Symplectic-Haantjes geometry of first order magnetic (super)integrable systems

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Motivation

Despite significant effort in the new millennium, integrable systems immersed in magnetic fields are not yet well understood, mainly because the 1:1 correspondence with separation of variables in the Hamilton-Jacobi (HJ) equation is broken. Moreover, the standard theory of separation of variables on the configuration space requires fixing of the gauge in a somewhat ad hoc manner.

In this talk we study the geometry of physically relevant integrable systems with magnetic fields using the recently proposed formalism based on symplectic-Haantjes ($\omega\mathscr{H}$) manifolds. In addition to theoretical insight, the Haantjes geometry naturally determines separation coordinates regardless of the choice of gauge. We also obtain a new family of integrable systems on curved manifolds by generalizing the obtained geometries.

Contents

Nijenhuis and Haantjes operators

Let M be a differentiable manifold and $\boldsymbol{K} : TM \to TM$ be a (1,1) tensor field.

Definition (Nijenhuis 1951)

The **Nijenhuis torsion** of $\boldsymbol{K}$ is the vector-valued 2-form defined by

$$\mathcal{T}_{\mathbf{K}}(X,Y) := \boldsymbol{K}^2[X,Y] + [\boldsymbol{K}X, \boldsymbol{K}Y] - \boldsymbol{K}\Big([X, \boldsymbol{K}Y] + [\boldsymbol{K}X, Y]\Big),$$

where $X, Y \in TM$ and $[\ ,\ ]$ denotes the commutator of two vector fields.

Definition (Haantjes, 1955)

The **Haantjes torsion** of $\boldsymbol{K}$ is the vector-valued 2-form defined by

$$\mathcal{H}_{\mathbf{K}}(X,Y) := \mathbf{K}^2\mathcal{T}_{\mathbf{K}}(X,Y) + \mathcal{T}_{\mathbf{K}}(\mathbf{K}X, \mathbf{K}Y) - \mathbf{K}\Big(\mathcal{T}_{\mathbf{K}}(X, \mathbf{K}Y) + \mathcal{T}_{\mathbf{K}}(\mathbf{K}X, Y)\Big).$$

Definition

A **Nijenhuis/Haantjes operator** is an operator whose Nijenhuis/Haantjes torsion identically vanishes.

Examples of Nijenhuis and Haantjes operators

- **Every operator** on a 2-dimensional manifold is a **Haantjes operator**. In general, it is not a Nijenhuis one.
- Let M be an n-dimensional manifold and $(x_1, \ldots, x_n)$ a local chart on M. Let us consider the **simple** operators

$$\mathbf{K}_1 := \begin{pmatrix} \lambda_1(x_1) & 0 & \cdots & 0 \\ 0 & \lambda_2(x_2) & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n(x_n) \end{pmatrix},$$

$$\mathbf{K}_2 := \begin{pmatrix} \lambda_1(x_1, \ldots, x_n) & 0 & \cdots & 0 \\ 0 & \lambda_2(x_1, \ldots, x_n) & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n(x_1, \ldots, x_n) \end{pmatrix}.$$

$\mathbf{K}_1$ is a **Nijenhuis** operator, $\mathbf{K}_2$ is a **Haantjes** operator.

Haantjes algebras: definition

An associative algebra of Haantjes operators of rank m, hereafter **Haantjes algebra**, is a pair $(M, \mathscr{H})$ where

- M is a differentiable manifold of dimension n;
- $\mathscr{H}$ is a set of Haantjes operators $\boldsymbol{K} : TM \to TM$, that generate i) a **free module** of rank m over the ring of smooth functions on M

  $$\mathcal{H}_{\left(f(\mathbf{x})\mathbf{K}_1+g(\mathbf{x})\mathbf{K}_2\right)}(X, Y) = \mathbf{0}, \quad \forall \boldsymbol{K}_1, \boldsymbol{K}_2 \in \mathscr{H}, \quad \forall X, Y \in TM,$$

  where $f(\boldsymbol{x}), g(\boldsymbol{x})$ are arbitrary smooth functions on M. **Rank m** means that the module is generated by a "basis" with m elements. ii) a **ring** w.r.t. the composition operation

  $$\mathcal{H}_{\left(\mathbf{K}_1\mathbf{K}_2\right)}(X, Y) = \mathcal{H}_{\left(\mathbf{K}_2\mathbf{K}_1\right)}(X, Y) = \mathbf{0}, \forall \boldsymbol{K}_1, \boldsymbol{K}_2 \in \mathscr{H}, \forall X, Y \in TM.$$

