

Renormalization of Point Scatterers in 2D & 3D, the Coincidence-Limit Problem and Dynamical Formulation of stationary scattering

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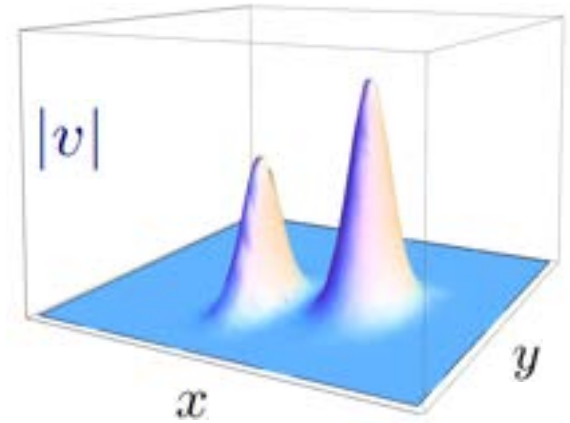


Outline:

- N^{th} -order Born approximation
- Multi-delta-function potentials in 2D & 3D, their renormalization, and coincidence-limit problem
- Dynamical formulation of stationary scattering
- Applications:
 - Consistent & finite treatment of multi-delta-function potentials in 2D & 3D
 - Potentials for which N^{th} -order Born approximation is exact.
- Conclusions

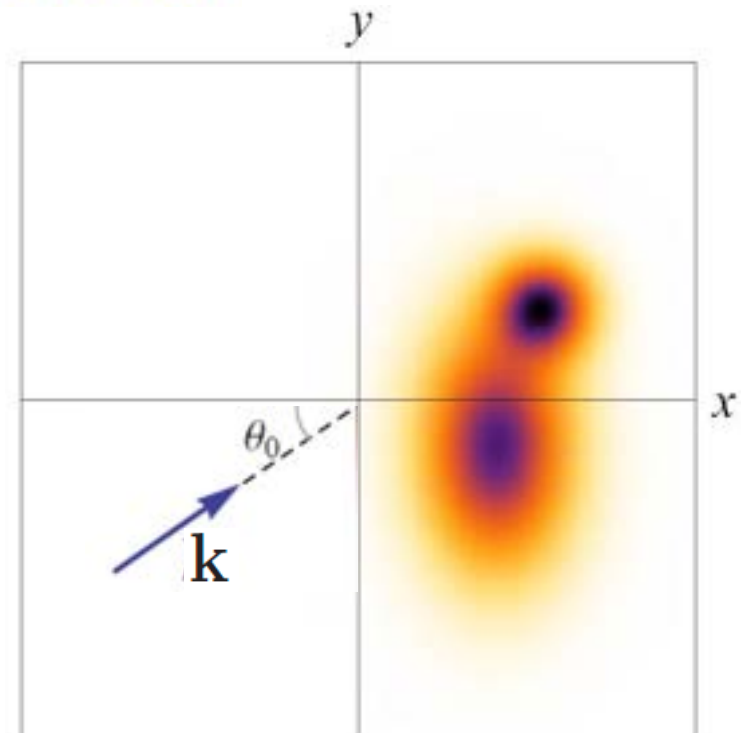
Scattering in 2D

$$[-\partial_x^2 - \partial_y^2 + v(x, y)] \psi(x, y) = k^2 \psi(x, y)$$



$\mathbf{k} = k(\cos \theta_0, \sin \theta_0)$: Incident wavevector

θ_0 : Incidence angle



Scattering in 2D

$$[-\partial_x^2 - \partial_y^2 + v(x, y)] \psi(x, y) = k^2 \psi(x, y)$$

Lippmann-Schwinger eq./Green's fn. method:

$$\left. \begin{aligned} (\hat{\mathbf{p}}^2 + \hat{v})|\psi\rangle &= k^2|\psi\rangle \\ \hat{\mathbf{p}}^2|\mathbf{k}\rangle &= k^2|\mathbf{k}\rangle \end{aligned} \right\} \Rightarrow |\psi\rangle = |\mathbf{k}\rangle + \hat{G}^+ \hat{v} |\psi\rangle$$

$$\hat{G}^+ := (-\hat{\mathbf{p}}^2 + k^2 + i\epsilon)^{-1}$$

Lippmann-Schwinger eq./Green's fn. method:

$$\langle \mathbf{x} | \psi \rangle = \langle \mathbf{x} | \mathbf{k} \rangle + \langle \mathbf{x} | \hat{G}^+ \hat{v} | \psi \rangle$$

$$\langle \mathbf{x} | \hat{G}^+ | \mathbf{x}' \rangle := -\frac{i}{4} H_0^{(1)}(k|\mathbf{x} - \mathbf{x}'|)$$

Lippmann-Schwinger eq./Green's fn. method:

$$\langle \mathbf{x} | \psi \rangle = \langle \mathbf{x} | \mathbf{k} \rangle + \langle \mathbf{x} | \hat{G}^+ \hat{v} | \psi \rangle \rightarrow \frac{1}{2\pi} \left[e^{i\mathbf{k} \cdot \mathbf{x}} + \frac{e^{ikr}}{\sqrt{r}} f(\mathbf{k}', \mathbf{k}) \right]$$

$$\langle \mathbf{x} | \hat{G}^+ | \mathbf{x}' \rangle := -\frac{i}{4} H_0^{(1)}(k|\mathbf{x} - \mathbf{x}'|) \quad \text{as } r := |\mathbf{x}| \rightarrow \infty$$

$$\mathbf{k}' := \frac{k\mathbf{x}}{r}$$

$$f(\mathbf{k}', \mathbf{k}) := -\pi \sqrt{2\pi i/k} \langle \mathbf{k}' | \hat{v} | \psi \rangle: \text{ Scattering amplitude}$$

Lippmann-Schwinger eq./Green's fn. method:

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$$f(\mathbf{k}', \mathbf{k}) := -\pi \sqrt{2\pi i/k} \langle \mathbf{k}' | \hat{v} | \psi \rangle \quad \text{as } r := |\mathbf{x}| \rightarrow \infty$$

$$|\psi\rangle = |\mathbf{k}\rangle + \hat{G}^+ \hat{v} |\psi\rangle$$

$$\Rightarrow |\psi\rangle = [1 - \hat{G}^+ \hat{v}]^{-1} |\mathbf{k}\rangle = \sum_{n=0}^{\infty} (\hat{G}^+ \hat{v})^n |\mathbf{k}\rangle$$

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Lippmann-Schwinger eq./Green's fn. method:

