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Analitic and Algebraic Methods in Physics (AAMP-2024)

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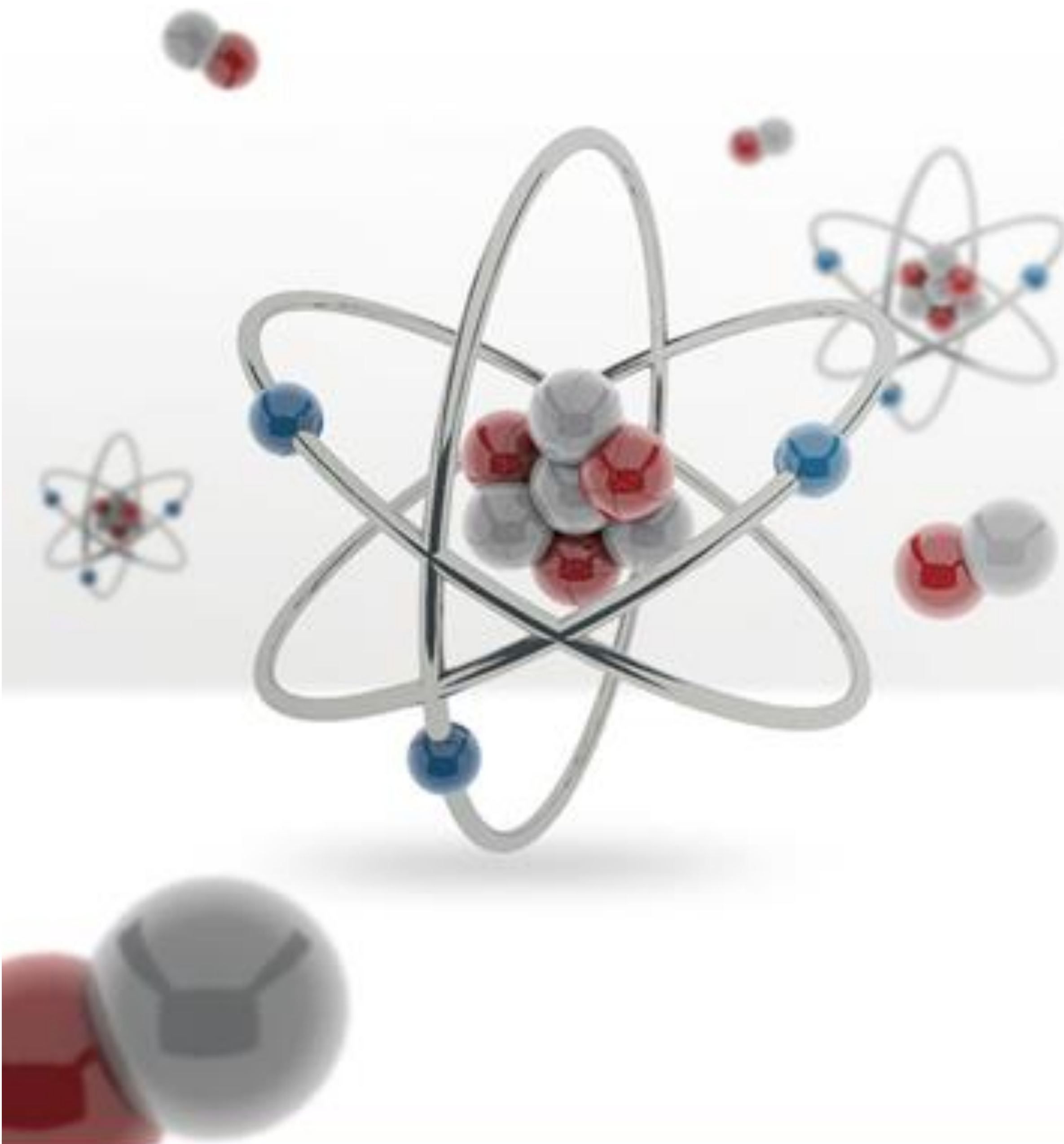
A conditionally exactly solvable 1D Dirac pseudoscalar interaction potential



Outline

- ✓ 1D Dirac Equation
- ✓ Potential
- ✓ Solution and Bound States
- ✓ Energy Spectrum
- ✓ Discussion