- In addition, if $\boldsymbol{K}_1\boldsymbol{K}_2 = \boldsymbol{K}_2\boldsymbol{K}_1$, $\forall \boldsymbol{K}_1, \boldsymbol{K}_2 \in \mathscr{H}$ the algebra $\mathscr{H}$ will be called an **Abelian Haantjes algebra**.

Basic examples

A natural Haantjes algebra is realized, in a local chart $\{U, \boldsymbol{x} = (x^1, \ldots, x^n)\}$, by any set of **diagonal operators** of the form

$$\mathbf{K} = \sum_{k=1}^{n} \lambda_k(\boldsymbol{x}) \frac{\partial}{\partial x^k} \otimes \mathrm{d}x^k, \tag{1}$$

where the smooth functions $\lambda_k(\boldsymbol{x})$ play the role of **eigenvalue fields** of $\mathbf{K}$. Such operators generate an algebraic structure that will be said to be a *diagonal* Haantjes algebra.

More generally, we shall say that a Haantjes algebra is **semisimple** if each operator $\mathbf{K} \in \mathscr{H}$ is semisimple, i.e. its (proper) eigenvectors form a frame in all open neighborhoods $U \subset M$.

$\omega\mathscr{H}$ manifolds

P. Tempesta & G. Tondo, *Ann. Mat. Pura Appl.* (2021)

Definition

A $\omega\mathscr{H}$ **(or symplectic–Haantjes) manifold** of class m is a triple $(\mathbf{M}, \omega, \mathscr{H})$ which satisfies the following properties:

i) (M, ω) is a symplectic manifold of dimension $2n$;

ii) ω is a symplectic form in M;

iii) $\mathscr{H}$ is an **Abelian Haantjes algebra** of rank m;

iv) $(\omega, \mathscr{H})$ are algebraically compatible, that is

$$\omega(X, \boldsymbol{K}Y) = \omega(\boldsymbol{K}X, Y) \qquad \forall \boldsymbol{K} \in \mathscr{H}.$$

For our purposes, the manifold M will be the phase space $M = T^*Q$.

Proposition 2 in D. Reyes et al. (2024) implies that compatibility with ω can only work with Abelian Haantjes algebras.

Diagonalization of Haantjes algebras

P. Tempesta & G. Tondo. J. Geom. Phys. (2021).

Theorem (Simultaneous diagonalization)

Let $(M, \mathscr{H})$ be an **Abelian** *Haantjes algebra.*
i) If $\mathscr{H}$ is a **semisimple Haantjes algebra***, then there exist local coordinate charts $\{(x_1, \ldots, x_n)\}$ where all $\boldsymbol{K} \in \mathscr{H}$ can be simultaneously diagonalized.*

Conversely, let $\{\boldsymbol{K}_1, \ldots, \boldsymbol{K}_m\}$ be a commuting set of m $C^\infty(M)$–linearly independent operators. If they share a set of local coordinate charts in which they take a **diagonal form***, then they generate a semisimple Abelian Haantjes algebra (of rank not smaller than m).*

ii) More generally, if the Abelian Haantjes algebra $(M, \mathscr{H})$ is **non-semisimple***, then there exist local coordinate charts $\{(x_1, \ldots, x_n)\}$ where all $\boldsymbol{K} \in \mathscr{H}$ take a* **block-diagonal form***.*

Haantjes chains, Darboux-Haantjes coordinates

Definition

Let $(M, \mathscr{H})$ be a Haantjes algebra of rank m. A smooth function H generates a **Haantjes chain** of 1-forms of length m if there exist a distinguished basis $\{\tilde{\boldsymbol{K}}_1, \ldots, \tilde{\boldsymbol{K}}_m\}$ of $\mathscr{H}$ such that $\tilde{\boldsymbol{K}}_\alpha^T \mathrm{d}H =: \mathrm{d}H_\alpha,\ \alpha = 1, \ldots, m$ are locally exact.