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$$N\text{-th Born approximation: } f(\mathbf{k}', \mathbf{k}) \approx \sum_{\ell=1}^N f_{\ell}(\mathbf{k}', \mathbf{k})$$

Multi-delta-function potential:

$$v(\mathbf{x}) = \sum_{j=1}^N \lambda_j \delta(\mathbf{x} - \mathbf{a}_j), \quad \mathbf{x} \in \mathbb{R}^d, \quad d \in \{2, 3\}$$

$$\Rightarrow \hat{v} := \sum_{j=1}^N \lambda_j |\mathbf{a}_j\rangle\langle\mathbf{a}_j|$$

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$$\Rightarrow \psi(\mathbf{x}) = \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{d/2}} + \sum_{n=1}^N G(\mathbf{x} - \mathbf{a}_n) X_n \quad (\star)$$

$$G(\mathbf{x} - \mathbf{x}') := \langle\mathbf{x}|\hat{G}|\mathbf{x}'\rangle, \quad X_n := \mathfrak{z}_n \psi(\mathbf{a}_n)$$

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$$\psi(\mathbf{x}) \rightarrow \frac{1}{(2\pi)^{d/2}} \left[e^{i\mathbf{k}\cdot\mathbf{x}} + \frac{f(\mathbf{k}', \mathbf{k}) e^{ikr}}{r^{\frac{d-1}{2}}} \right] \quad \text{for } r \rightarrow \infty$$

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$$G(\mathbf{x}) = \begin{cases} -\frac{i}{4} H_0^{(1)}(kr) & \text{for } d = 2 \\ -\frac{1}{4\pi} \frac{e^{ikr}}{r} & \text{for } d = 3 \end{cases}$$

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$$f(\mathbf{k}', \mathbf{k}) = \frac{c_d}{\sqrt{k^{3-d}}} \sum_{m,n=1}^N A_{mn}^{-1} e^{i(\mathbf{a}_n\cdot\mathbf{k} - \mathbf{a}_m\cdot\mathbf{k}')}$$

$$c_d := \begin{cases} -\sqrt{i/8\pi} & \text{for } d=2 \\ -1/4\pi & \text{for } d=3 \end{cases}$$

$$\mathfrak{f}(\mathbf{k}',\mathbf{k})=\frac{c_d}{\sqrt{k^{3-d}}}\sum_{m,n=1}^N A_{mn}^{-1}e^{i(\mathbf{a}_n\cdot\mathbf{k}-\mathbf{a}_m\cdot\mathbf{k}')}$$

$$A_{mn}:=\mathfrak{z}_n^{-1}\delta_{mn}-G(\mathbf{a}_m-\mathbf{a}_n)\qquad G(\mathbf{x})=\left\{\begin{array}{ll}-\frac{i}{4}H_0^{(1)}(kr) & \text{for } d=2\\[1ex]-\frac{1}{4\pi}\frac{e^{ikr}}{r} & \text{for } d=3\end{array}\right.$$

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But : $G(0) = \infty \Rightarrow A_{mm} = \mathfrak{z}_m^{-1} - G(0) = \infty !$

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Absorb the singularity of $G(\mathbf{x})$ at $\mathbf{x} = 0$ in \mathfrak{z}_m^{-1} .

Renormalize: $\mathfrak{z}_m \rightarrow \tilde{\mathfrak{z}}_m$

$$f(\mathbf{k}', \mathbf{k}) = \frac{c_d}{\sqrt{k^{3-d}}} \sum_{m,n=1}^N A_{mn}^{-1} e^{i(\mathbf{a}_n \cdot \mathbf{k} - \mathbf{a}_m \cdot \mathbf{k}')}$$

$$d = 2: A_{mn} = \begin{cases} \tilde{\mathfrak{z}}_m^{-1} + \frac{i}{4} & \text{for } m = n \\ \frac{i}{4} H_0^{(1)}(k|\mathbf{a}_m - \mathbf{a}_n|) & \text{for } m \neq n \end{cases}$$

$$d = 3: A_{mn} = \begin{cases} \tilde{\mathfrak{z}}_n^{-1} + \frac{ik}{4\pi} & \text{for } m = n \\ \frac{e^{ik|\mathbf{a}_m - \mathbf{a}_n|}}{4\pi|\mathbf{a}_m - \mathbf{a}_n|} & \text{for } m \neq n \end{cases}$$

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$$N = 1: v(\mathbf{x}) = \mathfrak{z}_1 \delta(\mathbf{x} - \mathbf{a}_1)$$

$$f(\mathbf{k}', \mathbf{k}) = \begin{cases} -\sqrt{\frac{i}{8\pi k}} \frac{e^{i\mathbf{a}_1 \cdot (\mathbf{k} - \mathbf{k}')}}{\tilde{\mathfrak{z}}_1^{-1} + \frac{i}{4}} & \text{for } d = 2 \\ -\frac{1}{4\pi} \frac{e^{i\mathbf{a}_1 \cdot (\mathbf{k} - \mathbf{k}')}}{\tilde{\mathfrak{z}}_1^{-1} + \frac{ik}{4\pi}} & \text{for } d = 3 \end{cases}$$

[Jackiw 1991]

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Coincidence-limit problem:

$$\mathbf{a}_i \rightarrow \mathbf{a}_j \text{ for some } i \text{ and } j \Rightarrow f(\mathbf{k}', \mathbf{k}) \rightarrow 0!$$

$$f(\mathbf{k}', \mathbf{k}) = \frac{c_d}{\sqrt{k^{3-d}}} \sum_{m,n=1}^N A_{mn}^{-1} e^{i(\mathbf{a}_n \cdot \mathbf{k} - \mathbf{a}_m \cdot \mathbf{k}')}$$

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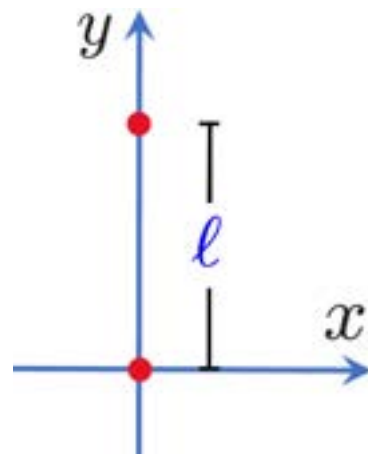
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WOLOG: Set $i = N - 1$ and $j = N$. Then $\mathbf{a}_i \rightarrow \mathbf{a}_j$ implies

$$v(\mathbf{x}) = \sum_{n=1}^N \tilde{z}_n \delta(\mathbf{x} - \mathbf{a}_n) \rightarrow \sum_{n=1}^{N-2} \tilde{z}_n \delta(\mathbf{x} - \mathbf{a}_n) + (\tilde{z}_{N-1} + \tilde{z}_N) \delta(\mathbf{x} - \mathbf{a}_{N-1})$$