Paul Adrien Maurice Dirac (1902-1984)

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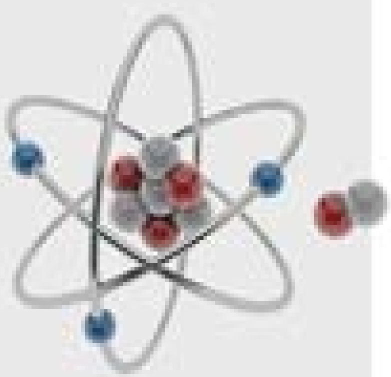


1D stationary Dirac equation

$$E\sigma_0\psi = \left(-ic\hbar\sigma_1 \frac{d}{dx} + W(x)\sigma_2 + mc^2\sigma_3 \right)\psi$$

$\sigma_{0,1,2,3}$ - *identity matrix, Pauli matrices*

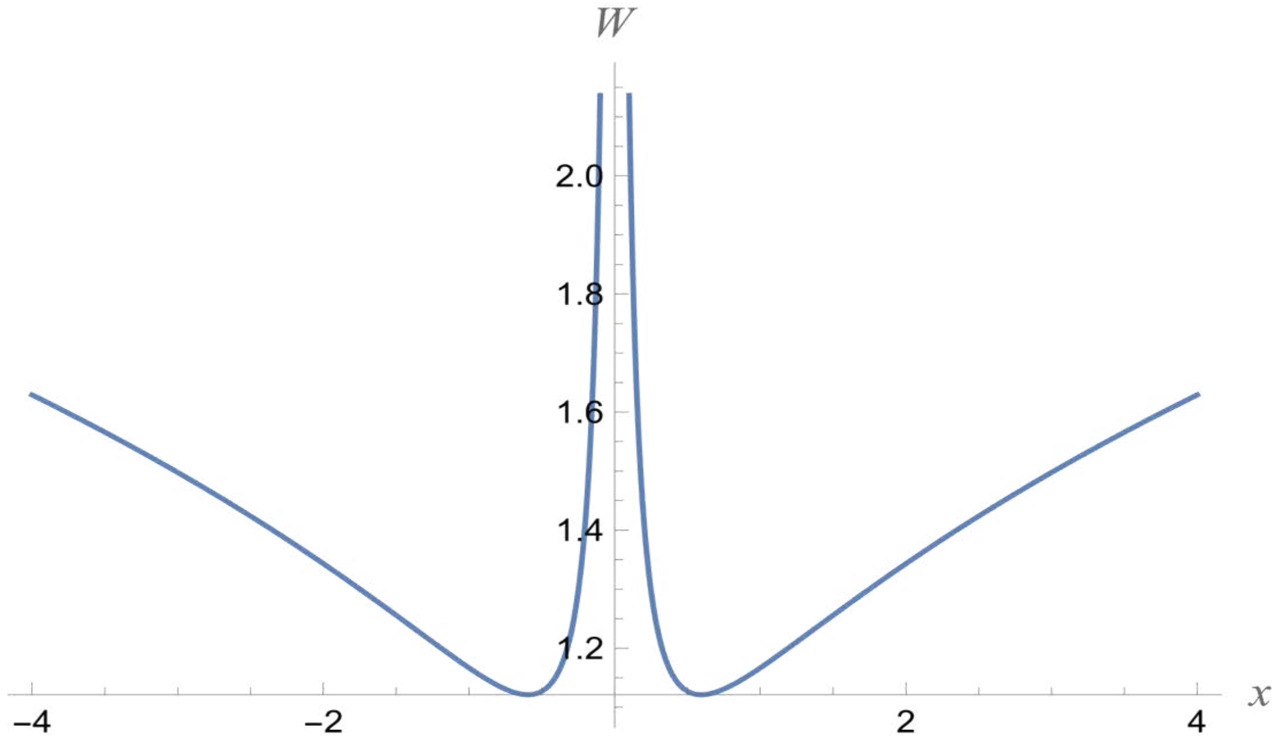
$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

