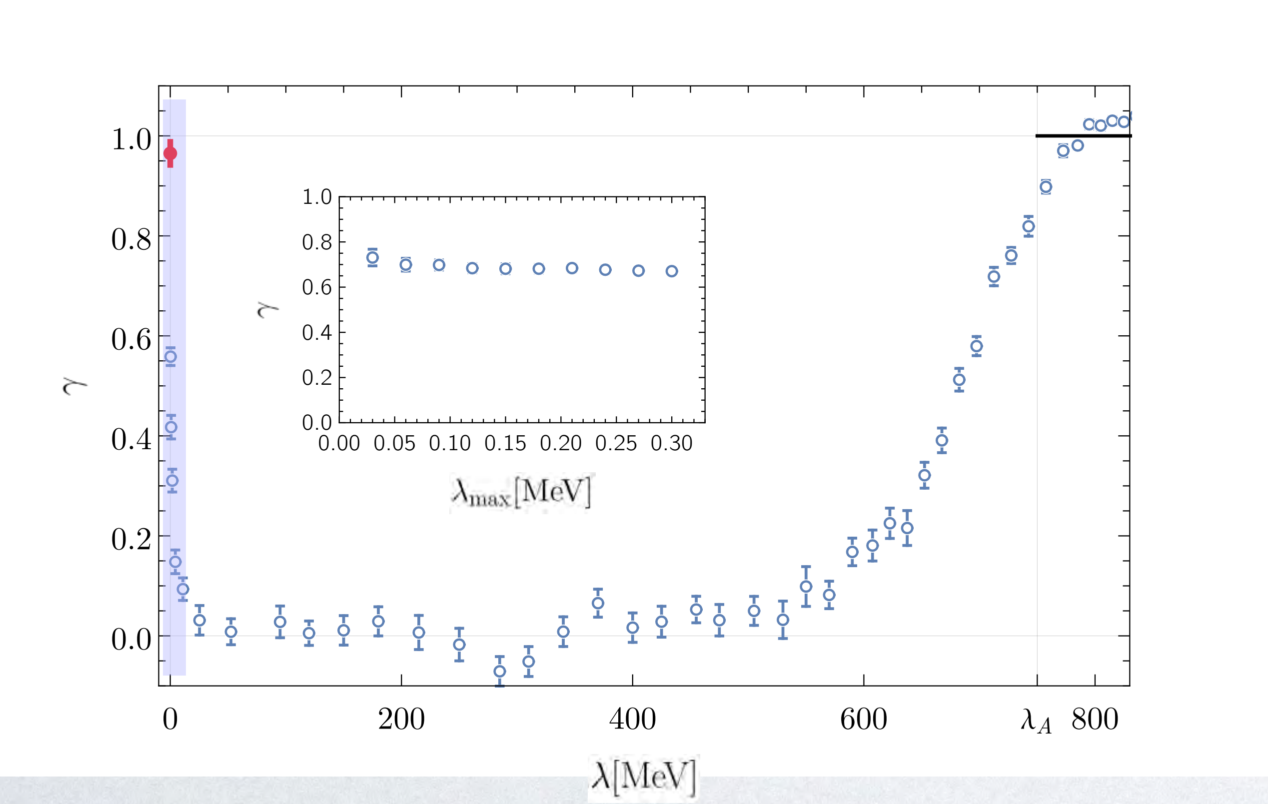
INDEX — THERMODYNAMIC LIMIT

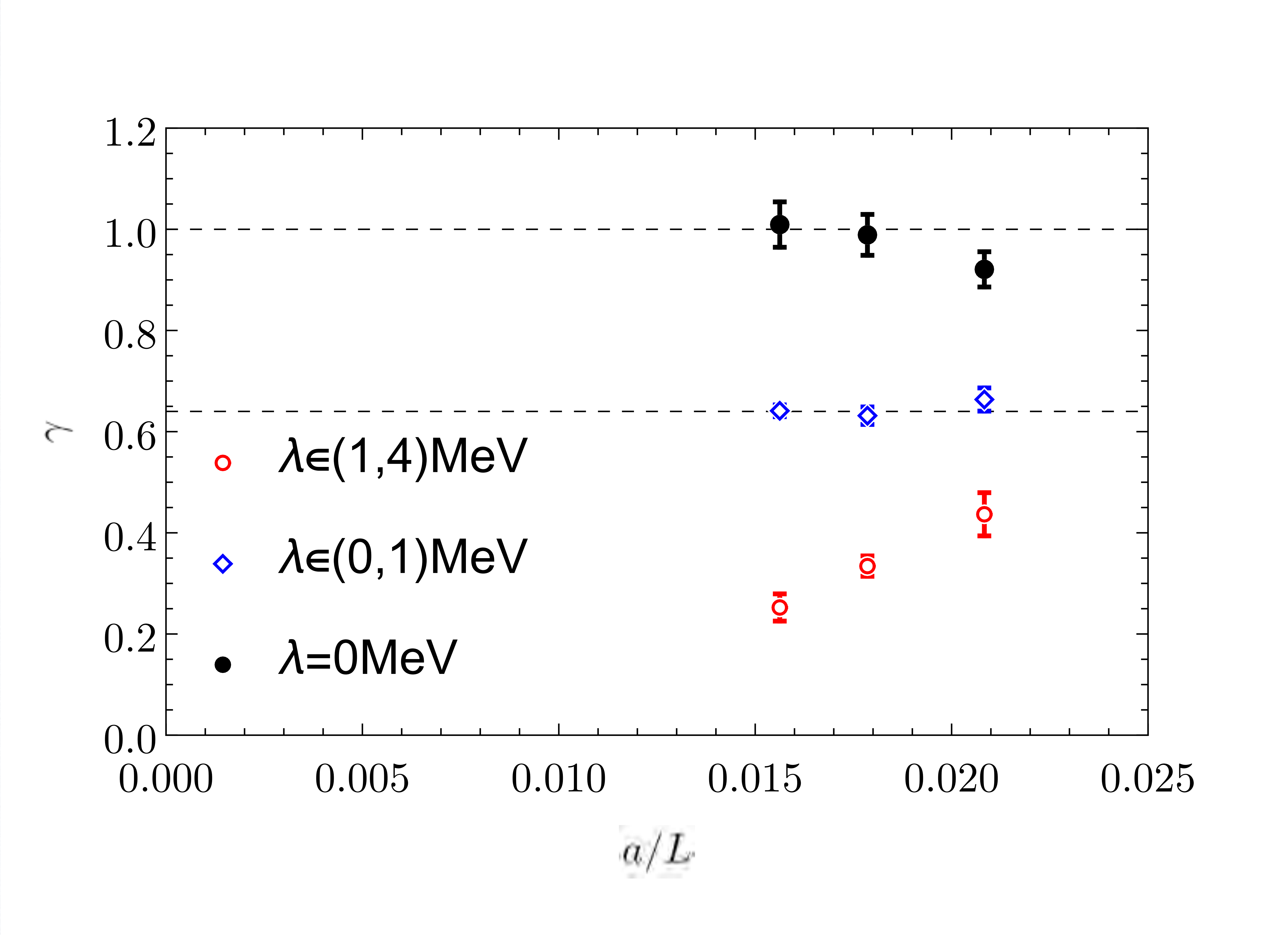
Gamma index from the ratio method

- To double-check our results we computed the index using ration method

$$\gamma \equiv \frac{1}{\log 2} \log \frac{\ell(2L)}{\ell(L)}$$

- We use three volume pairs (24,48), (28,56), and (32,64) and we found results that are compatible with the fitted value

