

Towards an Algebraic Approach to Gravitational Quantum Mechanics

Georg Junker

Joint work with

Won Sang Chung (Korea)
and

Hassan Hassanabadi (Iran)

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Abstract

Most approaches towards a quantum theory of gravitation indicate the existence of a minimal length scale of the order of the Planck length. Quantum mechanical models incorporating such an intrinsic length scale call for a deformation of Heisenberg's algebra resulting in a generalized uncertainty principle and constitute what is called gravitational quantum mechanics. Using an explicit position representation, the free time evolution of a Gaussian wave packet is investigated as well as the spectral properties of a particle bound by an external attractive potential. Here the cases of a box with finite width and infinite walls, and an attractive potential well of finite depth are considered.

Outline

- 1 Historical background
- 2 GUP and gravitational QM
- 3 GUP calculus
- 4 Examples

The two fundamental discoveries of 20th century

General relativity: Einstein 1915

Quantum mechanics: Heisenberg (1925) and Schrödinger (1926)

Matvei Petrovich Bronstein in 1936

Quantentheorie schwacher Gravitationsfelder,

Physikalische Zeitschrift der Sowjetunion 9 (1936), 140–157.

Hartland S. Snyder in 1947

Quantized Space-Time,

Phys. Rev. 71 (1947) 38-41

$$[\hat{X}, \hat{P}] = i\hbar \left(1 + \frac{\lambda^2}{\hbar^2} \hat{P}^2 \right), \quad \lambda = \text{'natural unit of length'}$$

Quantum gravity: Minimal length $\lambda = \lambda_P \approx 10^{-33} \text{ cm}$

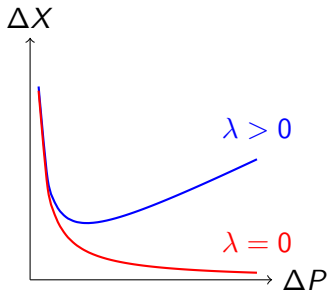
Generalised uncertainty principle (GUP)

Recall uncertainty relation:

$$\Delta X \Delta P \geq \frac{1}{2} \left\langle |[\hat{X}, \hat{P}]| \right\rangle_{\psi} = \frac{\hbar}{2} \left(1 + \frac{\lambda^2}{\hbar^2} \langle \hat{P}^2 \rangle_{\psi} \right)$$

With $\langle \hat{P} \rangle_{\psi} = 0$ (rest frame) we have the minimal uncertainty

$$\Delta X = \frac{1}{2} \left(\frac{\hbar}{\Delta P} + \frac{\lambda^2}{\hbar} \Delta P \right)$$



$$\min \Delta X = \lambda \quad \text{at} \quad \Delta P = \hbar / \lambda$$

Gravitational quantum mechanics (GQM)

Gravitational quantum mechanics

- Deformed Heisenberg algebra:

$$[\hat{X}, \hat{P}] = i\hbar \left(1 + \frac{\lambda^2}{\hbar^2} \hat{P}^2 \right)$$

with deformation parameter $\lambda > 0$ small [Length scale]

- Hamiltonian:

$$H = \frac{\hat{P}^2}{2M} + V(\hat{X})$$

for particle with mass $M > 0$

\implies Study effects for small non-vanishing λ

GUP momentum representation

From now on $\hbar = 1$: $[\hat{X}, \hat{P}] = i \left(1 + \lambda^2 \hat{P}^2 \right)$

Representation:

$$\hat{X} = i \left(1 + \lambda^2 P^2 \right) \frac{\partial}{\partial P}, \quad \hat{P} = P$$

Hamiltonian:

$$H_P = \frac{P^2}{2M} + V \left(i(1 + \lambda^2 P^2) \partial_P \right)$$

Scalar product:

$$\langle \Phi | \Psi \rangle = \int_{-\infty}^{\infty} dP \frac{1}{1 + \lambda^2 P^2} \Phi^*(P) \Psi(P)$$

Here \hat{P} is self-adjoint but \hat{X} is only symmetric.

For details see: A. Kempf et al, PRD 52 (1996) 1108-1118.