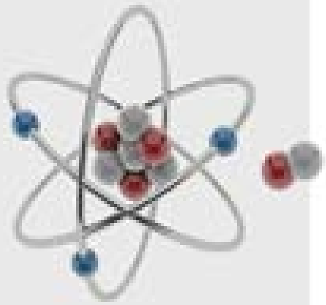
Dirac pseudoscalar potential

Basic Potential

$$W = \frac{c\hbar / 6}{|x|} + W_2 |x|^{1/3}$$



Symmetric with respect to the origin



Effective
1D stationary Schrödinger equation

$$\frac{d^2\psi}{dx^2}+\frac{2m}{\hbar^2}\big(E_S-V_S(x)\big)\psi=0$$

Effective Energy

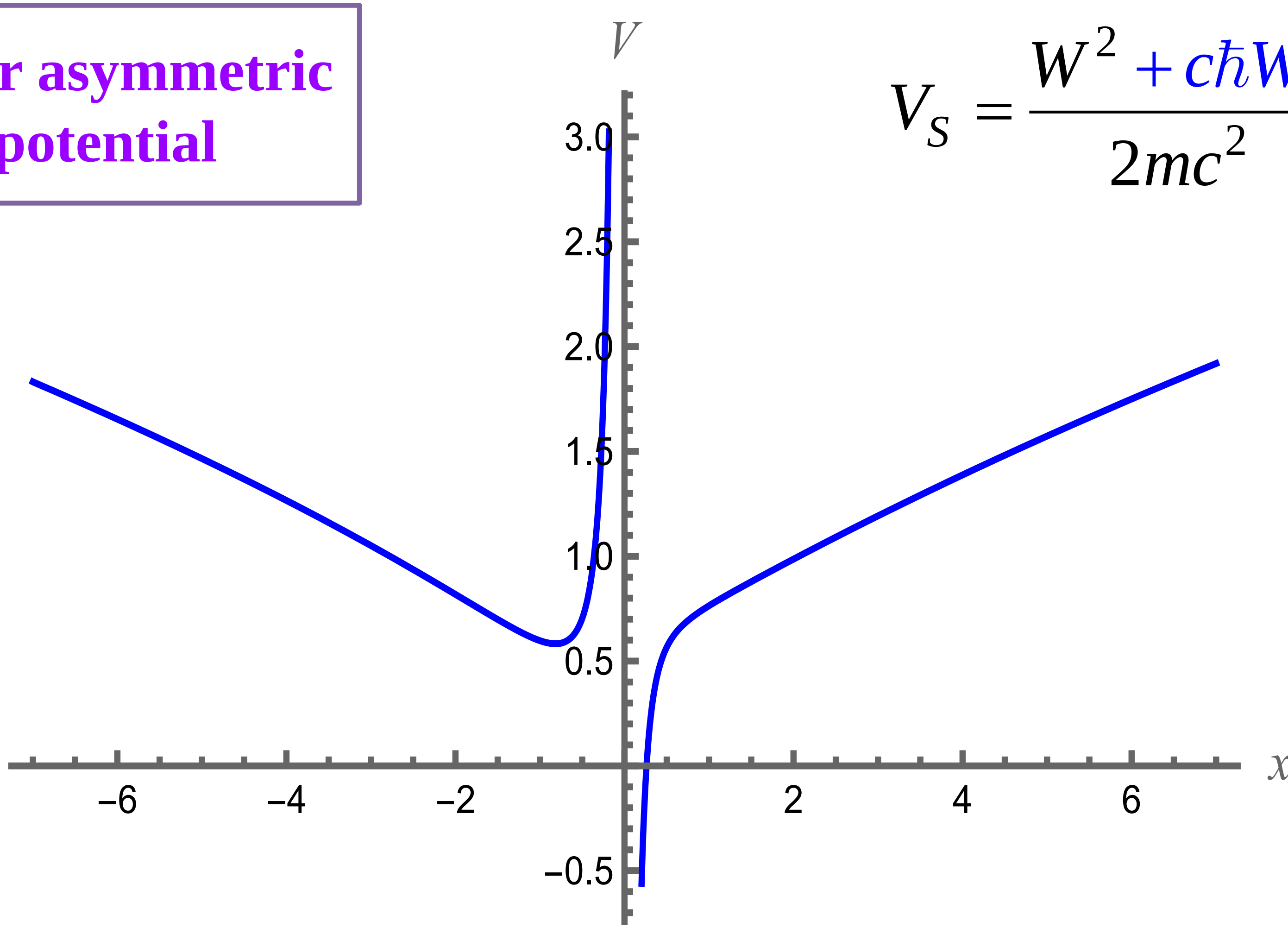
$$E_S=\frac{E^2-m^2c^4}{2mc^2}$$

Effective potential

$$V_S=\frac{W^2+c\hbar W'}{2mc^2}$$

Effective Schrödinger potential

Weakly singular asymmetric confining potential