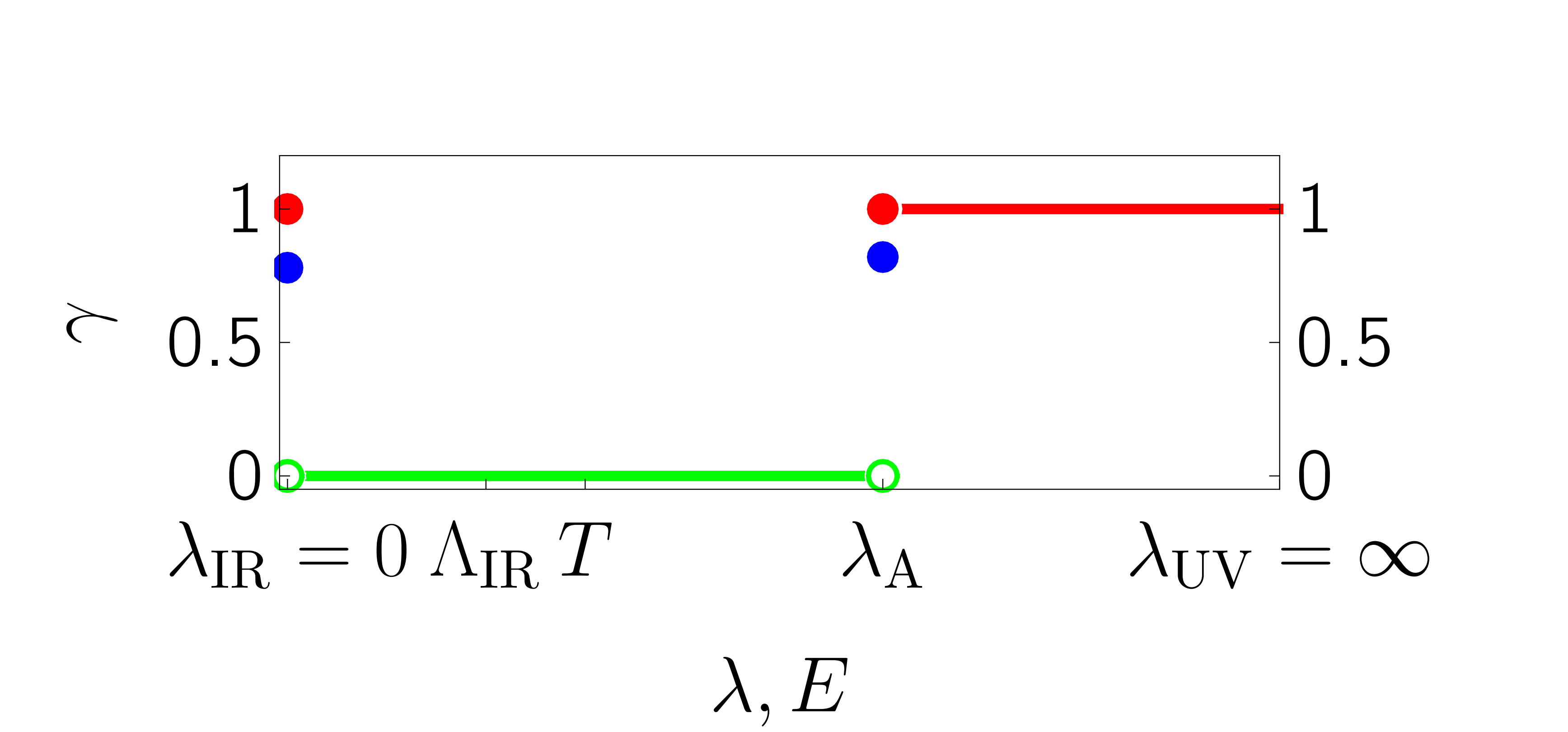


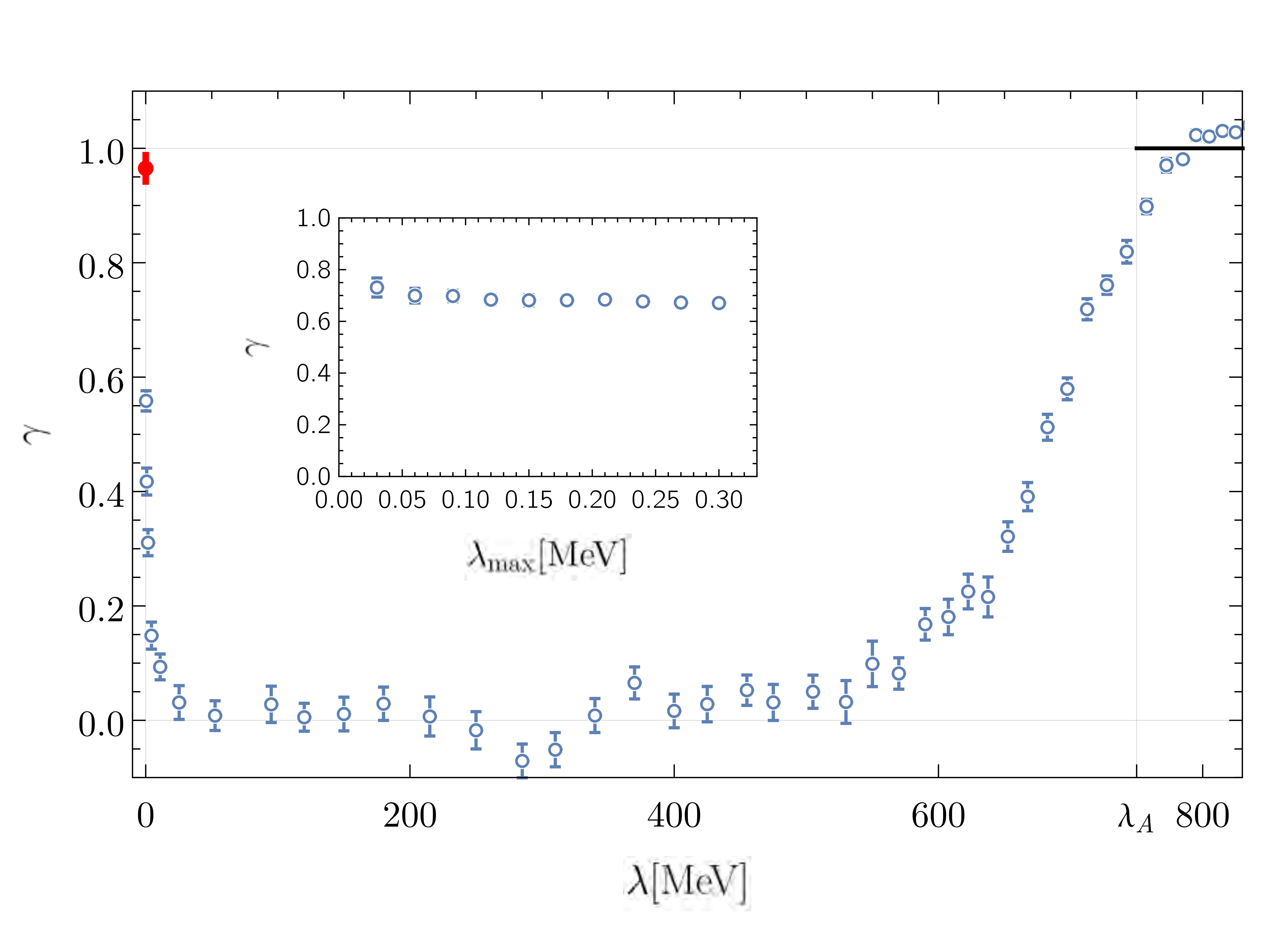


INDEX — THERMODYNAMIC LIMIT

Conjectured infinite volume limit

- The "bulk" modes and the zero-modes scale linearly with the size of the box
- The "gap" modes are localized, that is do not depend on the box size
- The critical regions at $\lambda = 0^+$ and $\lambda \approx \lambda_A$ seem to be delocalized but their radius scales with a power lower than 1.

