Weakly coupled Schrödinger operators with complex potentials

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Joint work with J. Behrndt, M. Holzmann, and P. Siegl

AAMP XXI
Prague, August 29, 2024

Content

The Operator

We are interested in eigenvalues of the m-sectorial operator

$$
\begin{aligned}
H_\beta &:= -\Delta - \beta V \\
\mathrm{dom}(H_\beta) &:= \{f \in H^1(\mathbb{R}^d) : (-\Delta - \beta V)f \in L^2(\mathbb{R}^d)\}
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Defined via the sectorial form

$$\mathfrak{h}_\beta[f] = \|\nabla f\|^2_{L^2(\mathbb{R}^d)} - \beta \int_{\mathbb{R}^d} V(x)\,|f(x)|^2 \,\mathrm{d}x$$

with $f \in \mathrm{dom}(\mathfrak{h}_\beta) = H^1(\mathbb{R}^d)$.

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Question: Does H_β have eigenvalues for $\beta \neq 0$ small?

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Then H_β has a negative eigenvalue for all sufficiently small $\beta > 0$ if and only if $\int_{\mathbb{R}} V(x) \, \mathrm{d}x \geq 0$. If it exists, it is unique. In 1D, this eigenvalue is simple.

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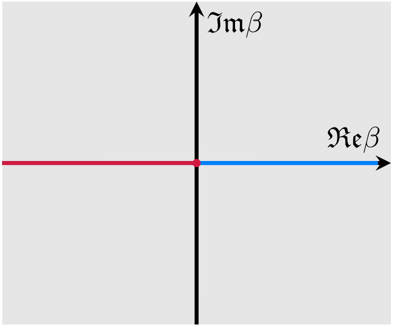

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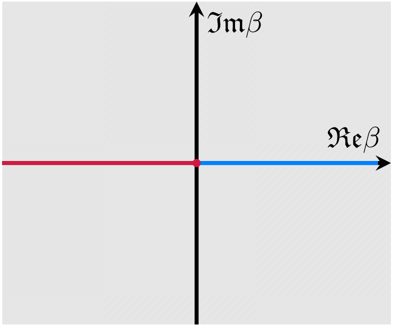

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If $\beta \to 0$ along the blue line (red line, resp.), then H_β has a unique neg. EV $\lambda_\beta \to 0$ (has no neg. EV, resp.) as $\beta \to 0$.

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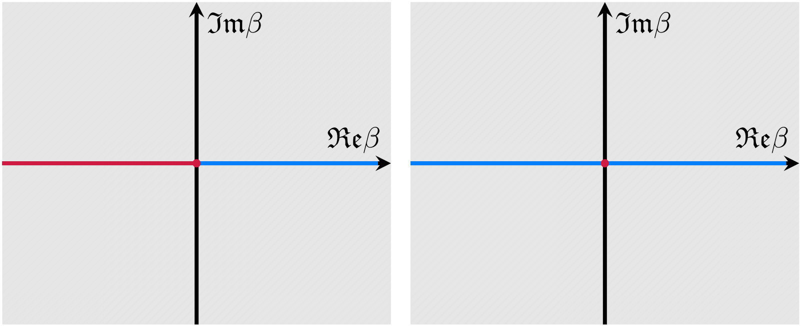

$\int_{\mathbb{R}} V(x)\,\mathrm{d}x > 0$ $\quad$ $\int_{\mathbb{R}} V(x)\,\mathrm{d}x = 0$

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More references for $\beta \in \mathbb{R}$

Weak coupling for self-adjoint Schrödinger operators:

- 1977: Klaus
- 1977: Blankenbecler-Goldberger-Simon
- 1979: Klaus
- 1980: Klaus-Simon
- 1985: Holden
- 1987: Gesztesy-Holden
- 1998: Fassari-Klaus
- 2014: Kondej-Lotoreichik
- 2018: Exner-Kondej-Lotoreichik

Question: Similar results for $\beta \in \mathbb{C}$?

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- $U \neq 0$ (w.l.o.g. $U > 0$)
- $U = 0 \implies U_1 < 0.$

The 1D Case - $\int_{\mathbb{R}} V \,\mathrm{d}x > 0$

For $\varepsilon > 0$ define $S_{1,\varepsilon} = \left\{ z \in \mathbb{C} : \mathfrak{Re}(z) > \left(-\frac{U_1}{U} + \varepsilon \right) \mathfrak{Im}(z)^2 \right\}$.