$$V_S = \frac{W^2 + c\hbar W'}{2mc^2}$$

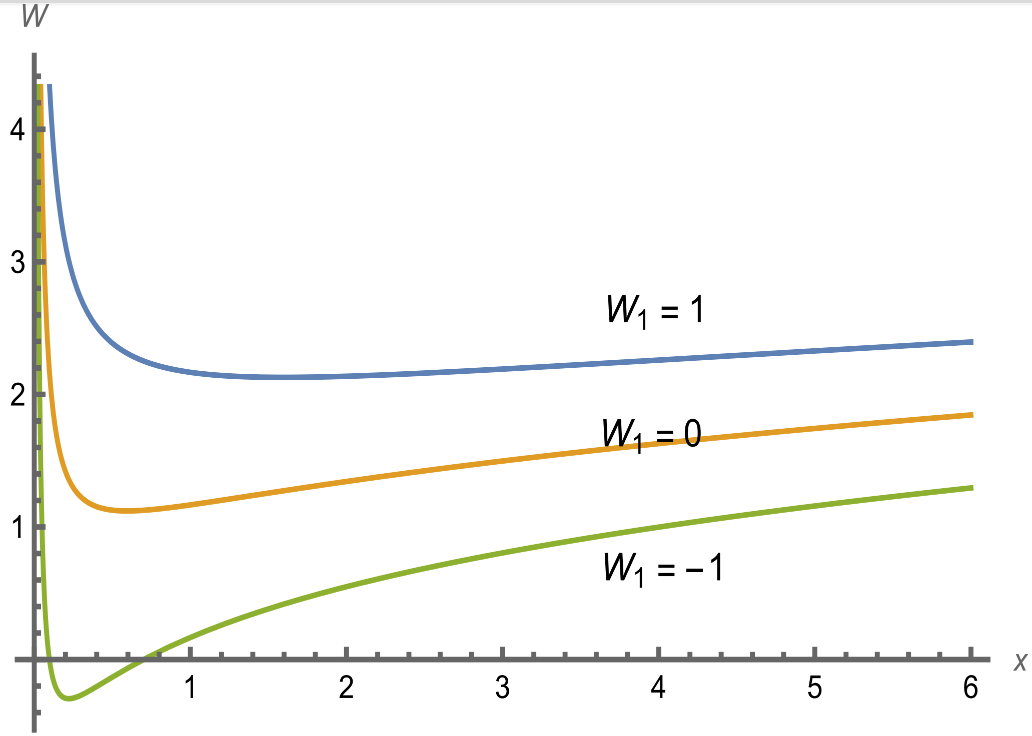
A.M. Ishkhanyan, V.P. Krainov, "Conditionally exactly solvable Dirac potential including $x^{1/3}$ pseudoscalar interaction", *Phys. Scr.* **98**

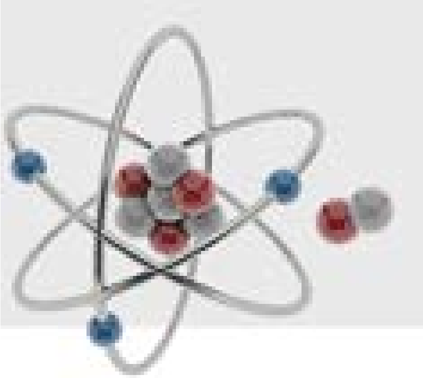
Dirac pseudoscalar interaction potential in the positive semi-axes $x > 0$

$$W = \frac{c\hbar / 6}{x} + \frac{W_1}{x^{1/3}} + W_2 x^{1/3}$$

For a positive W_2 *this potential forms a potential well.*
The strength of this term defined by the value of the parameter W_1