Canonical momentum representation $[\hat{X}, \hat{P}] = i(1 + \lambda^2 \hat{P}^2)$

Recall: $\hat{x} = i\partial_p$, $\hat{p} = p$, with $[\hat{x}, \hat{p}] = i$

Representation:

$$\hat{X} = i\partial_p, \quad \hat{P} = \frac{1}{\lambda} \tan(\lambda p)$$

Hamiltonian:

$$H_p = \frac{1}{2M\lambda^2} \tan^2(\lambda p) + V(i\partial_p)$$

Scalar product:

$$\langle \Phi | \Psi \rangle = \int_{-p_0}^{p_0} dp \Phi^*(p) \Psi(p), \quad p_0 := \frac{\pi}{2\lambda}$$

Deformation function:

$$p = \varphi(P) := \frac{1}{\lambda} \arctan(\lambda P) \in [-p_0, p_0]$$

Note:

$$\varphi'(P) = (1 + \lambda^2 P^2)^{-1}$$

GUP position representation $[\hat{X}, \hat{P}] = i(1 + \lambda^2 \hat{P}^2)$

Representation:

$$\hat{X} = x \quad \hat{P} = \frac{1}{\lambda} \tan(\lambda \hat{p}) = \frac{1}{\lambda} \tan(-i\lambda \partial_x) = \frac{1}{i\lambda} \tanh(\lambda \partial_x)$$

Hamiltonian:

$$H = -\frac{1}{2M\lambda^2} \tanh^2(\lambda \partial_x) + V(x)$$

Scalar product:

$$\langle \Phi | \Psi \rangle = \int_{-\infty}^{\infty} dx \Phi^*(x) \Psi(x)$$

Approximate solutions: $\hat{P} = -i\partial_x + i\frac{\lambda^2}{3}\partial_x^3 + O(\lambda^4)$

For example, particle in a box and free particle by

K. Nozari and T. Azizi, Gen. Relativ. Gravit. 38 (2006) 735-742

Our aim is to obtain exact solutions!!!

GUP derivative

Recall deformation function:

$$\varphi(z) = \frac{1}{\lambda} \arctan(\lambda z), \quad \varphi^{-1}(z) = \frac{1}{\lambda} \tan(\lambda z)$$

GUP derivate:

$$D_{\varphi} := -i\varphi^{-1}(i\partial_x) = \frac{1}{\lambda} \tanh(\lambda\partial_x)$$

To be understood as power series: $a_n = \frac{2^{2n+2}}{(2n+2)!} (2^{2n+2} - 1) B_{2n}$

$$D_{\varphi} = \partial_x - \frac{\lambda^2}{3} \partial_x^3 + \frac{2\lambda^4}{15} \partial_x^5 - \frac{17\lambda^6}{315} \partial_x^7 + \dots = \sum_{n=0}^{\infty} a_n \lambda^{2n} \partial_x^{2n+1}$$

GUP Hamiltonian:

$$H = -\frac{1}{2M\lambda^2} \tanh^2(\lambda\partial_x) + V(x) = -\frac{1}{2M} D_{\varphi}^2 + V(x)$$

Eigenvalue problem: $H\Psi = E\Psi$

Properties of GUP derivative

GUP exponential: $e^{ax} \implies E_\varphi(a; x)$

$$E_\varphi(a; x) := \exp \{ -i\varphi(ia)x \} \quad \text{with} \quad D_\varphi E_\varphi(a; x) = a E_\varphi(a; x)$$

GUP wave equation:

$$(D_\varphi^2 + \omega^2) y(x) = 0 \quad \omega > 0,$$

has solutions

$$E_\varphi(i\omega; x) = e^{i\varphi(\omega)x}, \quad E_\varphi(-i\omega; x) = e^{-i\varphi(\omega)x},$$

$$C_\varphi(i\omega; x) := \frac{1}{2} [e^{i\varphi(\omega)x} + e^{-i\varphi(\omega)x}] = \cos(\varphi(\omega)x),$$

$$S_\varphi(i\omega; x) := \frac{1}{2i} [e^{i\varphi(\omega)x} - e^{-i\varphi(\omega)x}] = \sin(\varphi(\omega)x)$$

with wave length

$$\ell := \frac{2\pi}{\varphi(\omega)} = \frac{2\pi\lambda}{\arctan(\lambda\omega)} \geq 4\lambda \quad \text{as} \quad E_\varphi(i\omega; x + \ell) = E_\varphi(i\omega; x)$$