In **2D**, WOLOG

N=2: $v(x, y) = \mathfrak{z}_1 \delta(x) \delta(y) + \mathfrak{z}_2 \delta(x) \delta(y - \ell), \ell \in \mathbb{R}^+$



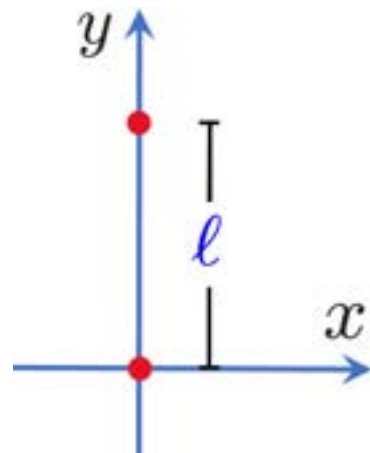
In **2D**, WOLOG

$$\mathbf{N=2}: v(x, y) = \beta_1 \delta(x) \delta(y) + \beta_2 \delta(x) \delta(y - \ell), \ell \in \mathbb{R}^+$$

$$f(\mathbf{k}', \mathbf{k}) = -\sqrt{\frac{i}{8\pi k}} \left[f_1(\mathbf{k}) + f_2(\mathbf{k}) e^{-ik\ell \sin \theta} \right]$$

$$f_1(\mathbf{k}) := \frac{4 \left[4\tilde{\beta}_2^{-1} + i - iH_0^{(1)}(k\ell) e^{ik\ell \sin \theta_0} \right]}{(4\tilde{\beta}_1^{-1} + i)(4\tilde{\beta}_2^{-1} + i) + H_0^{(1)}(k\ell)^2}$$

$$f_2(\mathbf{k}) := \frac{4 \left[(4\tilde{\beta}_1^{-1} + i) e^{ik\ell \sin \theta_0} - iH_0^{(1)}(k\ell) \right]}{(4\tilde{\beta}_1^{-1} + i)(4\tilde{\beta}_2^{-1} + i) + H_0^{(1)}(k\ell)^2}$$

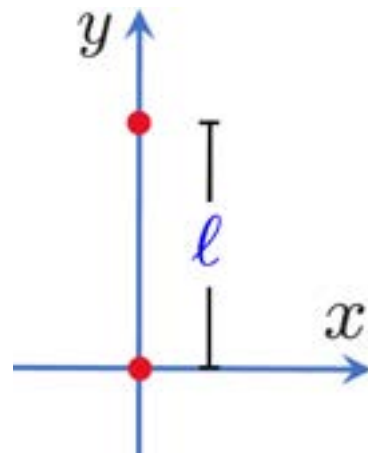


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$$f(\mathbf{k}', \mathbf{k}) = -\sqrt{\frac{i}{8\pi k}} \left[f_1(\mathbf{k}) + f_2(\mathbf{k}) e^{-ik\ell \sin \theta} \right]$$

$$\lim_{\ell \rightarrow 0} f_1(\mathbf{k}) = \lim_{\ell \rightarrow 0} f_2(\mathbf{k}) = 0 !$$

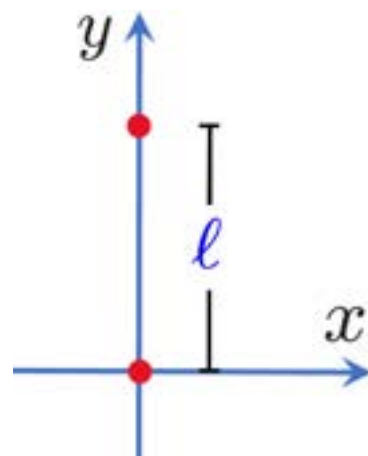


In **2D**, WOLOG

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For $\ell \rightarrow 0$, $v(x, y) \rightarrow \mathfrak{z} \delta(x) \delta(y)$, $\mathfrak{z} := \mathfrak{z}_1 + \mathfrak{z}_2$,

$$f(\mathbf{k}', \mathbf{k}) \rightarrow -\sqrt{\frac{i}{8\pi k}} \frac{1}{\tilde{\mathfrak{z}}^{-1} + \frac{i}{4}}$$

Dynamical formulation of stationary scattering (DFSS)

1D: Schrödinger eq.: $[-\partial_x^2 + v(x)]\psi(x) = k^2\psi(x)$

$v : \mathbb{R} \rightarrow \mathbb{C}$ is a short-range potential,

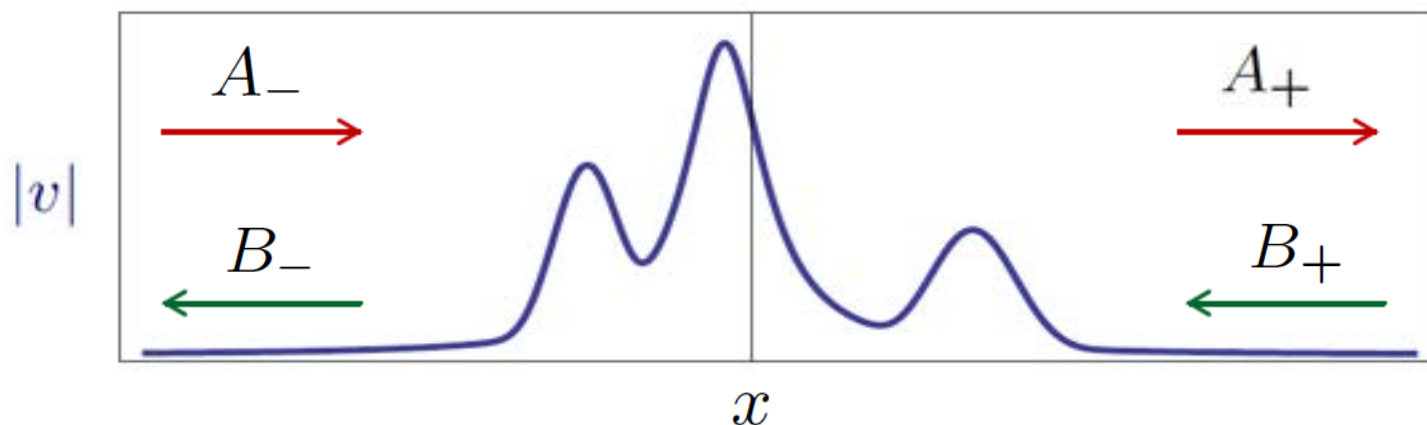
$$x v(x) \rightarrow 0 \quad \text{for} \quad x \rightarrow \pm\infty$$