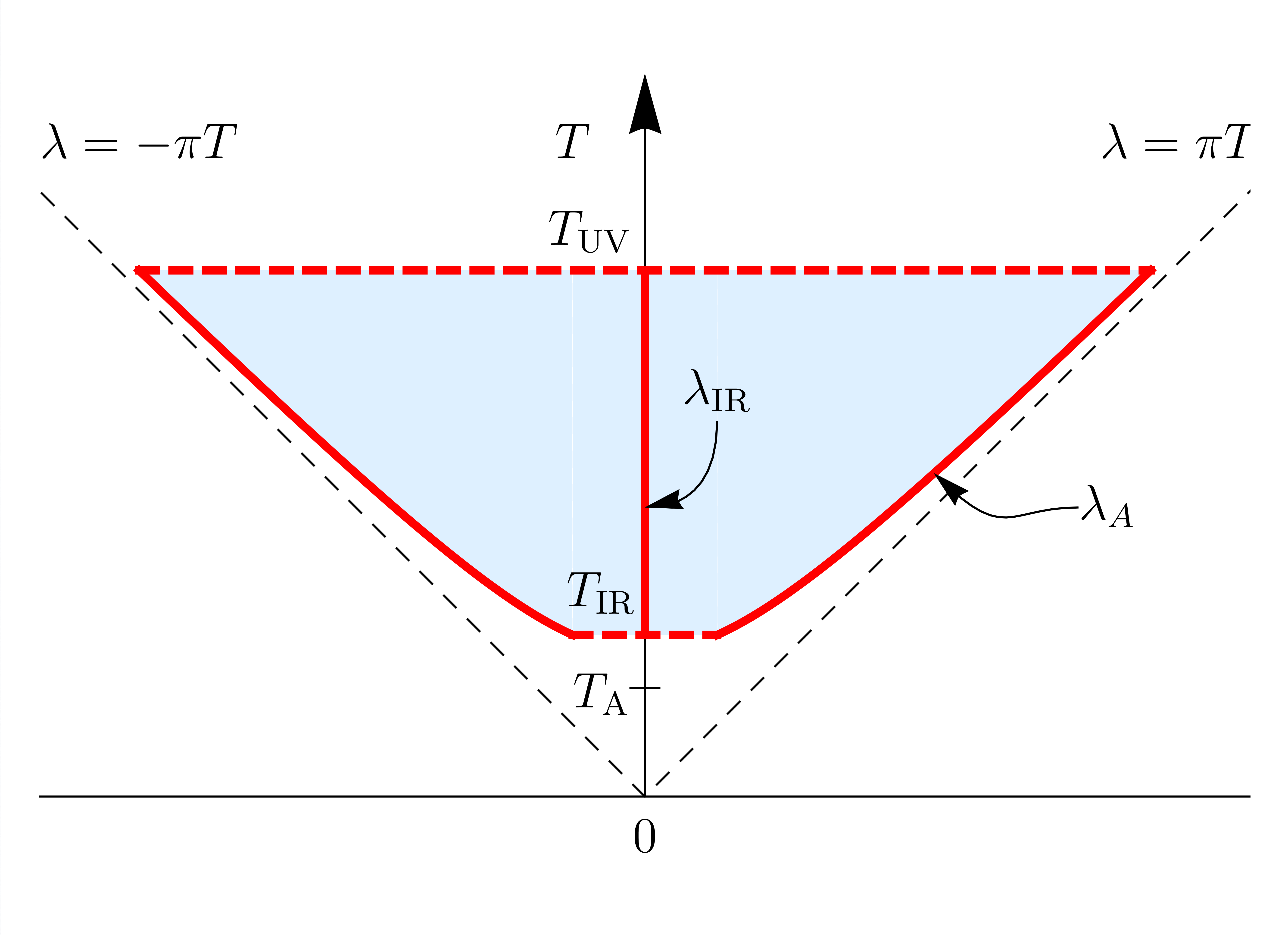




CONJECTURED PHASE DIAGRAM

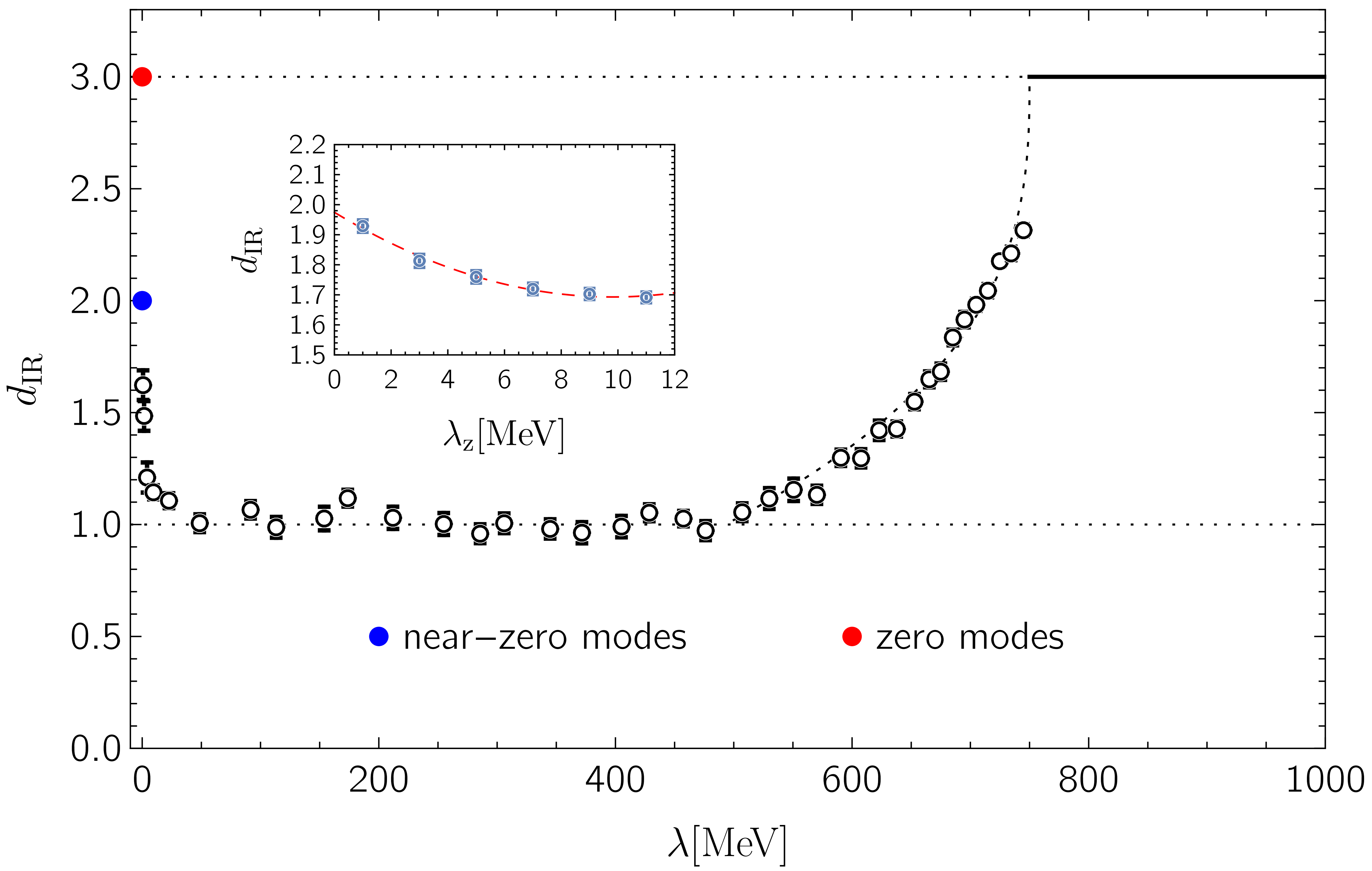
Localized/delocalized spectrum

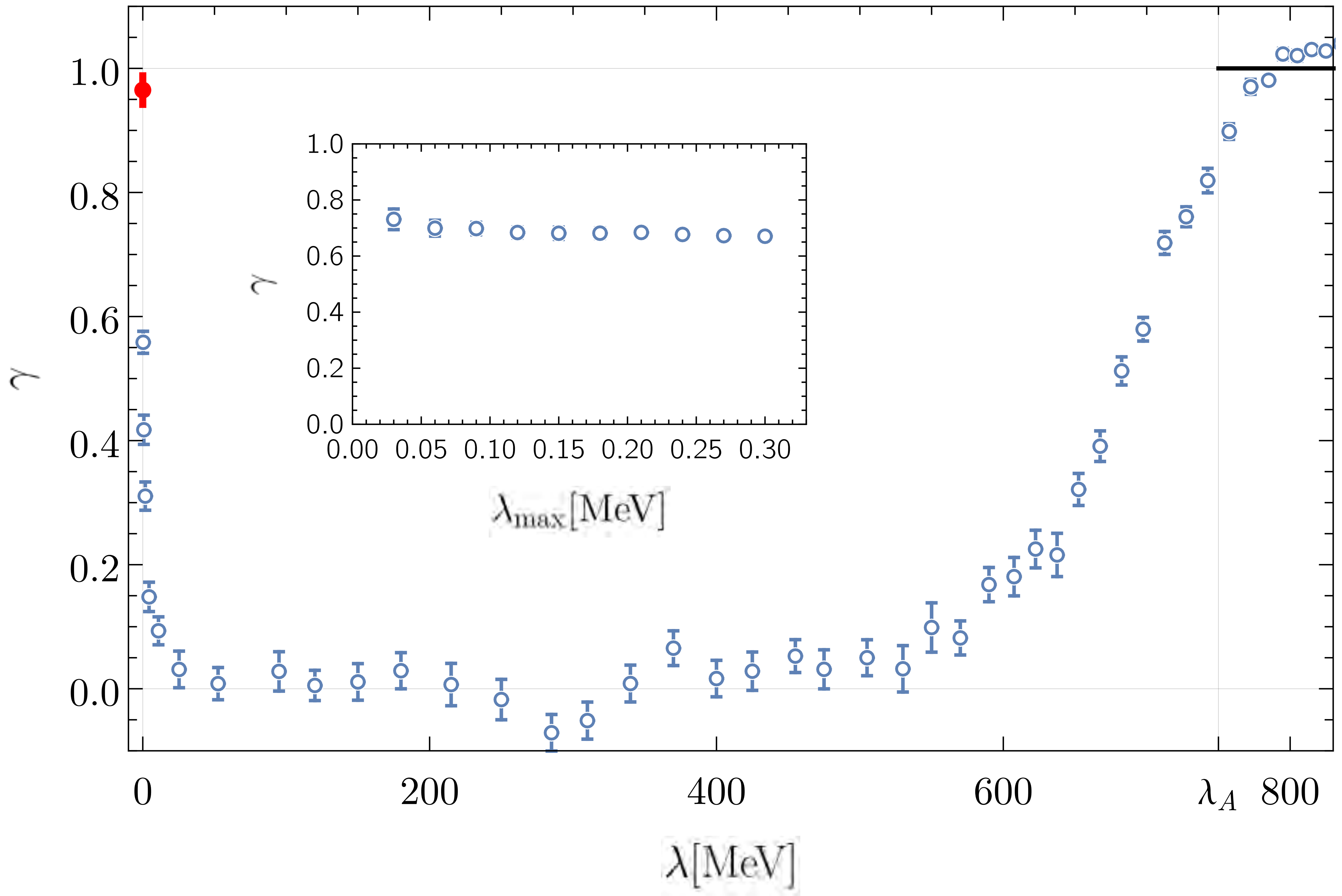
- At low temperature the eigenmodes of the Dirac operator are all delocalized
- For high temperature, above T_{IR}, localized modes appear
- The localized modes are below the mobility edge separated from the "bulk" modes by an Anderson like transition at $\lambda = \lambda_A$
- Our data indicates that there is a infinitesimal thin strip of delocalized modes also at $\lambda = 0^+$
- The localized modes are then separated from the delocalized modes by two edges: λ_A that increases with the temperature and the other one that stays in deep infrared at $\lambda = 0^+$
- It is not yet clear whether the localized modes disappear at a high temperature or whether they are present at all temperatures



AA and I. Horvath, Anderson Metal-to-Critical Transition in QCD, arXiv:2110.04833

PUZZLE FOR THE "GAP" MODES





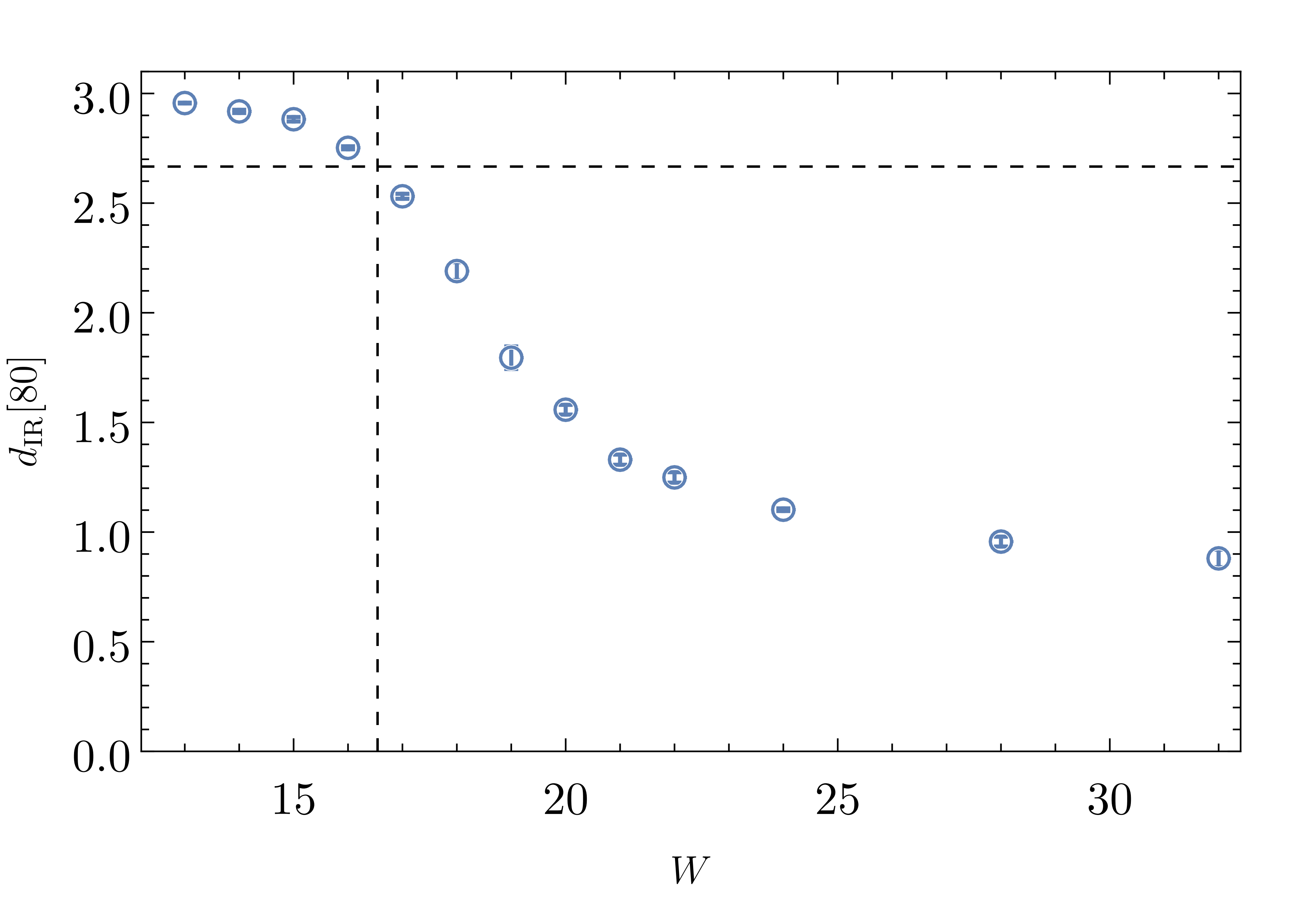
ANDERSON MODEL

Disorder induced metal-insulator transition

- Electron hopping in a random potential

$$H = \sum_{\langle x,y \rangle} (c_x c_y^\dagger + \text{hc}) + W\epsilon \sum_x c_x c_x^\dagger$$