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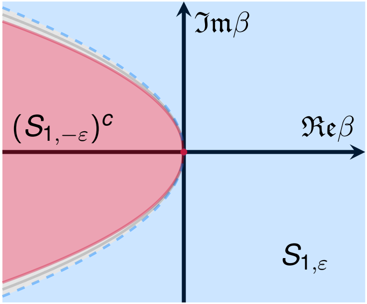

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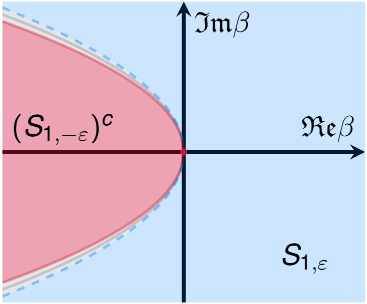

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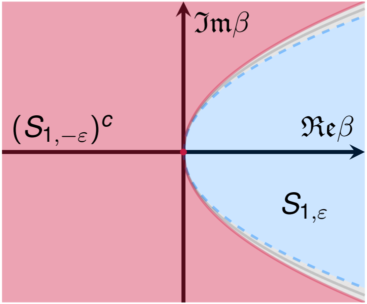

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$$\sqrt{-\lambda_\beta} = \frac{\beta}{2} U - \frac{\beta^2}{2} U_1 + \mathcal{O}(\beta^3) \quad \textit{as} \quad \beta \to 0 \quad \textit{in} \quad S_{1,\varepsilon}.$$

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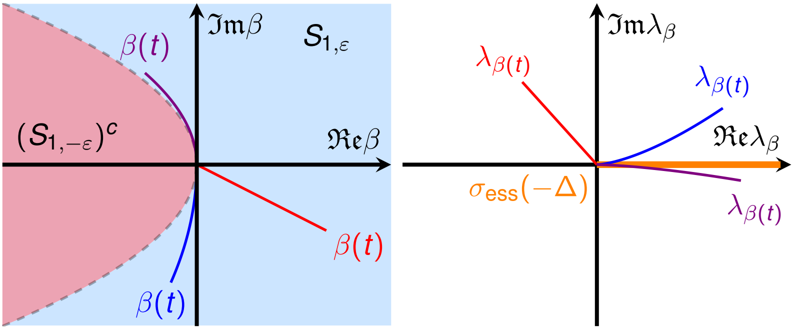
Visualization of the eigenvalue λ_β of H_β as $\beta \to 0$ if $U_1 > 0$.



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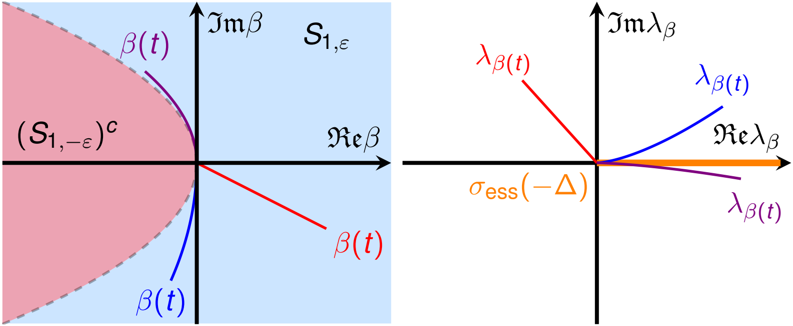

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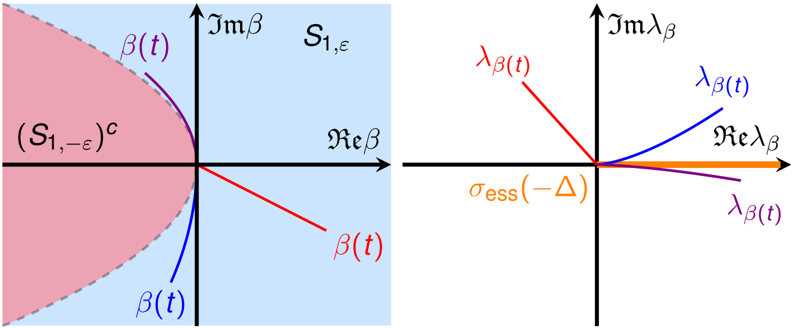

Remarkable: $\mathfrak{Re}(\beta)$ may be negative!

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Now: $\int_{\mathbb{R}} V(x) \, \mathrm{d}x = 0 \implies \frac{1}{2} \int_{\mathbb{R}^2} V(x) \, |x - y| \, V(y) \, \mathrm{d}(x, y) < 0$.

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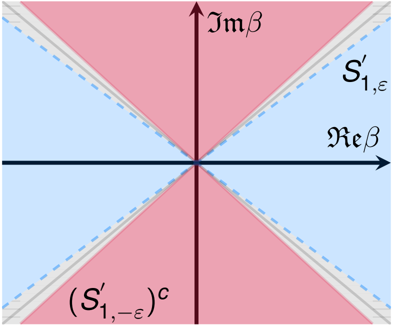

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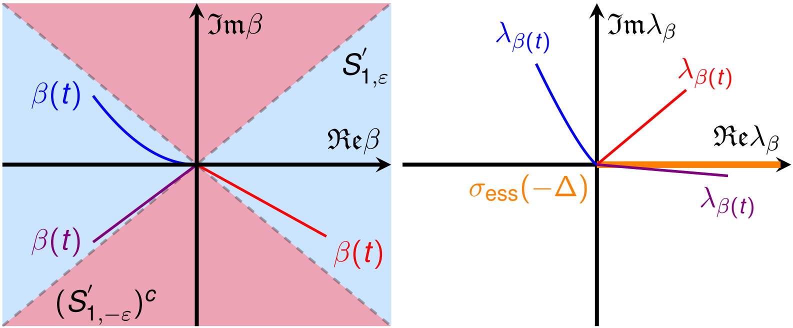
Visualization of the eigenvalue λ_β of H_β as $\beta \to 0$.