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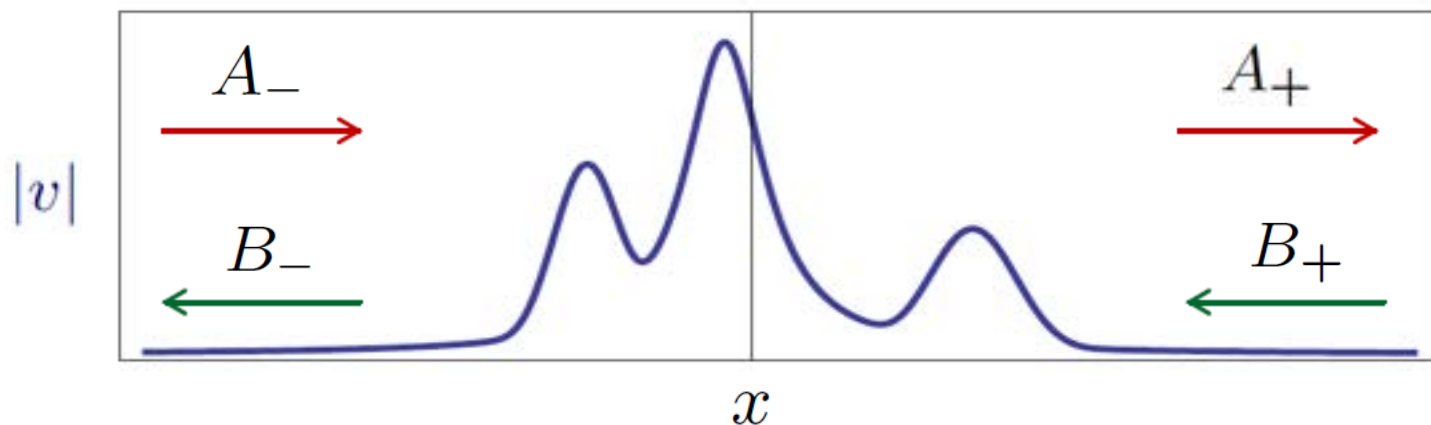


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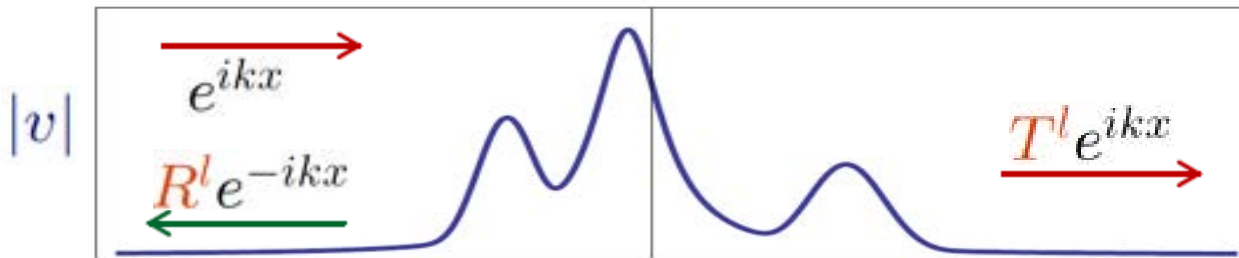
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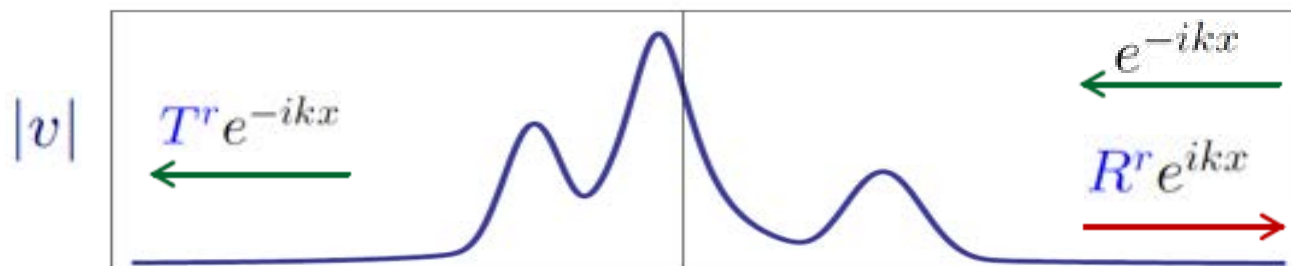
- Scattering from the left and right:

$$\psi^{\text{left}}(x) = \begin{cases} e^{ikx} + R^l e^{-ikx} & \text{for } x \rightarrow -\infty \\ T^l e^{ikx} & \text{for } x \rightarrow +\infty \end{cases}$$



- Scattering from the left and right:

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$$R^l = -\frac{M_{21}}{M_{22}}, \quad R^r = \frac{M_{12}}{M_{22}}, \quad T^l = T^r =: T = \frac{1}{M_{22}}.$$

Theorem: $\mathbf{M} = \mathbf{U}(+\infty, -\infty)$ where $\mathbf{U}(x, x_0)$ is the evolution operator for

$$\mathbf{H}(x) := \frac{v(x)}{2k} \begin{bmatrix} 1 & e^{-2ikx} \\ -e^{2ikx} & -1 \end{bmatrix}.$$

x plays the role of "time".

$$i\partial_x \mathbf{U}(x, x_0) = \mathbf{H}(x) \mathbf{U}(x, x_0)$$

[Ann. Phys. (NY), **341**, 77 (2014)]

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i.e.,

$$\begin{aligned} \mathbf{M} &= \mathcal{T} \exp \int_{-\infty}^{\infty} -i\mathbf{H}(x) dx \\ &= \mathbf{I} - i \int_{-\infty}^{\infty} dx_1 \mathbf{H}(x_1) \\ &\quad + (-i)^2 \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{x_2} dx_1 \mathbf{H}(x_2) \mathbf{H}(x_1) + \dots \end{aligned}$$