- The on-site *random* potential ϵ is uniformly distributed over [-1/2,1/2]
- For $W < W_c = 16.543$ the entire single-particle spectrum is delocalized
- For $W > W_c$ a localized band of modes opens up around $\lambda = 0$
- Using a pair of volume $L = 40/80$, we compute the scaling dimension d_{IR} for the eigenmodes at $\lambda = 0$

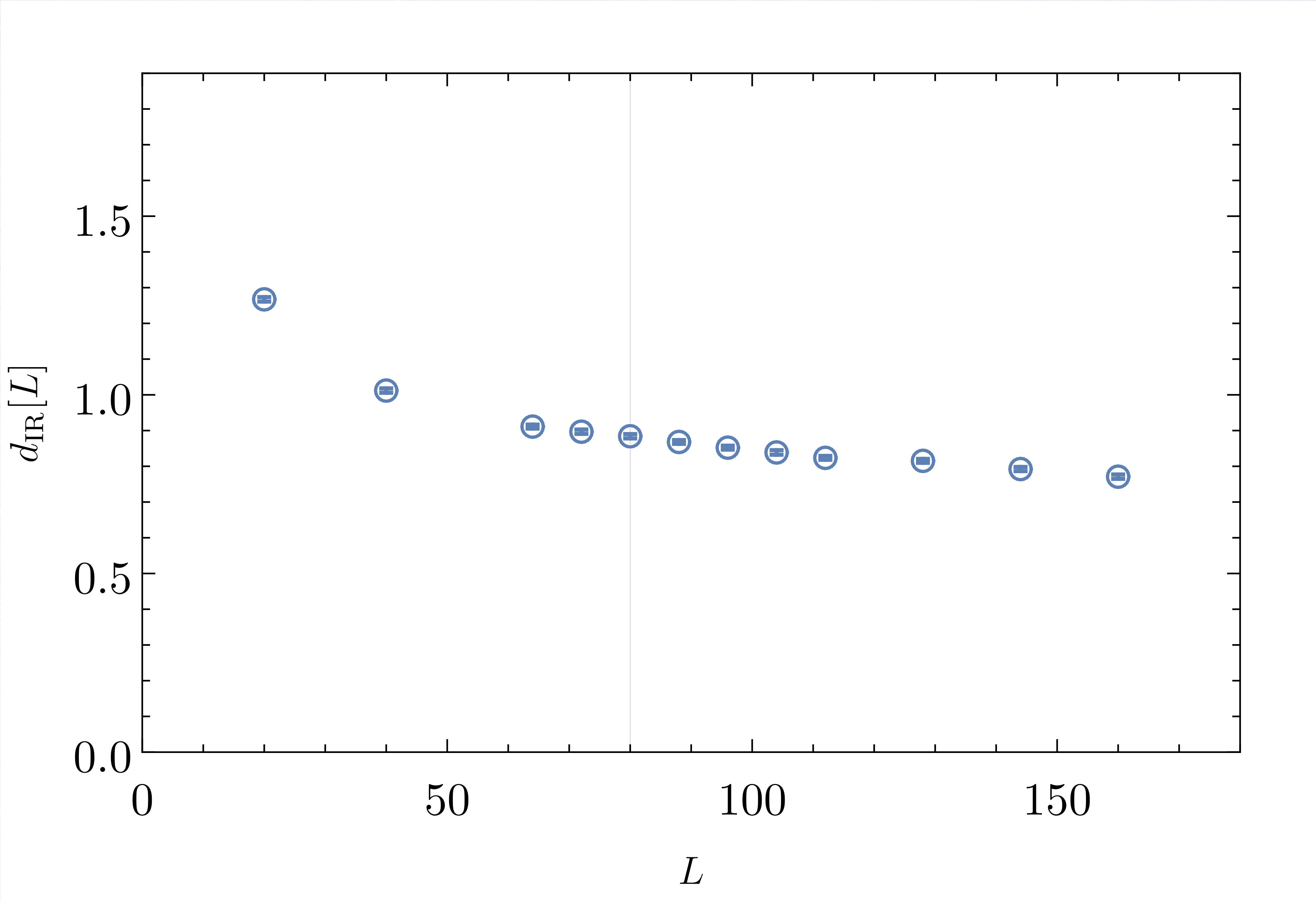


$$d_{\text{IR}}[L] = \frac{\log N_*(L)/N_*(L/2)}{\log 2}$$

ANDERSON MODEL

Mode dimension in localized region

- For $W > W_c$ we expect the modes to be localized and have $d_{\mathrm{IR}} \approx 0$
- For volume pair $L = 40/80$ we find the dimension $d_{\mathrm{IR}} \approx 1$
- One possibility is that this is a finite volume artifact
- For $W = 32 \gg W_c$ (deep in the localized region) we compute d_{IR} for larger volumes, in the range $L = 10 - 160$
- The dimension decreases with increasing volume but it is not clear that it will go to zero in the thermodynamic limit

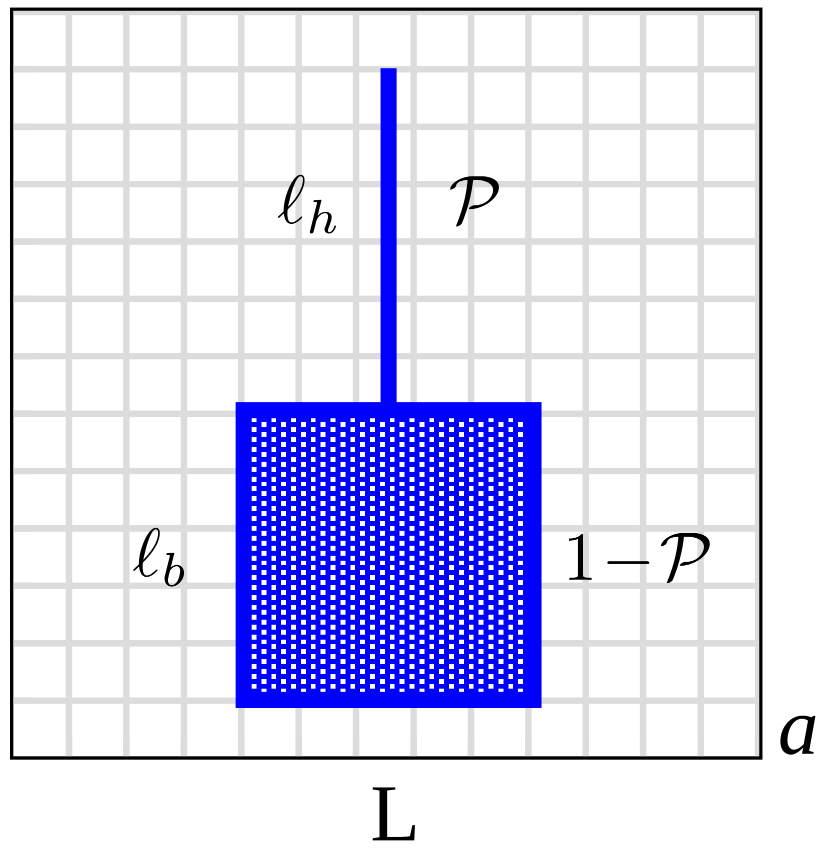


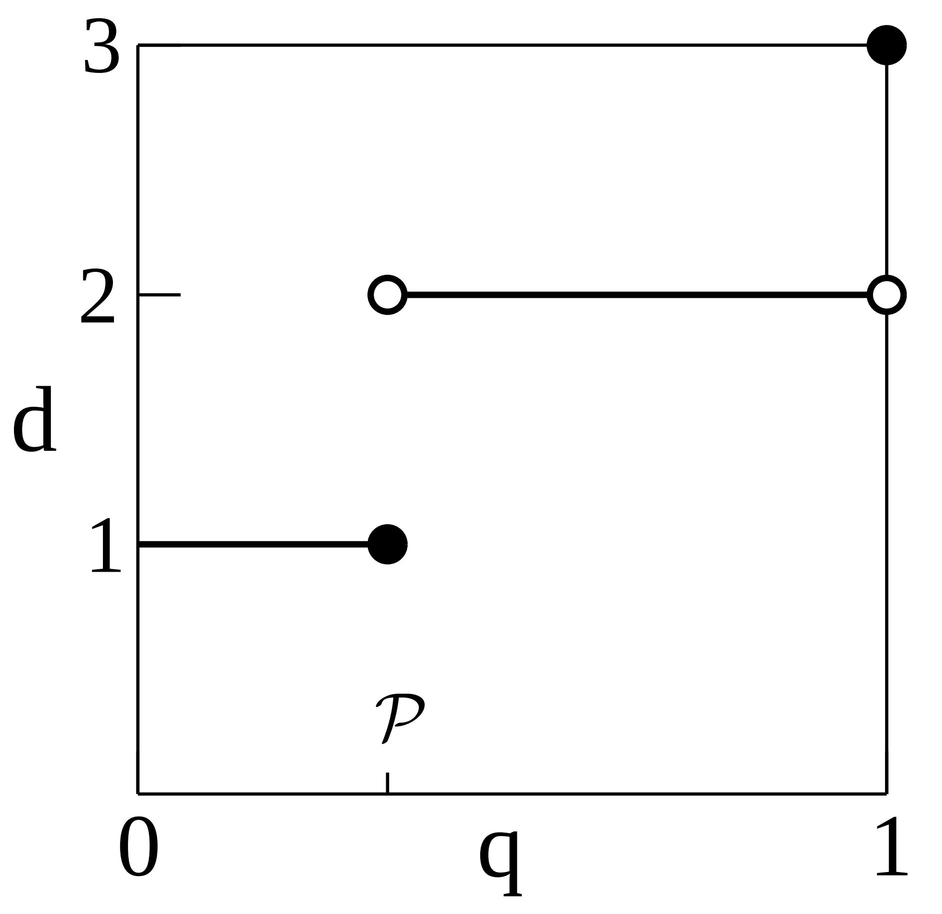
$$d_{\mathrm{IR}}[L] = \frac{\log N_*(L)/N_*(L/2)}{\log 2}$$