Theorem (D. Reyes, P. Tempesta & G. Tondo, 2022)

There is a 1:1 correspondence between complete **Louville integrability**, **separation of variables** *for the* **Hamilton–Jacobi equations** *associated with the integrals* $\{H_1, H_2, \ldots, H_n\}$, *and an* $\omega\mathcal{H}$ **manifold** *of class n with a Haantjes chain generated by* H_1 *with operators*

$$\boldsymbol{K}_\alpha = \sum_{i=1}^{n} \frac{\frac{\partial H_\alpha}{\partial p_i}}{\frac{\partial H_1}{\partial p_i}} \left(\frac{\partial}{\partial q^i} \otimes \mathrm{d}q^i + \frac{\partial}{\partial p_i} \otimes \mathrm{d}p_i \right) \qquad \alpha = 1, \ldots, n.$$

The **canonical coordinates** $(\boldsymbol{q}, \boldsymbol{p})$ *where* $\boldsymbol{K}_\alpha$ *take this diagonal form are called* **Darboux-Haantjes** *(DH) coordinates.*

Finding Darboux-Haantjes coordinates

P. Tempesta & G. Tondo. J. Geom. Phys. (2021).

1. Given an algebra $\mathscr{H}$, determine the nontrivial **joint eigen-distributions** $\mathcal{V}_a(\boldsymbol{x})$ as the intersection of (generalized) eigen-distributions $\mathcal{D}^{(j)}_{i_m}(\boldsymbol{x}) := ker\big(\boldsymbol{K}^{(j)} - \lambda_{i_m}\boldsymbol{I}\big)^{\rho_i}(\boldsymbol{x})$ corresponding to different operators $\boldsymbol{K}^{(j)}$

$$\mathcal{V}_a(\boldsymbol{x}) := \bigoplus_{i_1,\dots,i_m}^{s_1,\dots,s_m} \mathcal{D}^{(1)}_{i_1}(\boldsymbol{x}) \bigcap \dots \bigcap \mathcal{D}^{(m)}_{i_m}(\boldsymbol{x}), \qquad a = 1, \dots, v \leq n.$$

2. Construct a basis of closed 1-forms corresponding to each **annihilator** $(\bigoplus_{a=1,\dots,\hat{i},\dots,n} \mathcal{V}_a)^\circ$, where we omit one $\mathcal{V}_a(\boldsymbol{x})$ each time.
3. Find the local **characteristic and canonical coordinates** by integrating (suitable combinations of) the locally exact 1-forms.

Contents

Application to 3D constant magnetic field

Let us consider an electron in the **constant magnetic field**

$$\vec{B}(\vec{x}) = b\ \vec{e}_z$$

with no electric potential on 3D Euclidean space. The corresponding Hamiltonian reads

$$H = \frac{1}{2}\left(\vec{p} + \vec{A}(\vec{x})\right)^2 \equiv \frac{1}{2}\vec{\Pi}(\vec{x})^2.$$

$\vec{A}(\vec{x})$ is the vector potential generating the magnetic field $\vec{B} = \nabla \times \vec{A}$. The superintegrable system admits 4 independent **first-degree integrals**

$$H_1 = \Pi_x + by,\ \ H_2 = \Pi_y - bx,\ \ H_3 = \Pi_z,\ \ H_4 = (x\Pi_y - y\Pi_x) - \tfrac{b}{2}(x^2 + y^2)$$

and the fifth, nonpolynomial one (A. Marchesiello, L. Šnobl & P. Winternitz, 2015).