$W_2 = 1$





$$-ic\hbar\frac{d\psi_1}{dx}+iW\psi_1=(E+mc^2)\psi_2$$

$$-ic\hbar\frac{d\psi_2}{dx}-iW\psi_2=(E-mc^2)\psi_1$$

Expressing ψ_2 from the first equation

$$\psi_2=\frac{iW\psi_1-ic\hbar\psi_1'}{E+mc^2}$$

Substituting it into second equation, we obtain second-order equation for ψ_1

$$\frac{d^2\psi_1}{dx^2}+\frac{E^2-m^2c^4-c\hbar W'-W^2}{c^2\hbar^2}\psi_1=0$$

Effective Schrödinger-like equation

$$\frac{d^2\psi_1}{dx^2}+\frac{2m}{\hbar^2}\left(E_S-U_1(x)\right)\psi_1=0$$

Effective energy

$$E_S=\frac{E^2-m^2c^4}{2mc^2}$$

Effective potential

$$U_1=\frac{W^2+c\hbar W'}{2mc^2}$$

$$-ic\hbar\frac{d\psi_1}{dx}+iW\psi_1=(E+mc^2)\psi_2$$

$$-ic\hbar\frac{d\psi_2}{dx}-iW\psi_2=(E-mc^2)\psi_1$$

Expressing ψ_1 *from the second equation*

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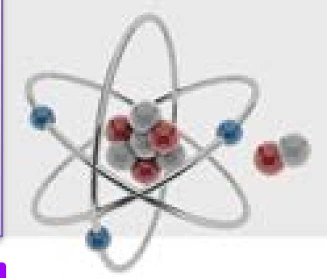
$$\psi_1=-\frac{iW\psi_2+ic\hbar\psi_2'}{E-mc^2}$$

Substituting it into first equation, we obtain second-order equation for ψ_2

$$\frac{d^2\psi_2}{dx^2}+\frac{2m}{\hbar^2}\left(E_S-U_2(x)\right)\psi_2=0$$

Effective energy $E_S=\dfrac{E^2-m^2c^4}{2mc^2}$ *Effective potential* $U_2=\dfrac{W^2-c\hbar W'}{2mc^2}$

1 Two effective Schrödinger potentials for the region $x > 0$



Effective Schrödinger Potential

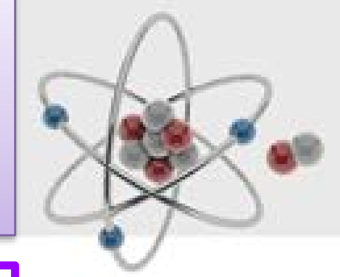
$$U_1 = -\frac{5\hbar^2}{72mx^2} + \frac{W_1^2 + 2c\hbar W_2/3}{2mc^2\, x^{2/3}} + \frac{W_1 W_2}{mc^2} + \frac{W_2^2}{2mc^2} x^{2/3}$$

Particular case of ***the first Stillinger potential***

$$V_{S1} = -\frac{5\hbar^2}{72mx^2} + \frac{V_1}{x^{2/3}} + V_0 + V_2 x^{2/3}$$

This potential is ***attractive*** **in the vicinity of the origin**

2 Two effective Schrödinger potentials for the region $x > 0$



Effective Schrödinger Potential

$$U_2 = \frac{7\hbar^2}{72mx^2} + \frac{\hbar W_1}{3mcx^{4/3}} + \frac{W_1^2}{2mc^2 x^{2/3}} + \frac{W_1 W_2}{mc^2} + \frac{W_2^2}{2mc^2} x^{2/3}$$

Particular case of *the second Exton potential*

$$V_{E2} = \frac{7\hbar^2}{72mx^2} + \frac{V_1}{x^{4/3}} + \frac{9mV_1^2}{2\hbar^2 x^{2/3}} + V_0 + V_2 x^{2/3}$$