Free particle - plane waves:

GQM

$$-D_{\varphi}^2 U(x) = P^2 U(x)$$

$$U_P(x) = \sqrt{\frac{\varphi'(P)}{2\pi}} e^{i\varphi(P)x}$$

Orthonormal

$$\int_{-\infty}^{\infty} dx U_P^*(x) U_{P'}(x) = \delta(P - P')$$

Almost Complete

$$\int_{-\infty}^{\infty} dP U_P^*(x) U_P(x') = \frac{\sin \frac{\pi}{2\lambda}(x - x')}{\pi(x - x')}$$

Same as in A. Kempf et al

SQM

$$-\partial_x^2 u(x) = p^2 u(x)$$

$$u_p(x) = \sqrt{\frac{1}{2\pi}} e^{ipx}$$

Orthonormal

$$\int_{-\infty}^{\infty} dx u_p^*(x) u_{p'}(x) = \delta(p - p')$$

Complete

$$\int_{-\infty}^{\infty} dp u_p^*(x) u_p(x') = \delta(x - x')$$

Gaussian wave packet

Gaussian: $\Phi(x) := \frac{1}{(2\pi\sigma^2)^{1/4}} \exp\left\{-\frac{x^2}{4\sigma^2}\right\}, \quad \|\Phi\|^2 = 1$

P -component:

$$\tilde{\Phi}(P) := \int_{-\infty}^{\infty} dx \Phi(x) U_P^*(x) = \sqrt{\frac{\sqrt{2}\sigma}{\sqrt{\pi}(1 + \lambda^2 P^2)}} e^{-\sigma^2[\varphi(P)]^2}$$

Almost complete:

$$\|\tilde{\Phi}\|^2 = 1 - \operatorname{erfc}\left(\frac{\pi\sigma}{\lambda\sqrt{2}}\right) = 1 - \sqrt{\frac{2}{\pi^3}} e^{-\frac{\pi^2\sigma^2}{2\lambda^2}} \frac{\lambda}{\sigma} \left[1 + O\left(\frac{\lambda^2}{\sigma^2}\right)\right]$$

Cross check:

$$\tilde{\tilde{\Phi}}(x) := \int_{-\infty}^{\infty} dP \tilde{\Phi}(P) U_P(x)$$

$$\Rightarrow |\Phi(x) - \tilde{\tilde{\Phi}}(x)| \leq \operatorname{erfc}\left(\frac{\pi\sigma}{\lambda\sqrt{2}}\right)$$

Gaussian as minimum uncertainty state

Recall $\Phi(x) := \frac{1}{(2\pi\sigma^2)^{1/4}} e^{-\frac{x^2}{4\sigma^2}}$ with $(\Delta x)^2 = \sigma^2$

and $\tilde{\Phi}(P) = \left(\frac{2\sigma^2}{\pi}\right)^{1/4} \sqrt{\varphi'(P)} e^{-\sigma^2 \varphi^2(P)}$

With $P = \frac{1}{\lambda} \tan(\lambda\varphi) = \varphi + \frac{1}{3}\lambda^2\varphi^3 + O(\lambda^4)$

and using

$$\langle \varphi^{2m} \rangle_{\tilde{\Phi}} := \sqrt{\frac{2\sigma^2}{\pi}} \int_{-p_0}^{p_0} d\varphi \varphi^{2m} e^{-2\sigma^2 \varphi^2} = \frac{(2m-1)!!}{2^{2m}\sigma^{2m}} + e^{-\frac{\pi^2\sigma^2}{2\lambda^2}} O\left(\frac{\sigma^{2m-1}}{\lambda^{2m-1}}\right)$$

$$\Rightarrow \langle P^2 \rangle = \langle \varphi^2 \rangle + \frac{2}{3}\lambda^2 \langle \varphi^4 \rangle + O(\lambda^4) = \frac{1}{4\sigma^2} \left(1 + \frac{\lambda^2}{2\sigma^2} + O(\lambda^4)\right)$$

$$\Delta x \Delta P = \frac{1}{2} \left(1 + \frac{\lambda^2}{4(\Delta x)^2} + O(\lambda^4)\right) = \frac{1}{2} (1 + \lambda^2 \langle P^2 \rangle + O(\lambda^4))$$