$$H_5 = -\Pi_x \cos\left(\frac{bz}{\Pi_z}\right) - \Pi_y \sin\left(\frac{bz}{\Pi_z}\right).$$

Cartesian $\omega\mathscr{H}$ structure I

In the following, we shall provide a set of **semisimple** Haantjes operators that solve the chain equations $\boldsymbol{K}_i^T dH = dH_i$, jointly with their spectral properties. We focus on the Cartesian $\omega\mathscr{H}$ structure $\mathscr{H}_1 = \{\boldsymbol{I}_{6\times 6}, \boldsymbol{K}_1, \boldsymbol{K}_3\}$.

$$\boldsymbol{K}_1 = \frac{1}{\Pi_x}\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & \partial_x A_y & \partial_x A_z & 1 & 0 & 0 \\ -\partial_x A_y & 0 & 0 & 0 & 0 & 0 \\ -\partial_x A_z & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\lambda_1^{(1)} = 0,\ \mathcal{D}_1 = \left\langle \partial_x A_z \frac{\partial}{\partial y} - \partial_x A_y \frac{\partial}{\partial z}, \frac{\partial}{\partial y} - \partial_x A_y \frac{\partial}{\partial p_x}, \frac{\partial}{\partial p_y}, \frac{\partial}{\partial p_z} \right\rangle,$$

$$\lambda_2^{(1)} = \frac{1}{\Pi_x},\ \mathcal{D}_2 = \left\langle \frac{\partial}{\partial x} - \partial_x A_y \frac{\partial}{\partial p_y} - \partial_x A_z \frac{\partial}{\partial p_z}, \frac{\partial}{\partial p_x} \right\rangle.$$

Cartesian $\omega\mathscr{H}$ structure II

$$
\boldsymbol{K}_3 = \frac{1}{\Pi_z}\left[\begin{array}{ccc|ccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
\hline
0 & 0 & -\partial_z A_x & 0 & 0 & 0 \\
0 & 0 & -\partial_z A_y & 0 & 0 & 0 \\
\partial_z A_x & \partial_z A_y & 0 & 0 & 0 & 1
\end{array}\right],
$$

$$
\lambda_1^{(3)} = 0,\ \mathcal{D}_3 = \left\langle \partial_z A_y \frac{\partial}{\partial x} - \partial_z A_x \frac{\partial}{\partial y}, \frac{\partial}{\partial x} - \partial_z A_x \frac{\partial}{\partial p_z}, \frac{\partial}{\partial p_x}, \frac{\partial}{\partial p_y} \right\rangle,
$$

$$
\lambda_2^{(3)} = \frac{1}{\Pi_z},\ \mathcal{D}_4 = \left\langle \frac{\partial}{\partial z} - \partial_z A_x \frac{\partial}{\partial p_x} - \partial_z A_y \frac{\partial}{\partial p_y}, \frac{\partial}{\partial p_z} \right\rangle.
$$

DH coordinates for the Cartesian $\omega\mathscr{H}$ structure I

We now determine the **Darboux-Haantjes coordinates** for $\mathscr{H}_1$.
Defining the distributions

$$\mathcal{V}_1 = \mathcal{D}_1 \cap \mathcal{D}_3, \quad \mathcal{V}_2 = \mathcal{D}_1 \cap \mathcal{D}_4, \quad \mathcal{V}_3 = \mathcal{D}_2 \cap \mathcal{D}_3, \quad \mathcal{V}_4 = \mathcal{D}_2 \cap \mathcal{D}_4 = \emptyset,$$

we can work out the annihilators

$$\begin{aligned}
(\mathcal{V}_1 \oplus \mathcal{V}_2)^\circ &= \langle dx, \partial_x A_y dy + \partial_z A_x dz + dp_x \rangle, \\
(\mathcal{V}_1 \oplus \mathcal{V}_3)^\circ &= \langle dz, \partial_x A_z dx + \partial_y A_z dy + dp_z \rangle, \\
(\mathcal{V}_2 \oplus \mathcal{V}_3)^\circ &= \langle dy, \partial_x A_y dx + \partial_z A_y dz + dp_y \rangle.
\end{aligned}$$