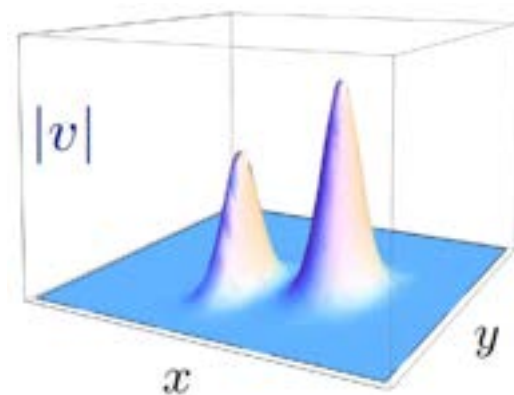
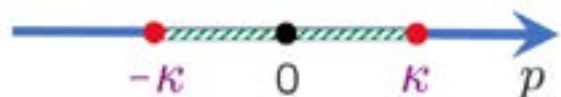
[Ann. Phys. (NY), **341**, 77 (2014)]

2D:

$$[-\partial_x^2 - \partial_y^2 + v(x, y)] \psi(x, y) = k^2 \psi(x, y)$$

\mathcal{F} : Vector space of functions $\phi : \mathbb{R} \rightarrow \mathbb{C}$

$$\mathcal{F}_k := \left\{ \phi \in \mathcal{F} \mid \phi(p) = 0 \text{ for } |p| \geq k \right\}$$



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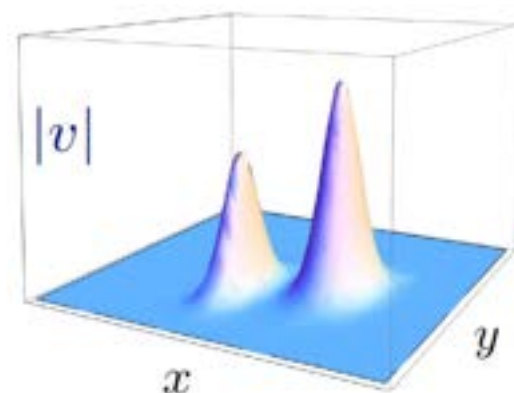
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$$\psi(x, y) \rightarrow \int_{-k}^k \frac{dp e^{ipy}}{4\pi^2 \varpi(p)} \left[A_{\pm}(p) e^{i\varpi(p)x} + B_{\pm}(p) e^{-i\varpi(p)x} \right] \text{ for } x \rightarrow \pm\infty$$

$$A_{\pm}, B_{\pm} \in \mathcal{F}_k \quad \varpi(p) := \sqrt{k^2 - p^2}$$

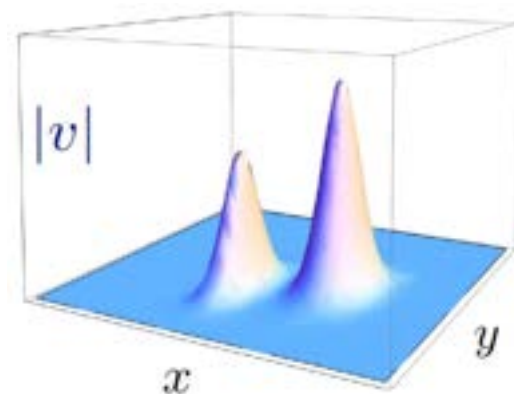


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$$\widehat{\mathbf{M}} : \mathcal{F}_k^2 \rightarrow \mathcal{F}_k^2$$

[PRA **93**, 042707 (2016)]

- $\widehat{\mathbf{M}}$ admits a formal **Dyson series** expansion:

$$\begin{aligned}\widehat{\mathbf{M}} &= \widehat{\Pi}_k \mathcal{T} \exp \left\{ -i \int_{-\infty}^{\infty} dx \widehat{\mathcal{H}}(x) \right\} \widehat{\Pi}_k \\ &= \widehat{\Pi}_k - i \int_{-\infty}^{\infty} dx_1 \widehat{\Pi}_k \widehat{\mathcal{H}}(x_1) \widehat{\Pi}_k \\ &\quad - \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{x_2} dx_1 \widehat{\Pi}_k \widehat{\mathcal{H}}(x_2) \widehat{\mathcal{H}}(x_1) \widehat{\Pi}_k + \dots\end{aligned}$$

Projection onto \mathcal{F}_k^2 : $(\widehat{\Pi}_k F)(p) := \begin{cases} F(p) & \text{for } p \in (-k, k) \\ 0 & \text{for } p \notin (-k, k) \end{cases}$

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$$\widehat{\mathcal{H}}(x) := \frac{1}{2} e^{-i\widehat{\varpi}x\sigma_3} v(x, \widehat{y}) \widehat{\varpi}^{-1} \mathcal{K} e^{i\widehat{\varpi}x\sigma_3},$$

$$(\widehat{\varpi}f)(p) := \varpi(p) f(p), \quad \sigma_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$(\widehat{y}f)(p) := i\partial_p f(p), \quad \mathcal{K} := \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}.$$

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$$= \frac{1}{2} \begin{bmatrix} e^{-ix \widehat{\varpi}} \widehat{u}(x) e^{ix \widehat{\varpi}} & e^{-ix \widehat{\varpi}} \widehat{u}(x) e^{-ix \widehat{\varpi}} \\ -e^{ix \widehat{\varpi}} \widehat{u}(x) e^{ix \widehat{\varpi}} & -e^{ix \widehat{\varpi}} \widehat{u}(x) e^{-ix \widehat{\varpi}} \end{bmatrix}$$

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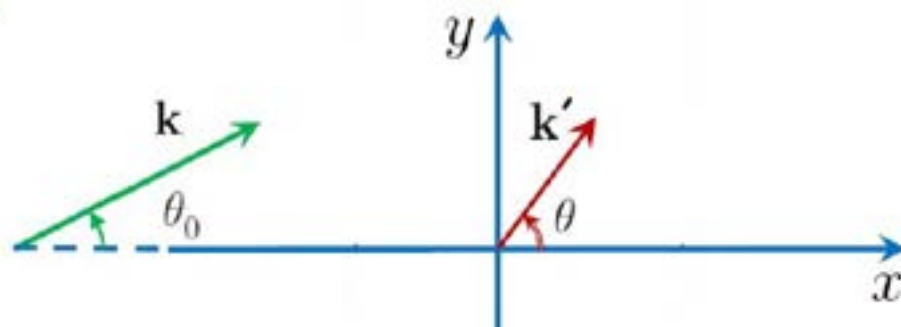
$$(v(x, \widehat{y}) f)(p) := \frac{1}{2\pi} \int_{-\infty}^{\infty} dq \tilde{v}(x, p - q) f(q)$$