This potential is *repulsive* in the vicinity of the origin

Effective Schrödinger potentials

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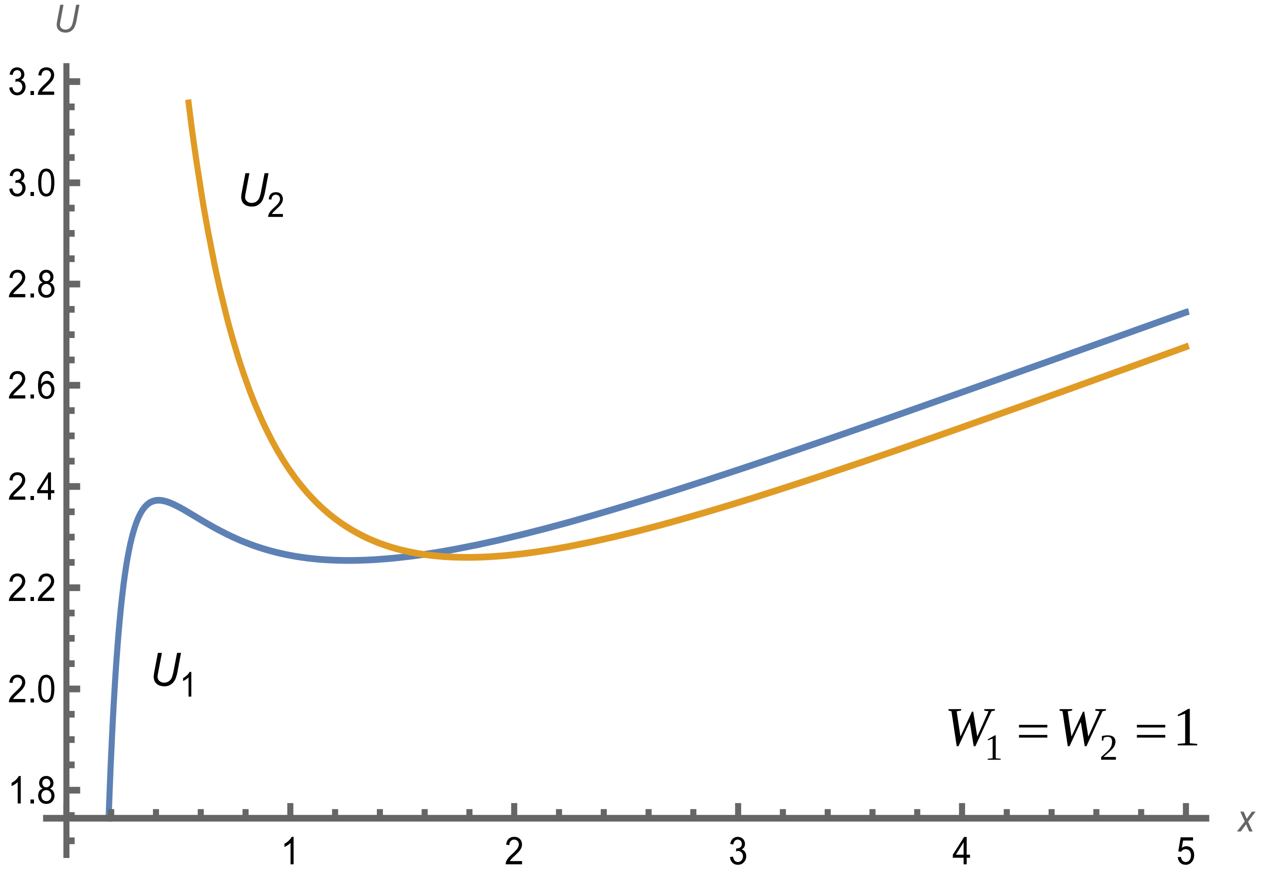
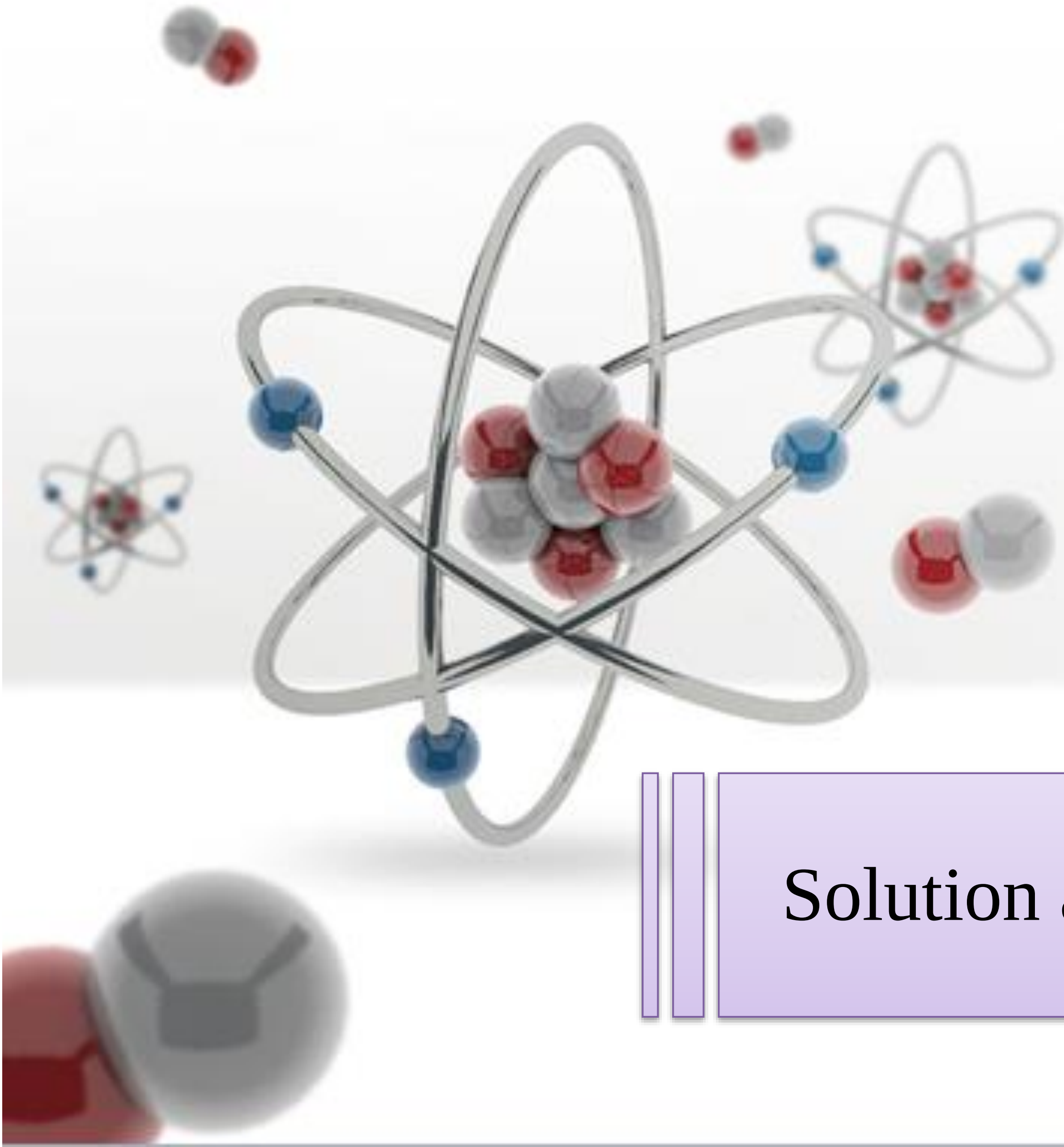
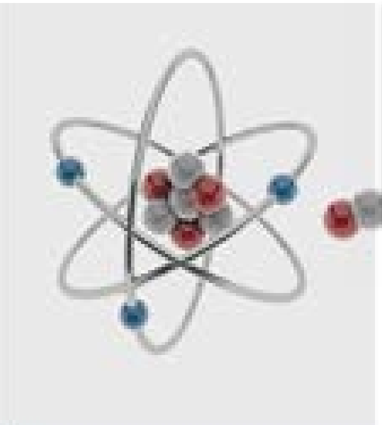