Integrating them, we find a set of Darboux-Haantjes coordinates for each choice of gauge

$$\begin{aligned}
q^1 &= x, & p_1 &= \Pi_x + by, \\
q^2 &= y, & p_2 &= \Pi_y, \\
q^3 &= z, & p_3 &= \Pi_z.
\end{aligned} \tag{2}$$

DH coordinates for the Cartesian $\omega\mathscr{H}$ structure II

In this chart, the operators of the algebra $\mathscr{H}_1$ are **indeed diagonal**

$$\boldsymbol{K}_1 = \frac{1}{p_1 - bq^2}\,\mathrm{diag}(1,0,0,1,0,0),$$
$$\boldsymbol{K}_3 = \frac{1}{p_3}\,\mathrm{diag}(0,0,1,0,0,1);$$

the corresponding Hamiltonians H, H_1, H_3 are **indeed separable**.

$$H = \frac{1}{2}\left[(p_1 - bq^2)^2 + p_2^2 + p_3^2\right],$$
$$H_1 = p_1, \quad H_2 = p_2 - bq^1, \quad H_3 = p_3,$$
$$H_4 = q^1 p_2 - q^2 p_1 + \frac{b}{2}\left((q^2)^2 - (q^1)^2\right).$$

The form of H is the one we would obtain by choosing the gauge $\vec{A} = (-bq^2, 0, 0)$ in the original Hamiltonian (13).

Contents

Stäckel geometry

In a suitable choice of coordinates, a large class of separable systems satisfy the **Stäckel equation**

$$\begin{bmatrix} \tilde{H}_1 \\ \tilde{H}_2 \\ \tilde{H}_3 \end{bmatrix} = S^{-1} \begin{bmatrix} f_1(q^1, p_1) \\ f_2(q^2, p_2) \\ f_3(q^3, p_3) \end{bmatrix}, \qquad S = \begin{bmatrix} S_{11}(q^1) & S_{12}(q^1) & S_{13}(q^1) \\ S_{21}(q^2) & S_{22}(q^2) & S_{23}(q^2) \\ S_{31}(q^3) & S_{31}(q^3) & S_{31}(q^3) \end{bmatrix},$$

where $\tilde{H}_i$ are the commuting Hamiltonians. Note the dependence for the rows of the **Stäckel matrix** S and of the generalized **Stäckel functions** f_i (Arnold, Kozlov and Neishtadt 1997).

The classical Stäckel functions for systems without magnetic field are always quadratic in momenta $f_k := \frac{1}{2} p_k^2 + W_k(q^k)$ and the associated Haantjes operator can be projected to a **Killing tensor**,

$$\boldsymbol{K}_{j-1} := \sum_{r=1}^{n} \frac{\tilde{S}_{jr}}{\tilde{S}_{1r}} \left(\frac{\partial}{\partial q^r} \otimes \mathrm{d}q^r + \frac{\partial}{\partial p_r} \otimes \mathrm{d}p_r \right) \xrightarrow{\pi} \tilde{\boldsymbol{K}}_{j-1} := \sum_{r=1}^{n} \frac{\tilde{S}_{jr}}{\tilde{S}_{1r}} \frac{\partial}{\partial q^r} \otimes \mathrm{d}q^r .$$

Here $\tilde{S}_{jk}$ denotes the cofactor of the element S_{kj}.