ANDERSON MODEL

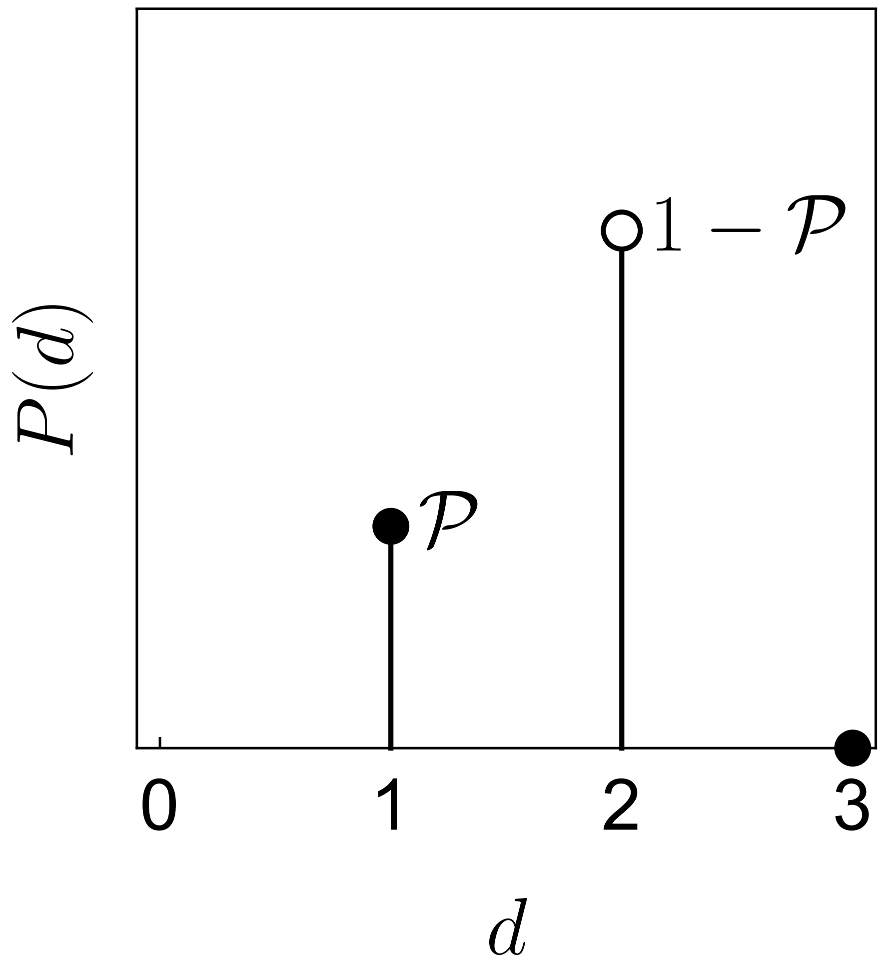
Spectral resolution for dimension

- The structure of the support for the probability distribution can be resolved component-by-component
- Consider the probability of each point in the entire volume $p_1 > p_2 > \ldots > p_N$ with $\sum p_i = 1$
- Lower dimensionality regions will have stronger components if they contribute a finite probability
- If we partition the total probability in bins, we define the number of points in the support $N_*(q, q + \Delta q)$ to correspond to the probability bin $[q, q + \Delta q]$
- The scaling with the volume of this support defines the dimension for each bin
- In the "shovel" example here we have a one dimensional component of probability $\mathscr{P}$ and a two-dimensional component of probability $1 - \mathscr{P}$
- The dimension $\mathrm{d}(q)$ can be turned into a spectral decomposition: $dq(\mathrm{d})/d\mathrm{d}$ vs d





$$P(d) = \frac{dq(\mathrm{d})}{d\mathrm{d}}$$

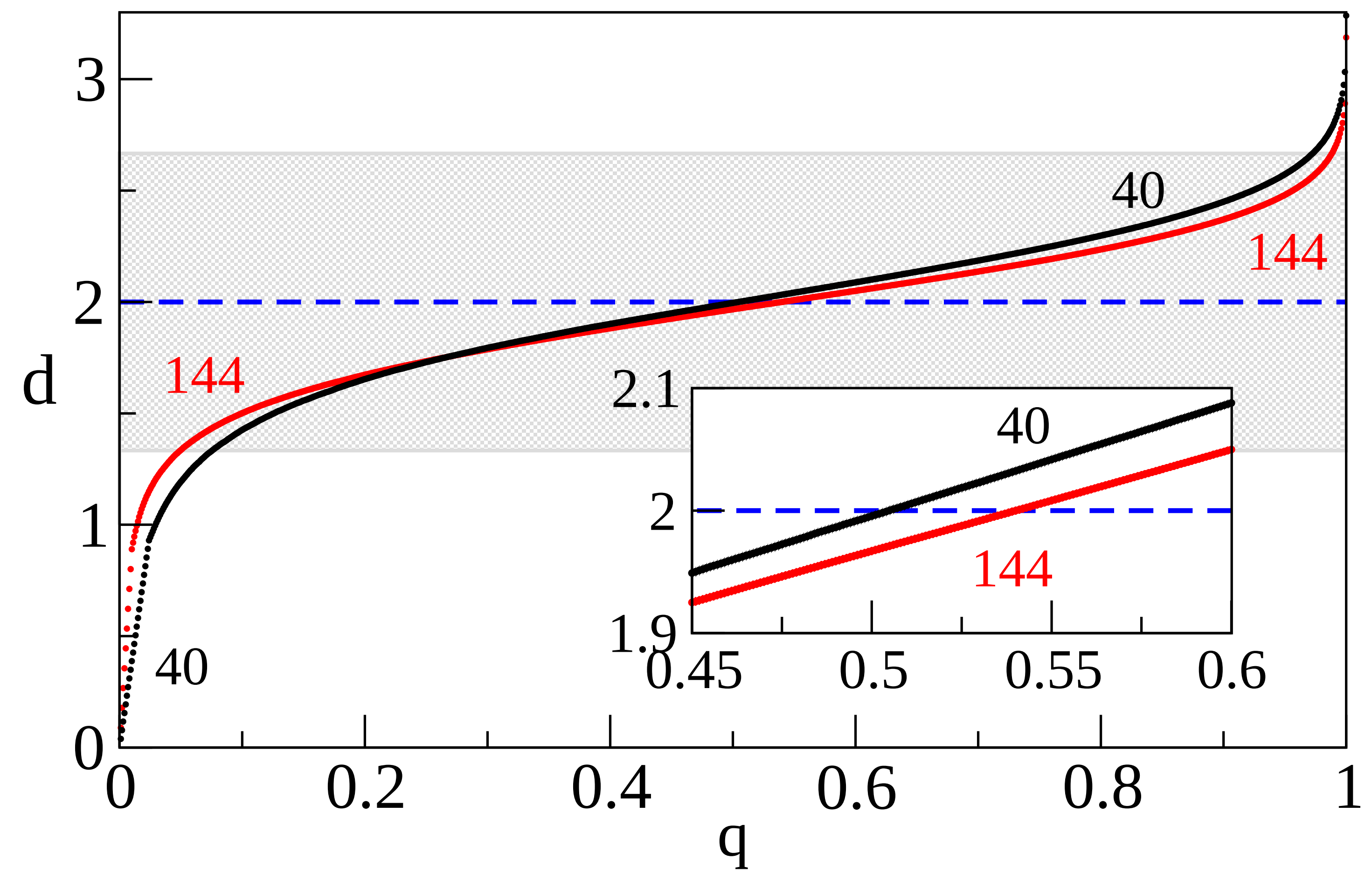


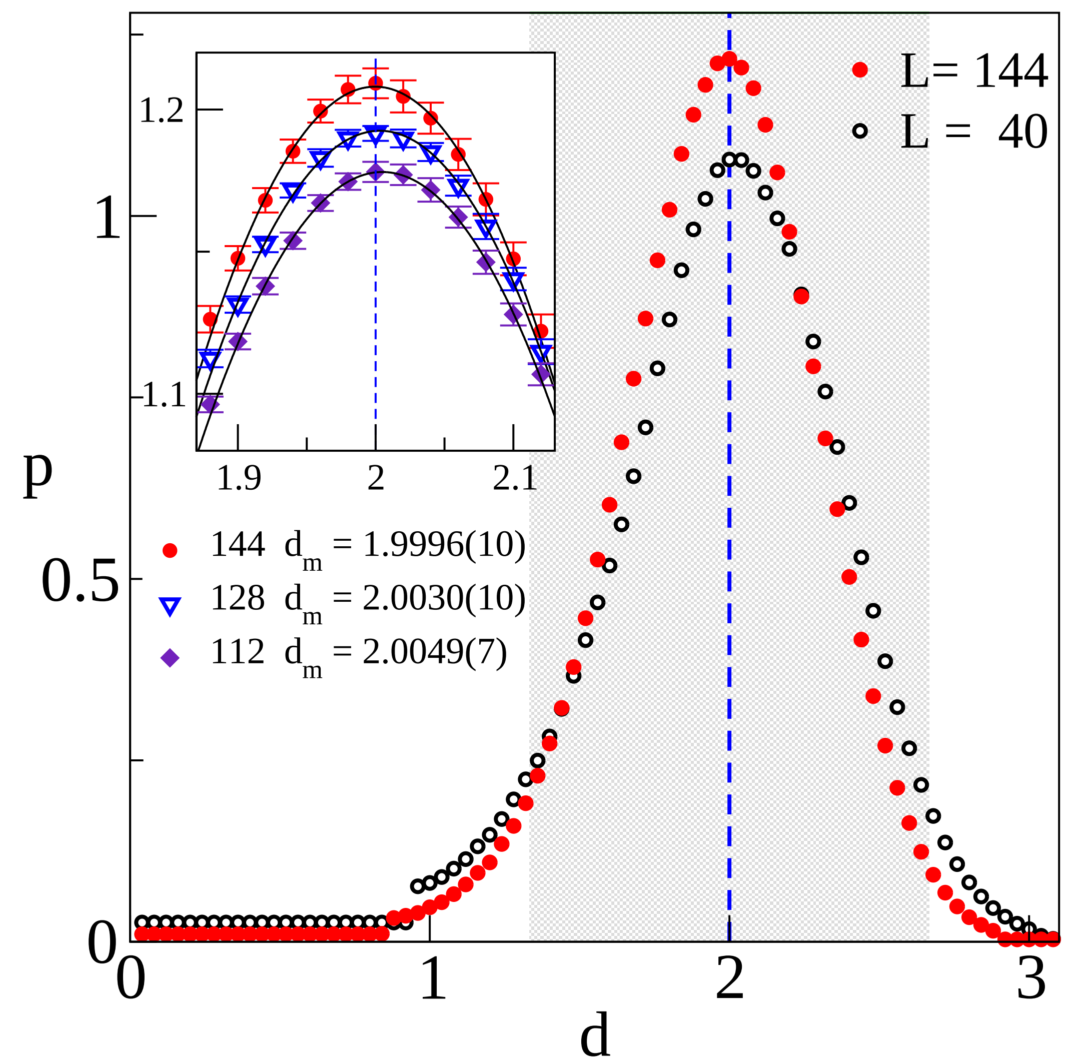
I. Horvath and P. Markoš, Topological Dimensions from Disorder and Quantum Mechanics?, arxiv:2212.09806