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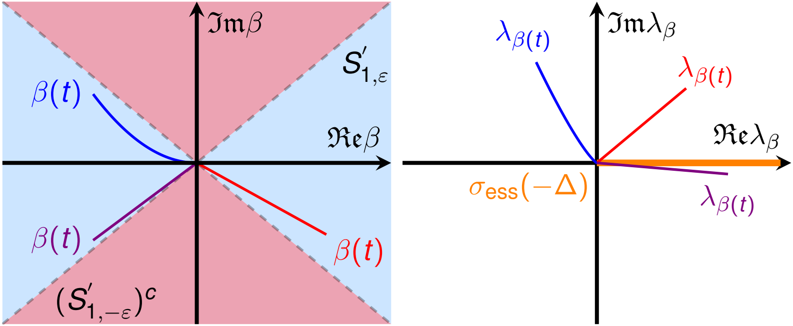

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and suppose $U > 0$. For $\varepsilon > 0$ define the parabolic region

$$S_{2,\varepsilon} = \left\{ z \in \mathbb{C} : \mathfrak{Re}(z) > 0,\ |\mathfrak{Im}(z)| < \left(\frac{U}{4} - \varepsilon\right) \mathfrak{Re}(z)^2 \right\}.$$

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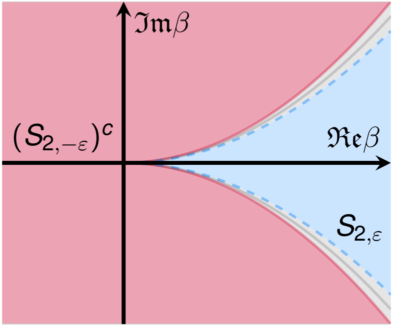

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Under the above assumptions on V there holds:

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$$\lambda_\beta = -\left(C_V + \mathcal{O}(\beta)\right)\exp\left(-\frac{4\pi}{\beta U}\right) \quad \text{as} \quad \beta \to 0 \quad \text{in} \quad S_{2,\varepsilon},$$

where $C_V = \exp\left(2\ln(2) - 2\gamma - 4\pi\frac{U_1}{U^2}\right)$.

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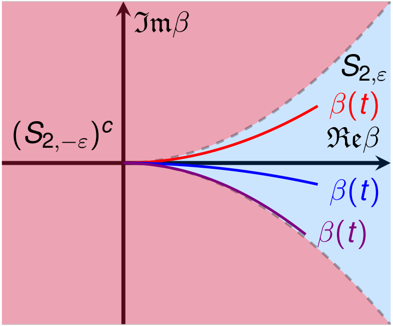
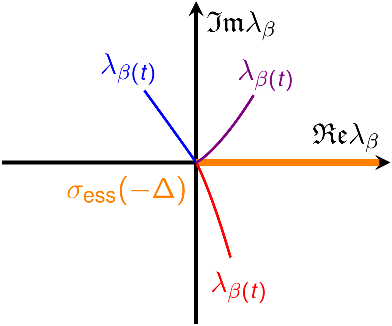
Visualization of the eigenvalue λ_β of H_β as $\beta \to 0$.





The 2D Case

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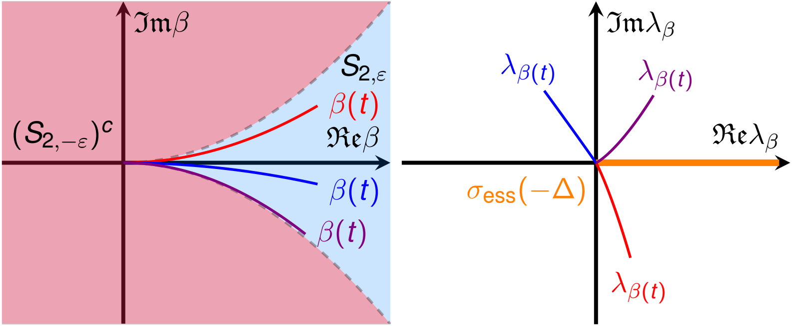

If $\beta \to 0$ in the blue area (red area, resp.), then H_β has an eigenvalue λ_β (has no eigenvalue, resp.) in $\mathbb{C} \setminus [0, \infty)$.

Eigenvalues in higher dimensions

Are there bound states if $d \geq 3$?

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If $d \geq 3$, there exists a constant $D_{0,d} > 0$, such that

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implies that $-\Delta - V$ has no eigenvalues in $\mathbb{C} \setminus [0, \infty)$.

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$\implies -\Delta - \beta V$ has no eigenvalues in $\mathbb{C} \setminus [0, \infty)$ if $d \geq 3$ and β is small enough.

Further references

Non-self-adjoint weak coupling:

- 2008: Borisov-Krejčiřík [Planar Waveguides]
- 2016: Novak [Waveguides]
- 2018: Cuenin-Siegl [1D Dirac]
- 2021: Cuenin-Ibrogimov [Indefinite Laplacian]

Thank you for your attention!