Gaussian is minimum uncertainty state up to $O(\lambda^4)$

GUP Fourier transformation

Let

$$\tilde{\phi}(P) := \mathcal{F}[\phi] = \int_{-\infty}^{\infty} dx \phi(x) U_P^*(x)$$

$$\tilde{\phi}(x) := \mathcal{F}^{-1}[\tilde{\phi}] = \int_{-\infty}^{\infty} dP \tilde{\phi}(P) U_P(x)$$

For smooth wave functions ϕ : $\sigma_{\phi}^2 := \langle (x - \langle x \rangle_{\phi})^2 \rangle_{\phi} \gg \lambda^2$

We expect $|\phi(x) - \tilde{\phi}(x)|$ to be very small

Approximate free time evolution of initial state ϕ_0 :

$$\tilde{\phi}_t(P) = \exp \left\{ -\frac{iP^2}{2M} t \right\} \tilde{\phi}_0(P)$$

Results in

$$\phi_t(x) \approx \tilde{\phi}_t(x) = \int_{-\infty}^{\infty} dP \tilde{\phi}_t(P) U_P(x)$$

Gaussian time evolution

$$\tilde{\Phi}_t(P) = \left(\frac{2\sigma^2}{\pi}\right)^{1/4} \sqrt{\varphi'(P)} e^{-\sigma^2 \varphi^2(P)} e^{-i\frac{P^2}{2M}t}, \quad |\tilde{\Phi}_t(P)|^2 = |\tilde{\Phi}(P)|^2$$

$$\Rightarrow \langle P^2 \rangle_{\tilde{\Phi}_t} = \frac{1}{4\sigma^2} \left(1 + \frac{\lambda^2}{2\sigma^2} + O(\lambda^4)\right)$$

$$\tilde{\Phi}_t(x) := \int_{-\infty}^{\infty} dP \tilde{\Phi}_t(P) U_P(x)$$

with

$$e^{-i\frac{P^2}{2M}t} = e^{-i\sigma^2 \varphi^2 \frac{t}{\tau}} \left(1 - \frac{i}{3} \lambda^2 \varphi^4 \sigma^2 \frac{t}{\tau} + O(\lambda^4)\right),$$

where $\tau := 2M\sigma^2$

Now let: $\sigma_t^2 := \sigma^2(1 + i\frac{t}{\tau}) \quad \Rightarrow \quad \text{Gaussian integral}$

Gaussian time evolution

Result:

$$\tilde{\Phi}_t(x) = \Phi_t^{\lambda=0}(x) \left(1 - i \frac{\lambda^2}{\sigma^2} \frac{t}{\tau} \left(\frac{1}{12} \frac{\sigma^8}{\sigma_t^8} \frac{x^4}{\sigma^4} - \frac{\sigma^6}{\sigma_t^6} \frac{x^2}{\sigma^2} + \frac{\sigma^4}{\sigma_t^4} \right) \right) + O\left(\frac{t^2}{\tau^2} \frac{\lambda^4}{\sigma^4}\right)$$

Is normalized up to and including $O(\frac{t^2}{\tau^2}) \Rightarrow \frac{\sigma^2}{\sigma_t^2} = 1 - i \frac{t}{\tau} + O(\frac{t^2}{\tau^2})$

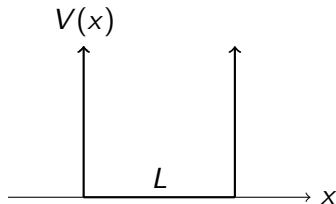
Time-dependent width:

$$\begin{aligned} \langle x^2 \rangle_{\tilde{\Phi}_t} &:= \int dx x^2 \left| \tilde{\Phi}_t(x) \right|^2 \\ &= \sigma^2 \left(1 + \frac{t^2}{\tau^2} + 8 \frac{\lambda^2}{\sigma^2} \frac{t^2}{\tau^2} + O\left(\frac{t^4}{\tau^4}\right) + O\left(\frac{t^2}{\tau^2} \frac{\lambda^4}{\sigma^4}\right) \right) \end{aligned}$$

Time-dependent uncertainty relation:

$$(\Delta x)_t^2 (\Delta P)_t^2 = \frac{1}{4} \left(1 + \frac{t^2}{\tau^2} + \frac{\lambda^2}{2\sigma^2} + \frac{17}{2} \frac{\lambda^2}{\sigma^2} \frac{t^2}{\tau^2} + \dots \right)$$