ANDERSON MODEL

Spectral resolution for dimension (critical modes)

- For criticality ($W = W_c$) the decomposition was worked out by Ivan and Peter Markoš [arxiv:2212.09806]
- They found that the support has components that scale with different dimensions
- A large component scales with dimension 2, but there seems to be other components that scale with powers between 4/3 and 8/3
- In this case $d_{\rm IR}$ represents that maximal dimension in the range, $d_{\rm IR} \approx 8/3$ as determined by Ivan and Peter Markoš earlier [arxiv: 2110.11266]

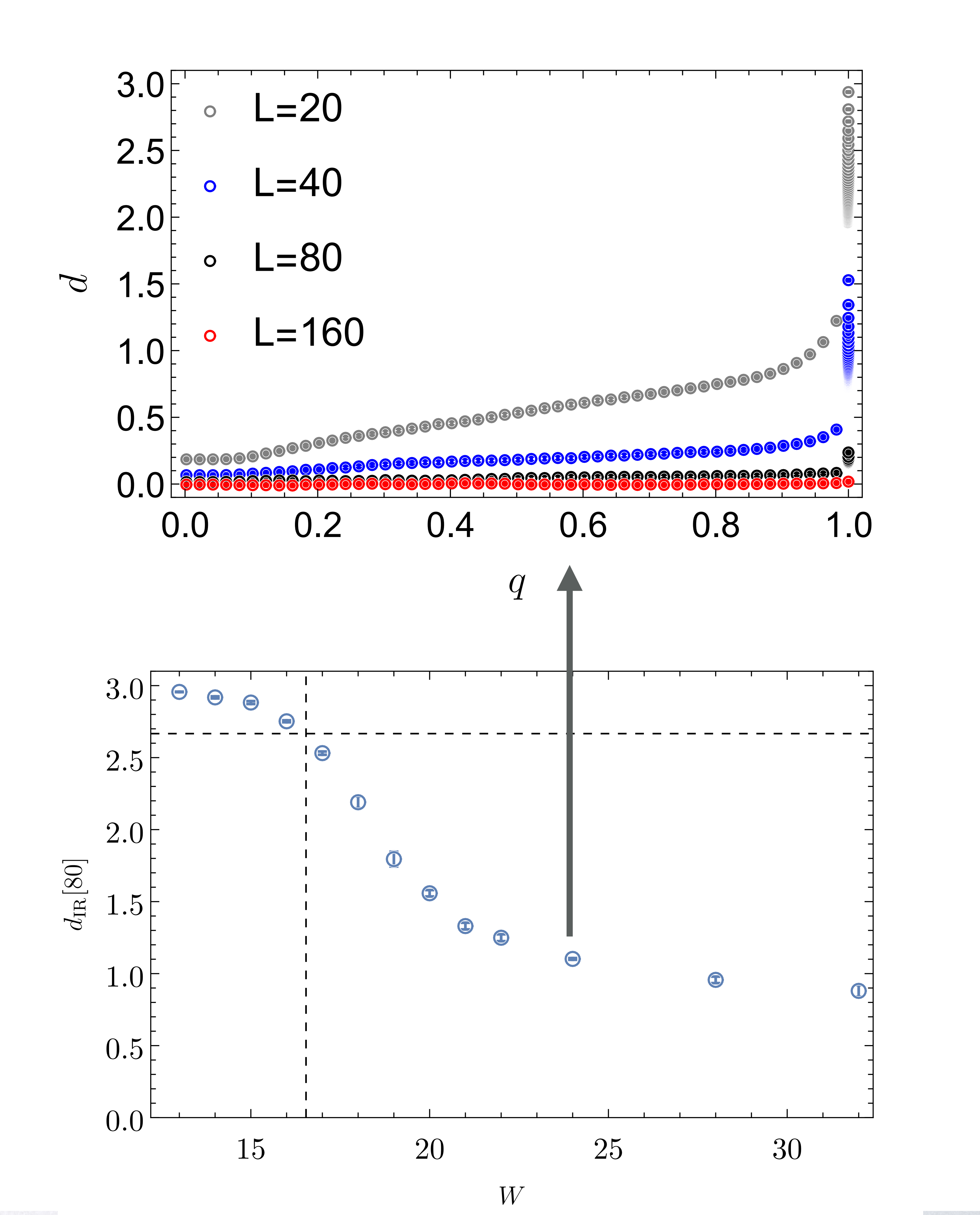




ANDERSON MODEL

Spectral resolution for dimension (localized modes)

- If we focus on the modes in the localized phase ($W = 24$) we find that the spectral resolution implies that the entire support is zero dimensional
- The dimension for all but the last probability bin seems to converge to zero in the thermodynamic limit
- This implies that the $d_{\mathrm{IR}} = 0$ in the thermodynamic limit, as expected for the localized modes



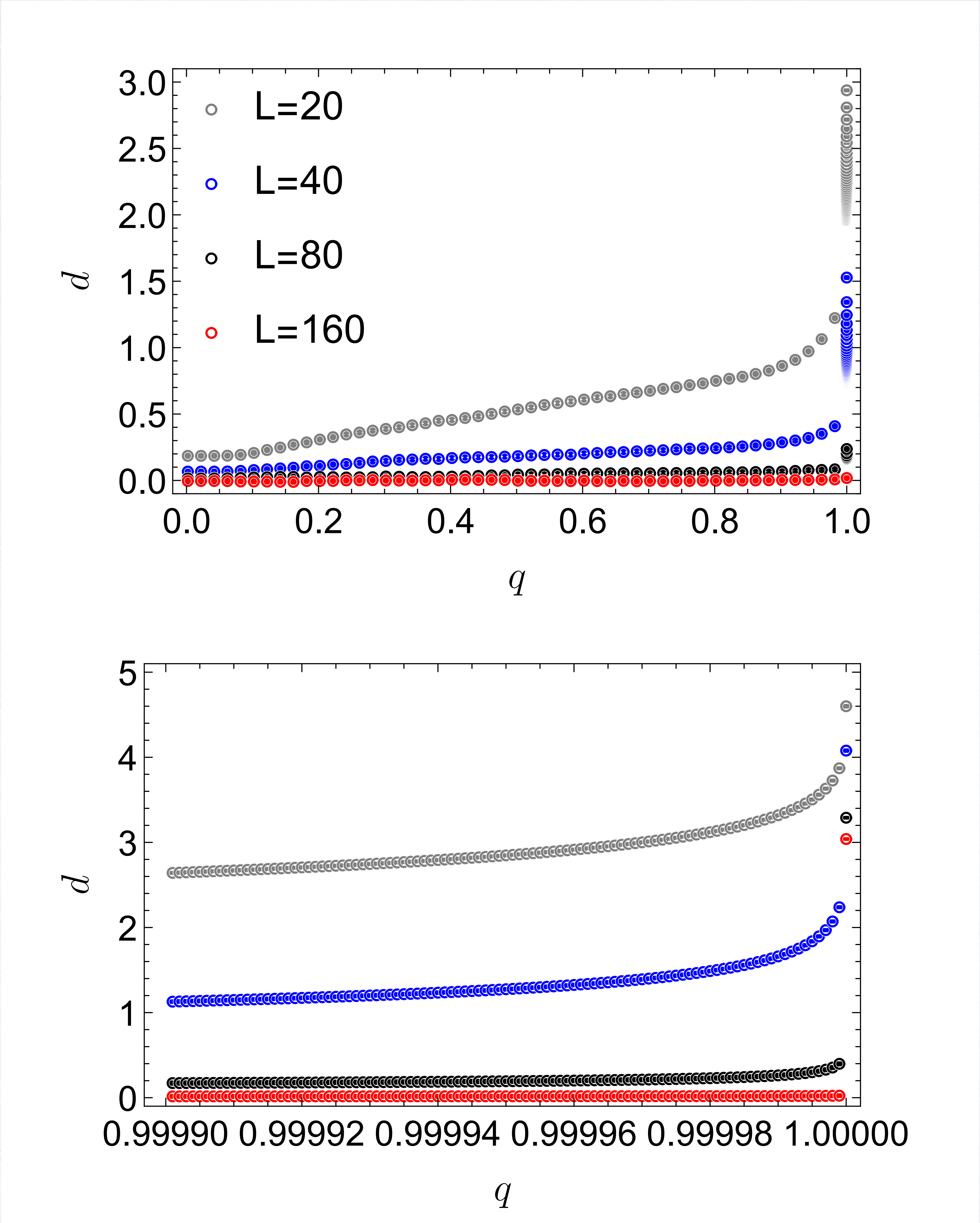
ANDERSON MODEL

Spectral resolution for dimension (localized modes)

- If we focus on the modes in the localized phase ($W = 24$) we find that the spectral resolution implies that the entire support is zero dimensional

- The dimension for all but the last probability bin seems to converge to zero in the thermodynamic limit