$$\tilde{v}(x, p) := \int_{-\infty}^{\infty} dy e^{-ipy} v(x, y)$$

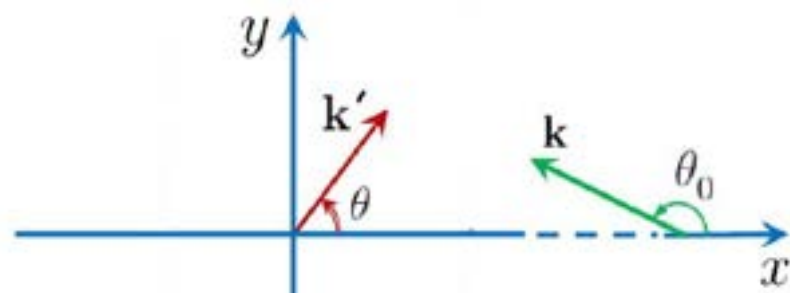
- \widehat{M} can be used to compute $f(k', k)$.

$$f(k', k) = \frac{e^{-i\pi/4}}{\sqrt{2\pi k}} \times \begin{cases} A_+(k \sin \theta) - 2\pi\delta(\theta - \theta_0) & \text{for } \cos \theta > 0 \\ B_-(k \sin \theta) & \text{for } \cos \theta < 0 \end{cases}$$

Left incidence: $\theta_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$



Right incidence: $\theta_0 \in (\frac{\pi}{2}, \frac{3\pi}{2})$

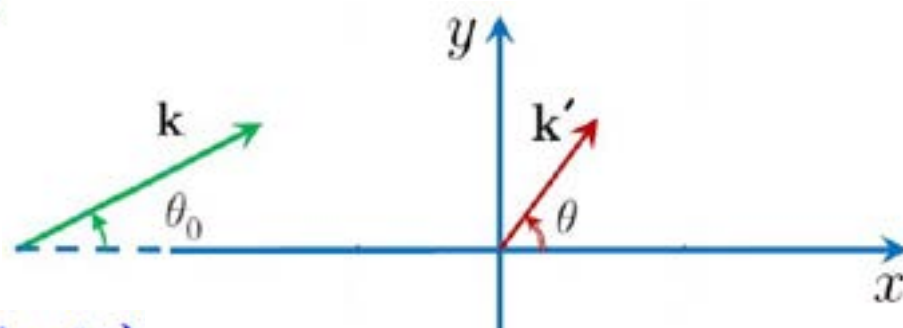


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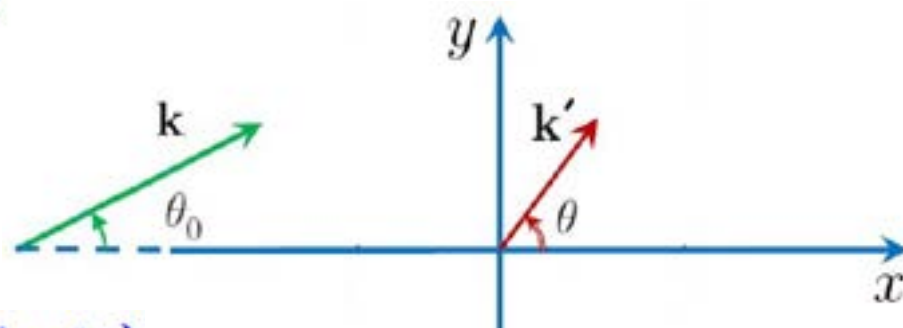
$$\Delta(p) := 2\pi k \cos \theta_0 \delta(p - k \sin \theta_0)$$

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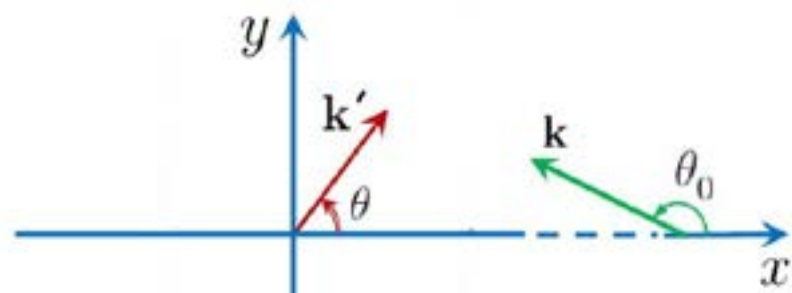
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For $v(x, y) = g(y)\delta(x)$,

$$\begin{aligned}\hat{\mathcal{H}}(x) &:= \frac{1}{2} e^{-i\hat{\varpi}x\sigma_3} v(x, \hat{y}) \hat{\varpi}^{-1} \mathcal{K} e^{i\hat{\varpi}x\sigma_3} \\ &= \frac{1}{2} \delta(x) g(\hat{y}) \hat{\varpi}^{-1} \mathcal{K}\end{aligned}$$

[Ann. Phys. **443**, 168966 (2022), arXiv: 2204.09554]

For $v(x, y) = g(y)\delta(x)$, $\hat{\mathcal{H}}(x) = \frac{1}{2}\delta(x)g(\hat{y})\hat{\omega}^{-1}\kappa$

$$\kappa := \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \Rightarrow \kappa^2 = 0 \Rightarrow \hat{\mathcal{H}}(x_1)\hat{\mathcal{H}}(x_2) = \hat{0}$$

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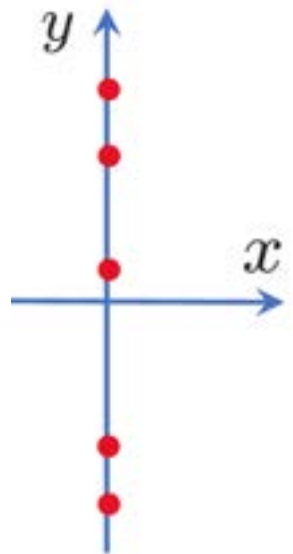
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For

$$v(x, y) = \sum_{n=1}^N \mathfrak{z}_n \delta(x) \delta(y - a_n)$$

we can use \hat{M}_{ij} to compute $\mathfrak{f}(\mathbf{k}', \mathbf{k})$.