Particle in a box



GQM

$$\Psi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi}{L} x$$

$$E_n(x) = \frac{1}{2M\lambda^2} \tan^2 \left(\frac{\lambda}{L} n\pi \right)$$

$$n = 1, 2, 3, \dots, n_{\max} < \frac{L}{2\lambda}$$

SQM

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi}{L} x$$

$$\varepsilon_n(x) = \frac{n^2 \pi^2}{2ML^2}$$

$$n = 1, 2, 3, \dots$$

Agrees to first order in λ^2 with K. Nozari and T. Azizi
(but no n_{\max})

Potentials with finite steps

Consider potential $V(x)$ with a finite jump at x_0 in GUP-SE

$$D_\varphi^2 \Psi(x) = 2M(V(x) - E)\Psi(x)$$

Consider "left-inverse of GUP derivative":

$$\int dx \lim_{\lambda \rightarrow 0} (D_\varphi f)(x) = \int dx f'(x) = f(x)$$

"GUP-integrate" $f = D_\varphi \Psi$ over interval $[x_0 - \varepsilon, x_0 + \varepsilon]$:

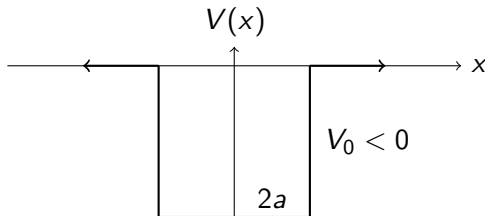
$$(D_\varphi \Psi)(x) \Big|_{x_0 - \varepsilon}^{x_0 + \varepsilon} = \int_{x_0 - \varepsilon}^{x_0 + \varepsilon} dx [2M(V(x) - E)\Psi(x)]$$

Limit $\varepsilon \rightarrow 0$ implies boundary condition at x_0 :

$$(D_\varphi \Psi)(x_0 - 0) = (D_\varphi \Psi)(x_0 + 0)$$

GUP-derivative expected to be continuous at x_0

Particle in a finite square well



Ansatz for bound states:

$$\Psi^{\pm}(x) = \pm \Psi(-x), \quad \kappa^2 = -2ME > 0, \quad q^2 = 2M(E - V_0) > 0$$

$$\Psi^{+}(x) = \begin{cases} Ae^{i\varphi(i\kappa)|x|} & |x| > a \\ B \cos(\varphi(q)x) & |x| < a \end{cases}$$

$$\Psi^{-}(x) = \begin{cases} A \frac{x}{|x|} e^{i\varphi(i\kappa)|x|} & |x| > a \\ C \sin(\varphi(q)x) & |x| < a \end{cases},$$

Particle in a finite square well continued

Continuity of Ψ and $D_\varphi \Psi$ at $x = \pm a$ results in

GQM

SQM

$$\kappa = q \tan(\varphi(q)a)$$

$$\kappa = q \tan(qa) \quad \text{sym.}$$

$$\kappa = -q \cot(\varphi(q)a)$$

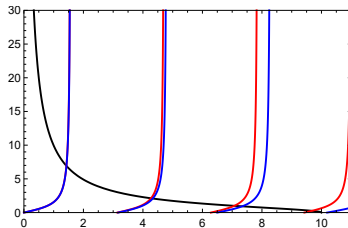
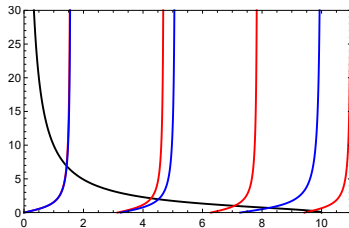
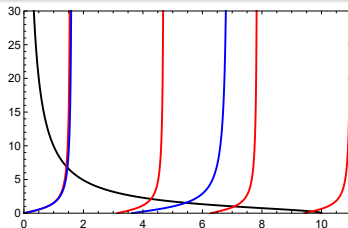
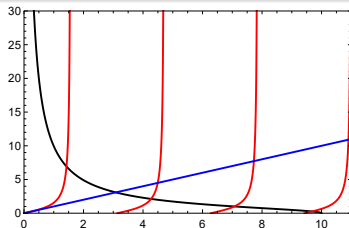
$$\kappa = -q \cot(qa) \quad \text{anti-sym.}$$