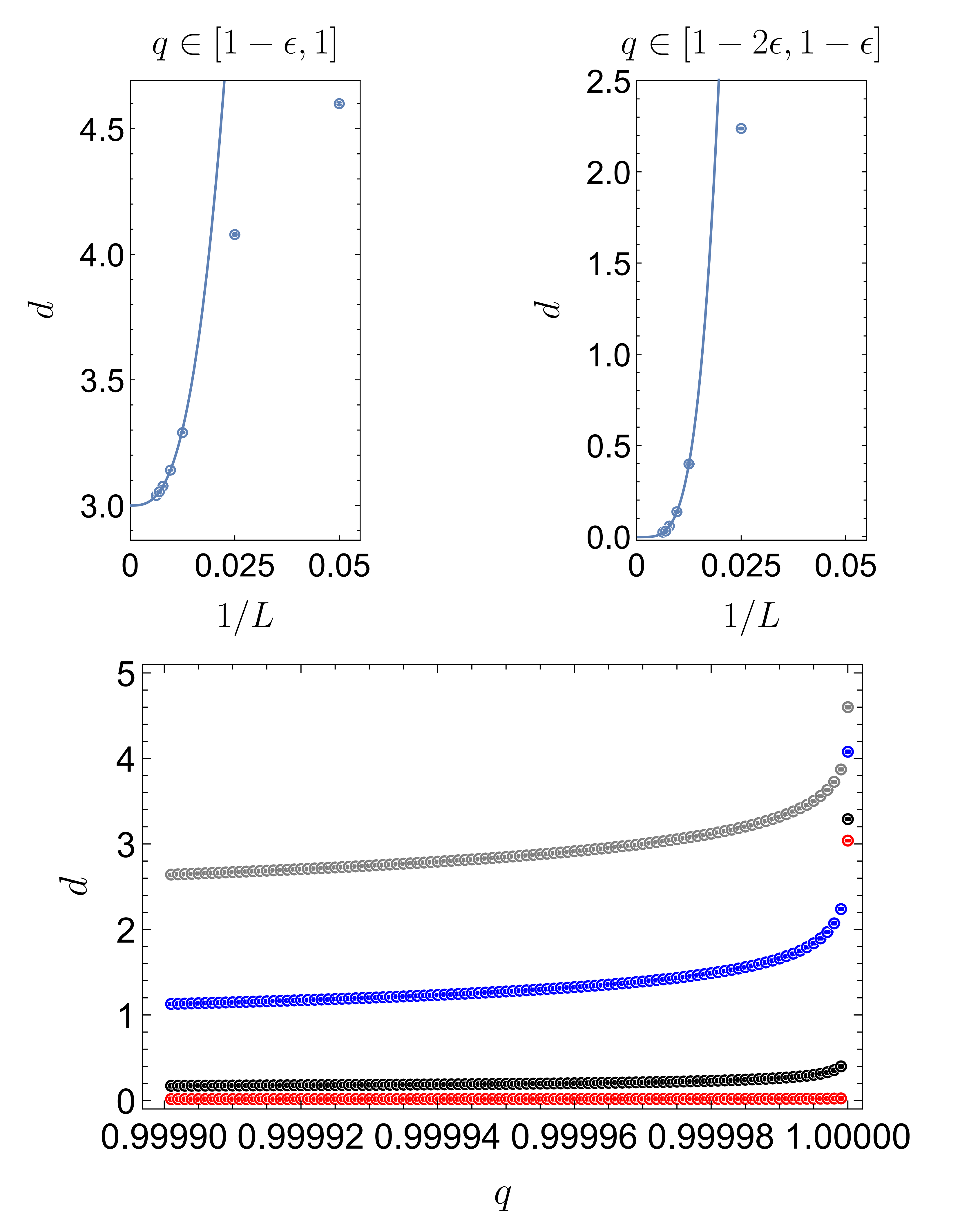
- This implies that the $d_{\mathrm{IR}} = 0$ in the thermodynamic limit, as expected for the localized modes



ANDERSON MODEL

Spectral resolution for dimension (localized modes)

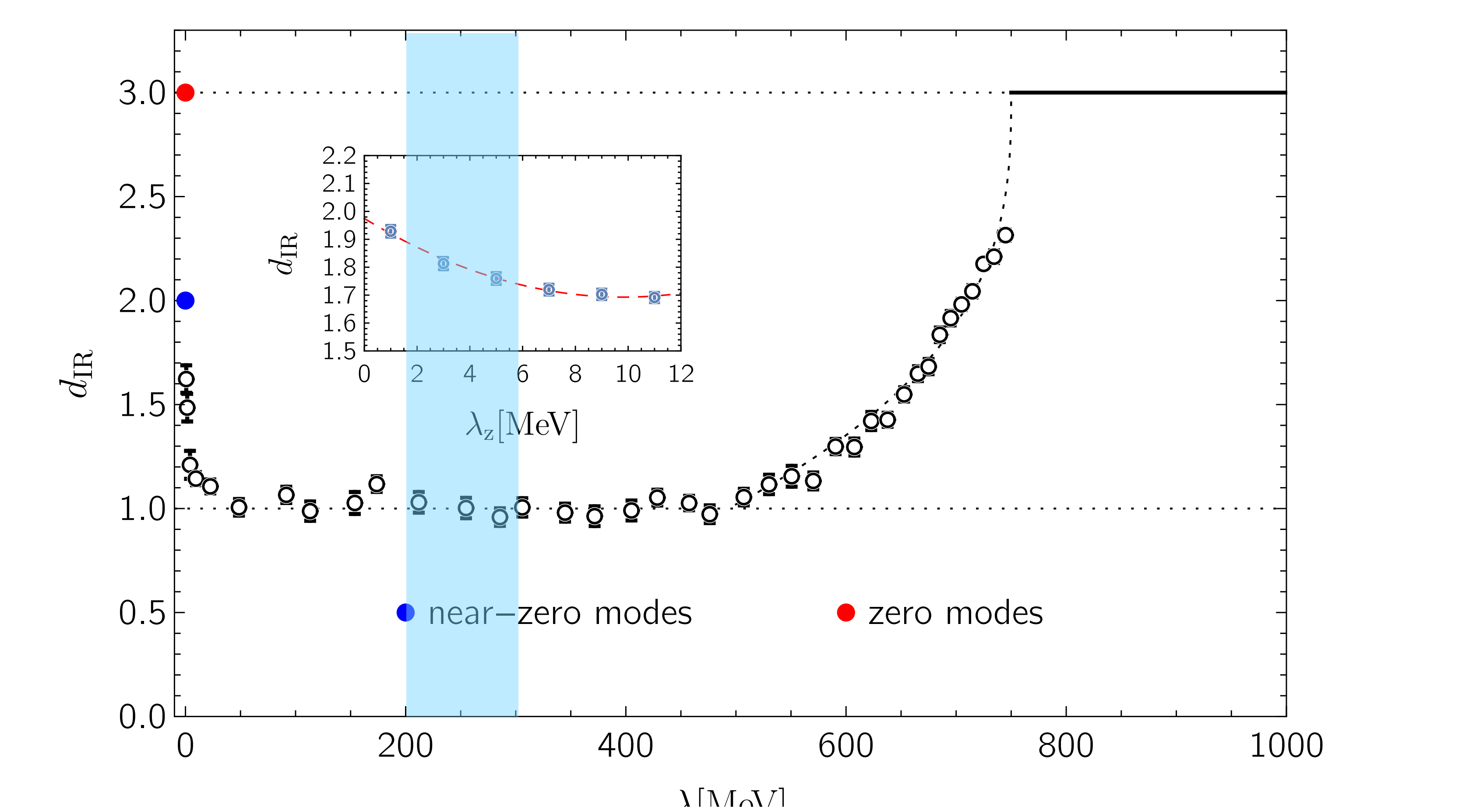
- If we focus on the modes in the localized phase ($W = 24$) we find that the spectral resolution implies that the entire support is zero dimensional
- The dimension for all but the last probability bin seems to converge to zero in the thermodynamic limit
- This implies that the $d_{\text{IR}} = 0$ in the thermodynamic limit, as expected for the localized modes
- The finite volume effects are much milder for the spectral resolution

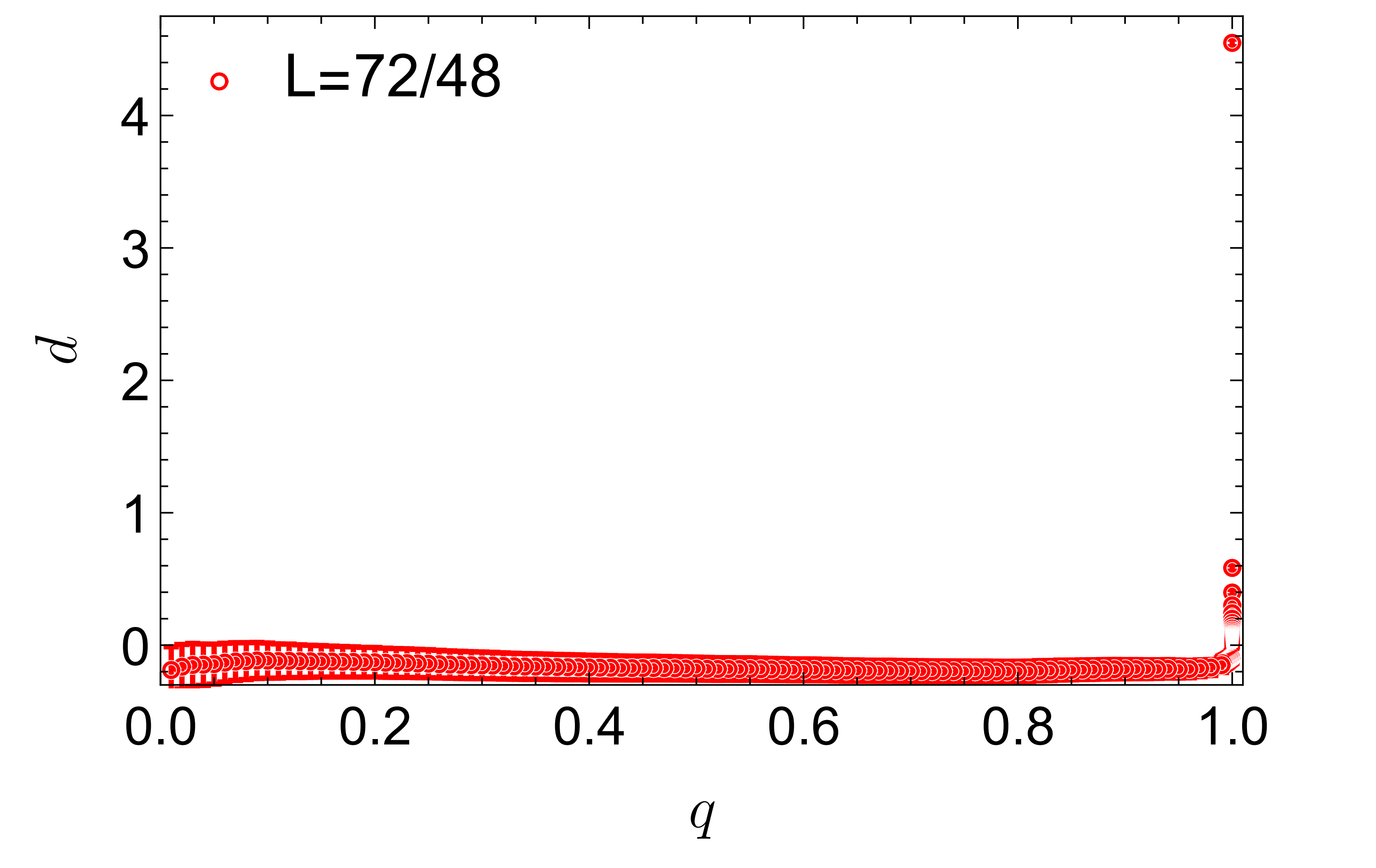


QCD IN THE IR PHASE

Spectral resolution for dimension (localized modes)