Stäckel analysis for the system in a constant magnetic field

To show that for our constant magnetic field, we shall work in the coordinates (2) that diagonalize the Haantjes algebra $\mathscr{H}_1$ and use a suitable functional combination of the Hamiltonians:

$$\begin{aligned} \tilde{H}_1 &= 2H - H_1^2 = -2bq^2 p_1 + b^2 (q^2)^2 + p_2^2 + p_3^2, \\ \tilde{H}_2 &= H_1 = p_1, \quad \tilde{H}_3 = H_3^2 = p_3^2. \end{aligned} \tag{3}$$

(This does not change the underlying foliations by invariant tori.) Then it follows that the Stäckel equation is satisfied by choosing

$$S = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2bq^2 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \tag{4}$$

and therefore $f_1(q^1, p_1) = p_1$, $f_2(q^2, p_2) = p_2^2 + b^2 (q^2)^2$ and $f_3(q^3, p_3) = p_3^2$. Note that f_1 is **linear, i.e. generalized.**

New integrable system I

We obtain a **new integrable system with magnetic field on a curved background** by generalizing the Stäckel matrix and functions above. Namely, let us consider the Stäckel matrix

$$S = \begin{bmatrix} 0 & 1 & 0 \\ 1 & b\xi_1(q^2) & \xi_2(q^2) \\ 0 & 0 & 1 \end{bmatrix}, \tag{5}$$

where ν_1, ν_2 are arbitrary functions, and the Stäckel functions

$$f_1 = \mu_1(q^1)p_1, \quad f_2 = \mu_2(q^2)p_2^2 + b^2\mu_4(q^2), \quad f_3 = \mu_3(q^3)p_3^2$$

where functions $\mu_1, \ldots, \mu_4$ are also arbitrary. This yields the following **integrable system**

$$\hat{H}_1 = \mu_3(q^3)p_3^2 + b^2\mu_4(q^2), \tag{6}$$

$$\hat{H}_2 = \mu_1(q^1)p_1, \tag{7}$$

$$\hat{H}_3 = \mu_3(q^3)p_3^2.$$

New integrable system - magnetic version

In order to construct the "magnetic" version of the system (6), we first represent it in an algebraically equivalent way

$$(\hat{H}_1 + \hat{H}_2^2, \hat{H}_2, \hat{H}_3) \tag{8}$$

and use the transformation

$$\begin{aligned} q^1 &= x, & p_1 &= \Pi_x + \frac{b\xi_1(y)}{2\mu_1(x)}, \quad \mu_1(x) \neq 0 \\ q^2 &= y, & p_2 &= \Pi_y, \\ q^3 &= z, & p_3 &= \Pi_z, \end{aligned}$$

These coordinates are canonical if we take into account the magnetic field

$$B = \frac{b}{2}\xi_1'(y)\sqrt{\mu_2(y)}\vec{e}_z.$$

New integrable system - magnetic version

With this transformation, system (8) provides us with

$$H = \mu_1^2(x)\Pi_x^2 + \mu_2(y)\Pi_y^2 + \xi_2(y)\mu_3(z)\Pi_z^2 + b^2\left(\mu_4(y) - \frac{\nu_1^2(y)}{4}\right),$$

$$H_1 = \mu_1(x)\left(\Pi_x + \frac{b}{2}\frac{\nu_1(y)}{\mu_1(x)}\right), \qquad H_2 = \mu_3(z)\Pi_z^2.$$

i.e. a family of integrable magnetic models with a nontrivial (pseudo)Riemannian metric

$$g = \mathrm{diag}[\mu_1^2(x), \mu_2(y), \xi_2(y)\mu_3(z)].$$

Summary

- We have reviewed the notion of $\omega\mathscr{H}$ **manifold** as needed for finite-dimensional Hamiltonian systems.
- The geometry enables search for **Darboux-Haantjes coordinates**, where the Haantjes algebra diagonalizes and the system is Liouville integrable through separation of variables.
- For a magnetic system, we have found **separation coordinates including the gauge** by an algorithm for the first time.
- We have also used the connection to **Stäckel geometry** to generate a **new family of integrable systems** with magnetic field on curved (pseudo)Riemannian spaces.
- The procedure, leading to new separable coordinates on phase space (work in progress on helical undulator), should be general (no assumptions on the integrals), but is computationally demanding (Haantjes condition is quartic in tensor components).

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