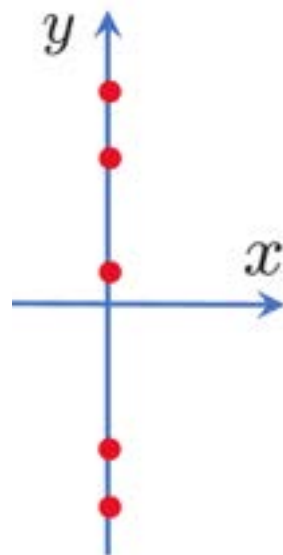


This again gives

$$f(k', k) = -\sqrt{\frac{i}{8\pi k}} \sum_{m,n=1}^N A_{mn}^{-1} e^{i(a_n \cdot k - a_m \cdot k')},$$

but now

$$A_{mn} := \begin{cases} \delta_m^{-1} + \frac{i}{4} & \text{for } m = n \\ \frac{i}{4} J_0(k|a_m - a_n|) & \text{for } m \neq n \end{cases}$$

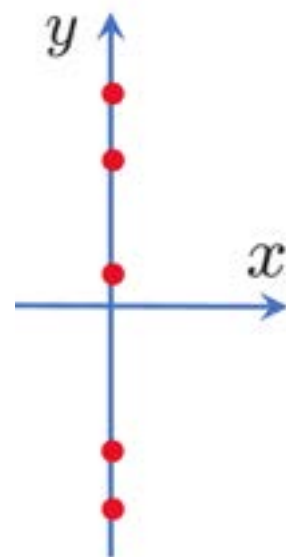


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Previously ([Lippmann-Schwinger approach](#)) we had:

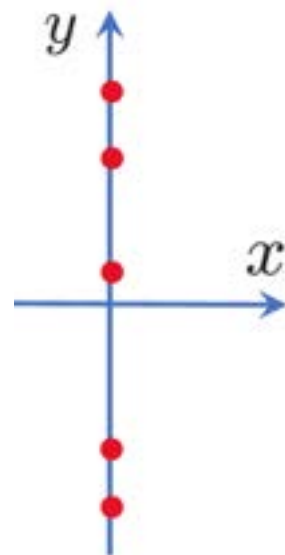
$$A_{mn} := \begin{cases} \tilde{\mathfrak{z}}_m^{-1} + \frac{i}{4} & \text{for } m = n \\ \frac{i}{4} H_0^{(1)}(k|a_m - a_n|) & \text{for } m \neq n \end{cases}$$

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- Nothing blows up \Rightarrow No need for renormalizing \mathfrak{z}_n

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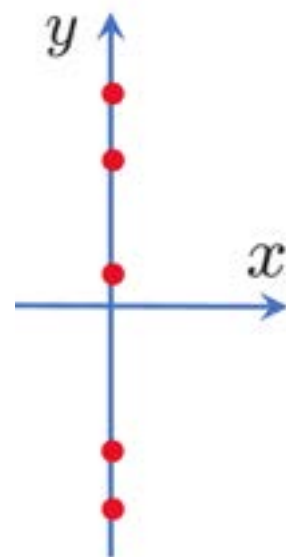
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- **This formula has correct coincidence limit.**

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Generalization to 3D

$$f(\mathbf{k}', \mathbf{k}) = -\frac{1}{4\pi} \sum_{m,n=1}^N A_{mn}^{-1} e^{i(\mathbf{a}_n \cdot \mathbf{k} - \mathbf{a}_m \cdot \mathbf{k}')}$$

LS+Renormalization gives:

$$A_{mn} = \begin{cases} \tilde{\mathfrak{J}}_n^{-1} + \frac{ik}{4\pi} & \text{for } m = n, \\ \frac{e^{ik|\mathbf{a}_m - \mathbf{a}_n|}}{4\pi|\mathbf{a}_m - \mathbf{a}_n|} & \text{for } m \neq n. \end{cases}$$

[Ann. Phys. **443**, 168966 (2022), arXiv: 2204.09554]

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$$f(k', k) = -\frac{1}{4\pi} \sum_{m,n=1}^N A_{mn}^{-1} e^{i(\mathbf{a}_n \cdot \mathbf{k} - \mathbf{a}_m \cdot \mathbf{k}')}$$

LS+Renormalization gives:

$$A_{mn} = \begin{cases} \tilde{\mathfrak{z}}_n^{-1} + \frac{ik}{4\pi} & \text{for } m = n, \\ \frac{e^{ik|\mathbf{a}_m - \mathbf{a}_n|}}{4\pi|\mathbf{a}_m - \mathbf{a}_n|} & \text{for } m \neq n. \end{cases}$$

DFSS gives:

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[Ann. Phys. **443**, 168966 (2022), arXiv: 2204.09554]

Generalization to 3D

$$f(k', k) = -\frac{1}{4\pi} \sum_{m,n=1}^N A_{mn}^{-1} e^{i(\mathbf{a}_n \cdot \mathbf{k} - \mathbf{a}_m \cdot \mathbf{k}')}$$

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Again nothing blows up, so **no need for renormalizing** \mathfrak{z}_n ,
and we obtain the **correct coincidence limit**.

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Exactness of the N -th-order Born Approximation

$$\hat{\mathcal{H}}(x) := \frac{1}{2} e^{-i\hat{\varpi}x\sigma_3} v(x, \hat{y}) \hat{\varpi}^{-1} \mathcal{K} e^{i\hat{\varpi}x\sigma_3}$$

$$\begin{aligned} \hat{\mathbf{M}} = \hat{\Pi}_k &- i \int_{-\infty}^{\infty} dx_1 \hat{\Pi}_k \hat{\mathcal{H}}(x_1) \hat{\Pi}_k \\ &- \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{x_2} dx_1 \hat{\Pi}_k \hat{\mathcal{H}}(x_2) \hat{\mathcal{H}}(x_1) \hat{\Pi}_k + \cdots \end{aligned}$$

Qxn: Are there potentials for which this series truncates?

Exactness of the N -th-order Born Approximation

$$\hat{\mathcal{H}}(x) := \frac{1}{2} e^{-i\hat{\varpi}x\sigma_3} v(x, \hat{y}) \hat{\varpi}^{-1} \kappa e^{i\hat{\varpi}x\sigma_3}$$

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Qxn: Are there potentials for which this series truncates?