- We focus on the "gap" region, the depleted spectral region
- The spectral resolution for $\lambda \in [200{,}300]\,\mathrm{MeV}$ is similar to the localized mode phase in the Anderson model
- This suggests that the $d_{\mathrm{IR}} = 0$ in the thermodynamic limit and the modes are localized

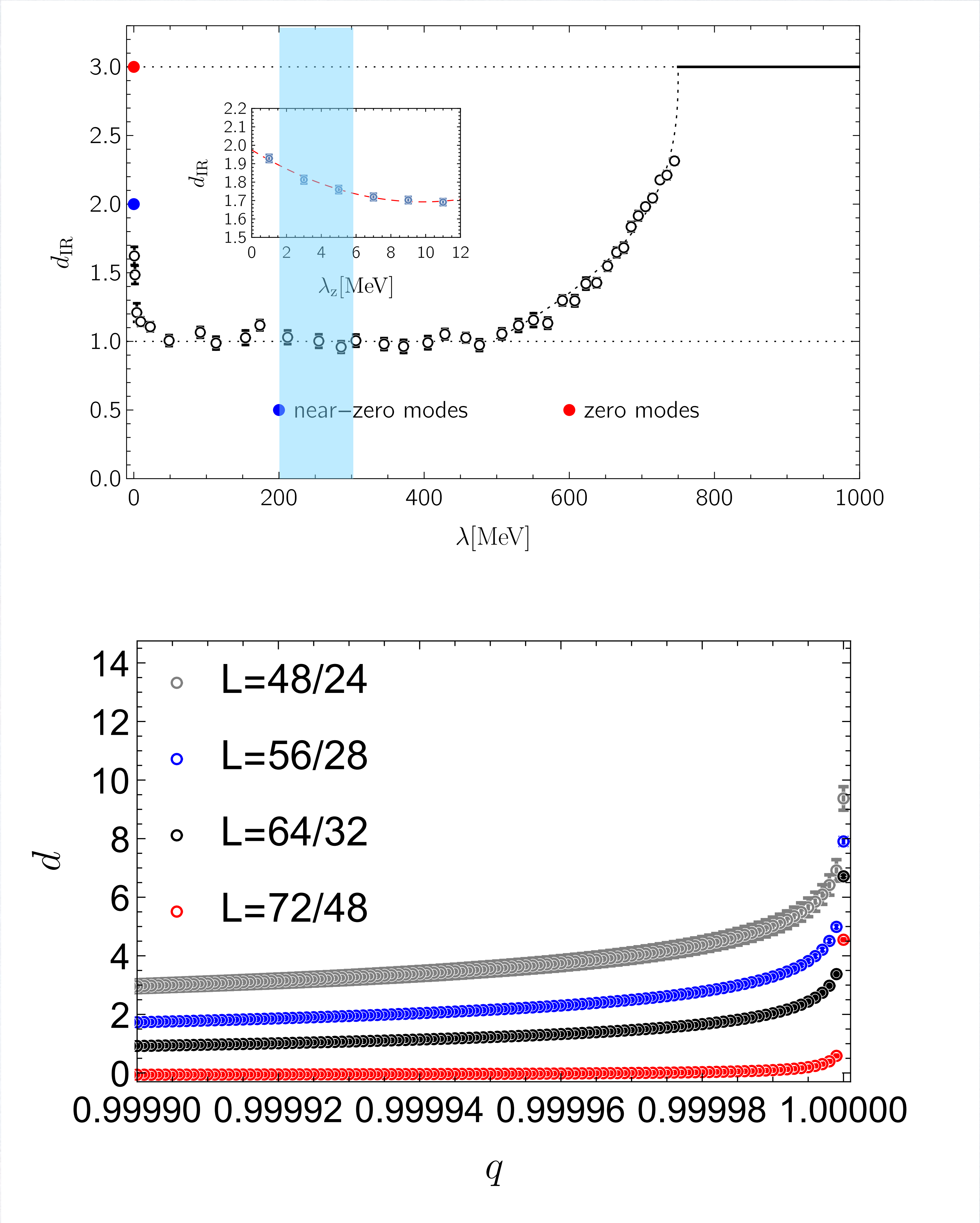




QCD IN THE IR PHASE

Spectral resolution for dimension (localized modes)

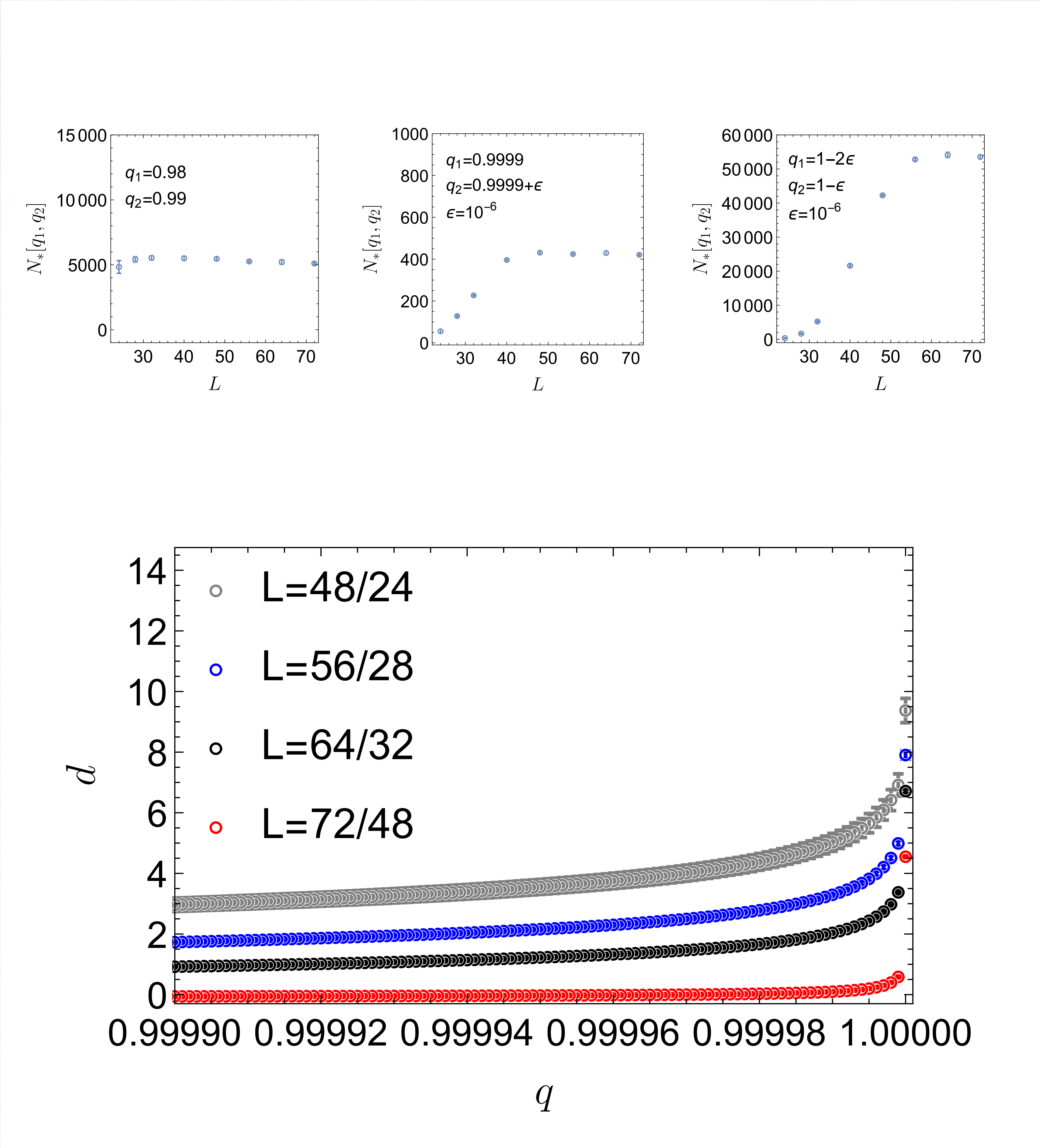
- We focus on the "gap" region, the depleted spectral region
- The spectral resolution for $\lambda \in [200{,}300]\,\mathrm{MeV}$ is similar to the localized mode phase in the Anderson model
- This suggests that the $d_{\mathrm{IR}} = 0$ in the thermodynamic limit and the modes are localized



QCD IN THE IR PHASE

Spectral resolution for dimension (localized modes)

- We focus on the "gap" region, the depleted spectral region
- The spectral resolution for $\lambda \in [200,300]\,\mathrm{MeV}$ is similar to the localized mode phase in the Anderson model
- This suggests that the $d_{\mathrm{IR}} = 0$ in the thermodynamic limit and the modes are localized



TAKE HOME

- At high temperature, in the IR phase, the deep infrared modes of the Dirac operator are delocalized
- The transition from low to high temperature for the IR Dirac spectrum is not delocalized-localized (metal-insulator in Anderson language)
- The IR modes remain delocalized, but their nature is more akin with the eigenvectors at the mobility edge
- The modes in the peak are delocalized and are likely to support long range correlations in glue fluctuations
- We carried out this calculation for pure glue system where we can control the parameters accurately, but there are strong indications that this happens for other QCD like systems (see later talks)
- The modes in the "gap" are localized with $d_{\text{IR}} = 0$ — the discrepancy between scaling of eigenmodes' size and their support is a finite volume artifact