With $\gamma := a\sqrt{q^2 + \kappa^2} = \sqrt{2Ma^2|V_0|}$ and $y := aq$

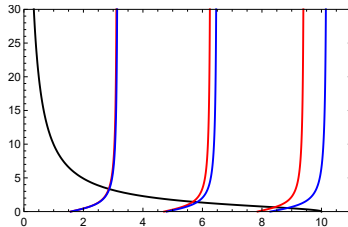
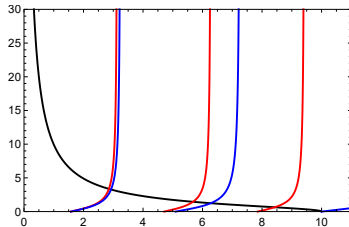
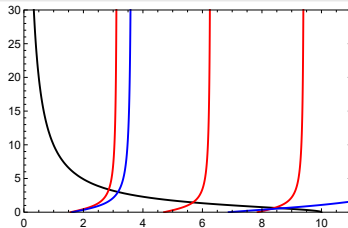
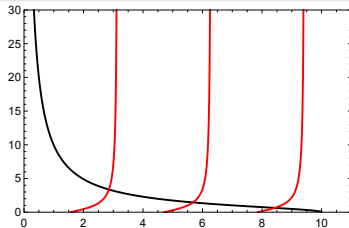
$$\frac{\sqrt{\gamma^2 - y^2}}{y} = \begin{cases} \tan\left(\frac{a}{\lambda} \arctan\left(\frac{\lambda}{a}\right)y\right) & \text{symmetric solution} \\ -\cot\left(\frac{a}{\lambda} \arctan\left(\frac{\lambda}{a}\right)y\right) & \text{anti-sym. solution} \end{cases}$$

Graphical solutions for $\gamma = 10$ on next slides.

Sym. solutions $a/\lambda = 1, 5, 10, 20$ for GQM and SQM



Anti-sym. sol. $a/\lambda = 1, 5, 10, 20$ for GQM and SQM



Discussion - Special cases

- For all $a > 0$ at least one bound state exists
- No excited states occur when $\gamma < \frac{a}{\lambda} \tan(\frac{a}{\lambda}\pi)$
- Case $a = \lambda$: Single bound state with

$$E_1 = V_0 + \frac{1}{4M\lambda^2} \left(\sqrt{1 - 8V_0M\lambda^2} - 1 \right)$$

- Case $a \rightarrow 0$ such that $\alpha := a|V_0| = \text{const.}$

$$V(x) \rightarrow 2\alpha\delta(x) \quad \text{but} \quad E_1 \approx -\frac{a\alpha\pi^2}{4\lambda^2} \rightarrow 0$$

Alternative GUP-delta function defined via GUP integral ???

Summary: Gravitational vs. Standard QM

GQM

$$[\hat{X}, \hat{P}] = i \left(1 + \lambda^2 \hat{P}^2 \right)$$

$$\Delta X \Delta P \geq \frac{1}{2} (1 + \lambda^2 \langle \hat{P}^2 \rangle)$$

$$H = \frac{\hat{P}^2}{2M} + V(\hat{X})$$

$$\hat{P} = \frac{1}{i} \frac{1}{\lambda} \tanh(\lambda \partial_x)$$

$$\left[-\frac{1}{2M\lambda^2} \tanh^2(\lambda \partial_x) + V(x) \right] \psi = E\psi$$

Highly non-trivial diff. eq.

few exact solutions
with many open questions

SQM

$$[\hat{x}, \hat{p}] = i$$

$$\Delta x \Delta p \geq \frac{1}{2}$$

$$H = \frac{\hat{p}^2}{2M} + V(\hat{x})$$

$$\hat{p} = \frac{1}{i} \partial_x$$

$$\left[-\frac{1}{2M} \partial_x^2 + V(x) \right] \psi = E\psi$$

Well-studied diff. eq.

many exact solutions
see any QM textbook

Prague's most famous writer: Franz Kafka 1883 - 1924



Georg Junker

Franz Kafka Das Schloß



Gravitational quantum mechanics