Born series:
$$f(\mathbf{k}', \mathbf{k}) = \sum_{\ell=1}^{\infty} f_{\ell}(\mathbf{k}', \mathbf{k})$$

$$f_{\ell}(\mathbf{k}', \mathbf{k}) := -\pi \sqrt{2\pi} \langle \mathbf{k}' | \hat{v} (\hat{G}^+ \hat{v})^{\ell-1} | \mathbf{k} \rangle$$

Theorem: Let $d \in \{2, 3\}$ and $v : \mathbb{R}^d \rightarrow \mathbb{C}$ be a short-range potential with Fourier transform $\tilde{v} : \mathbb{R}^d \rightarrow \mathbb{C}$. Suppose that

$$\tilde{v}(\mathbf{p}) = 0 \quad \text{for} \quad \hat{\mathbf{u}} \cdot \mathbf{p} < \alpha$$

for some $\alpha \in \mathbb{R}^+$ and $\hat{\mathbf{u}} \in \mathbb{R}^d$ such that $|\hat{\mathbf{u}}| = 1$.

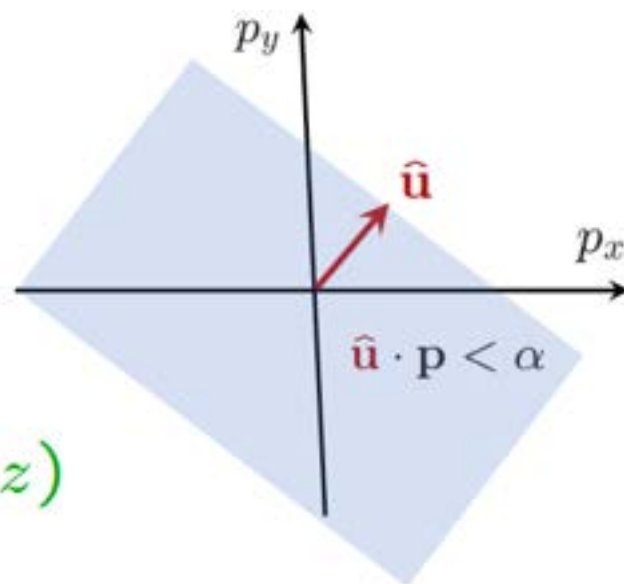
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For $\hat{\mathbf{u}}$ along x axis,

$$v(x, y, z) = e^{i\alpha x} \int_0^\infty d\kappa e^{i\kappa x} f(\kappa, y, z)$$



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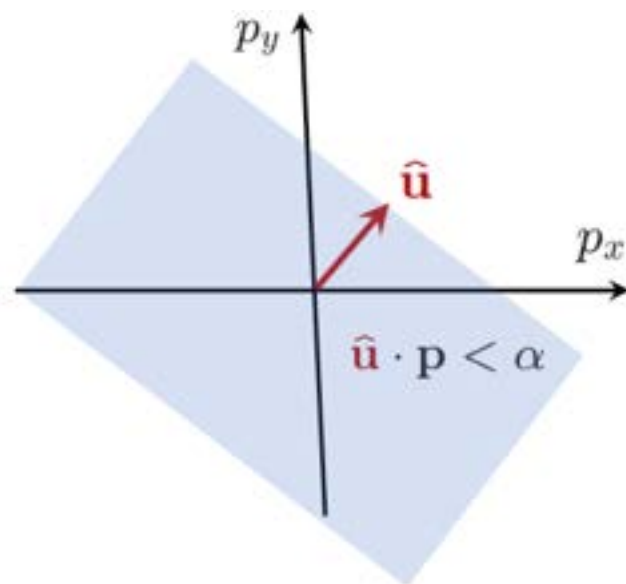
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Let

$$N := \lfloor 2k/\alpha \rfloor$$

i.e., N is the largest integer that is not larger than $2k/\alpha$, so that $k < \frac{1}{2}(N+1)\alpha$.



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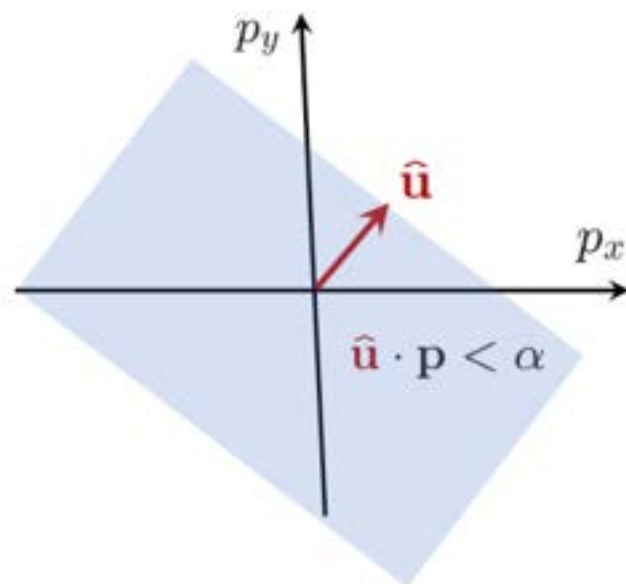
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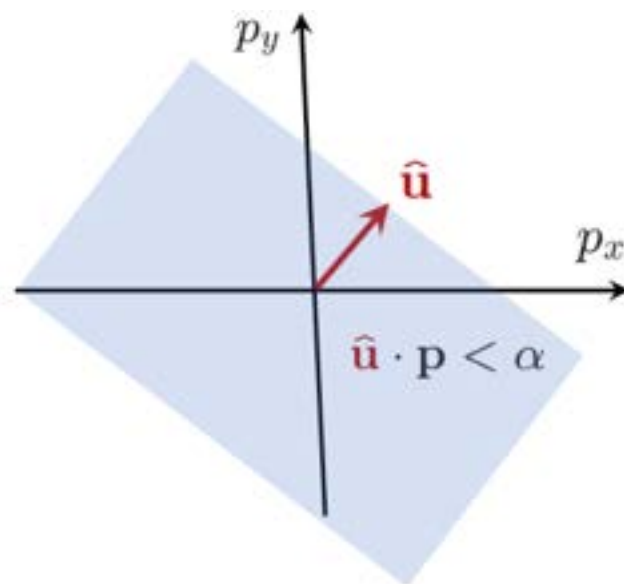
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In particular, v **does not scatter waves with** $k < \alpha/2$, and the **1st Born approximation is exact for** $k < \alpha$.

Concluding Remarks:

- Standard treatment of multi-delta-function point scatterers, which involves renormalization of coupling-constants, suffers from the **coincidence-limit problem**.
- This can be remedied if one allows the **renormalized coupling constants** also depend on the distances between the point scatterers, but **the method does not give a clue as to how this dependence should be**.
- **DFSS** on the contrary produces **finite results** with a **consistent coincidence limit**.
- **DFSS** has found a number of other remarkable applications such as the discovery of **complex potentials for which the N-th Born approximation is exact**. This result also **extends to EM scattering**.

References:

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Thank you for your attention.