Solution and bound states



Solution of the Schrödinger equation for the first Stillinger potential

$$\frac{d^2\psi_1}{dx^2}+\frac{2m}{\hbar^2}\left(E_S-U_1(x)\right)\psi_1=0$$

$$\psi_1(x)=x^{1/6}e^{a-(z-A)^2/2}\left(C_1H_{a-1}(z-A)+\right.$$

$$\left.+C_2\cdot {}_1F_1\left(\frac{1-a}{2},\frac{1}{2},(z-A)^2\right)\right)$$

$$A=\sqrt{\frac{3\left(E^2-2W_1W_2-m^2c^4\right)^2}{8c\hbar W_2^3}}$$

$$a=\frac{3\left(E^2-m^2c^4\right)\left(E^2-4W_1W_2-m^2c^4\right)}{16c\hbar W_2^3}$$

$$z=\sqrt{\frac{3W_2}{2c\hbar}}x^{2/3}$$

Wave function for bound states on the interval $x \in (0, +\infty)$

$$\psi_{1R}\Big|_{x\to+\infty} = 0 \Rightarrow C_2 = 0$$

$$\psi_1 = C_1 x^{1/6} e^{a-(z-A)^2/2} H_{a-1}(z-A)$$

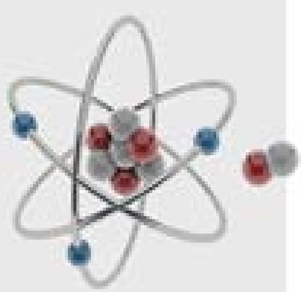
$x \to 0:$

$$\psi_1 = C_1 e^{a-A^2/2}\left(A_1 x^{1/6} + B_1 x^{5/6} + O\left(x^{3/2}\right)\right)$$

$$A_1 = H_{a-1}(-A)$$

$$B_1 = -\sqrt{\frac{3W_2}{2c\hbar}}\left(H_a(-A) + AH_{a-1}(-A)\right)$$

$$A = \sqrt{2a}$$



Two wave function components are not independent
They are connected via the following relations

$$\psi_1 = -\frac{iW\psi_2 + ic\hbar\psi_2'}{E - mc^2} \qquad \psi_2 = \frac{iW\psi_1 - ic\hbar\psi_1'}{E + mc^2}$$

The relation between wave function components does not allow

$$A_1 = 0$$

It is impossible !!!
Why?

Wave component ψ_2

$$\psi_2 = i\frac{C_1 e^{a-A^2/2}}{E+mc^2}\left(\frac{A_2}{x^{1/6}} + B_2 x^{1/2} + O\left(x^{7/6}\right)\right)$$

$$A_2 = -\frac{2c\hbar}{3}B_1 + W_1 A_1$$

$$B_2 = W_2\left(2a - A^2\right)A_1 + W_1 B_1$$

$$A_2 \neq 0 \;\Rightarrow\; \psi_2\left(x\right)$$

has a singularity at the origin

A fundamental requirement of quantum mechanics is that the wave function is finite everywhere

$\boxed{A_2}$ **must be zero**

Exact eigenvalue equation for the bound state energy spectrum

$$A_2 = 0:$$

$$F = H_a(-A) + \left(A + \frac{\sqrt{3}W_1}{\sqrt{2c\hbar W_2}} \right) H_{a-1}(-A) = 0$$

Approximate energy spectrum when $W_1 = 0$

$$E_n \approx \pm\sqrt{m^2c^4 + W_2^{3/2}\sqrt{\frac{16c\hbar}{3}\left(n - \frac{1}{6}\right)}}$$

Maslov index

$$\gamma_{\mathrm{M}} = -\frac{1}{6}$$

Approximation of the eigenvalue equation

*For **high-lying** levels, one can apply the Airy-function approximation to the Hermite function for the **left transition region*** $-A \approx -\sqrt{2a}$

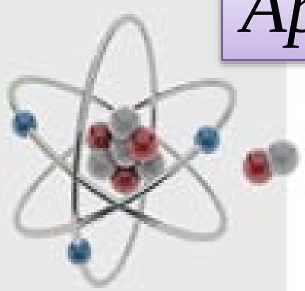
$$F \approx f(a)\sin\left(\pi\left(a-\gamma_M\right)\right)=0$$

$$\gamma_M = -\frac{1}{\pi}\tan^{-1}\left(\frac{Ai(u_1)-\sqrt[12]{\frac{2a+1}{2a-1}}\,\frac{A+s}{\sqrt{2a}}\,Ai(u_2)}{Bi(u_1)-\sqrt[12]{\frac{2a+1}{2a-1}}\,\frac{A+s}{\sqrt{2a}}\,Bi(u_2)}\right)$$

$$u_{1,2} = 2^{1/3}(2a\pm 1)^{1/6}\left(A-\sqrt{2a\pm 1}\right)$$

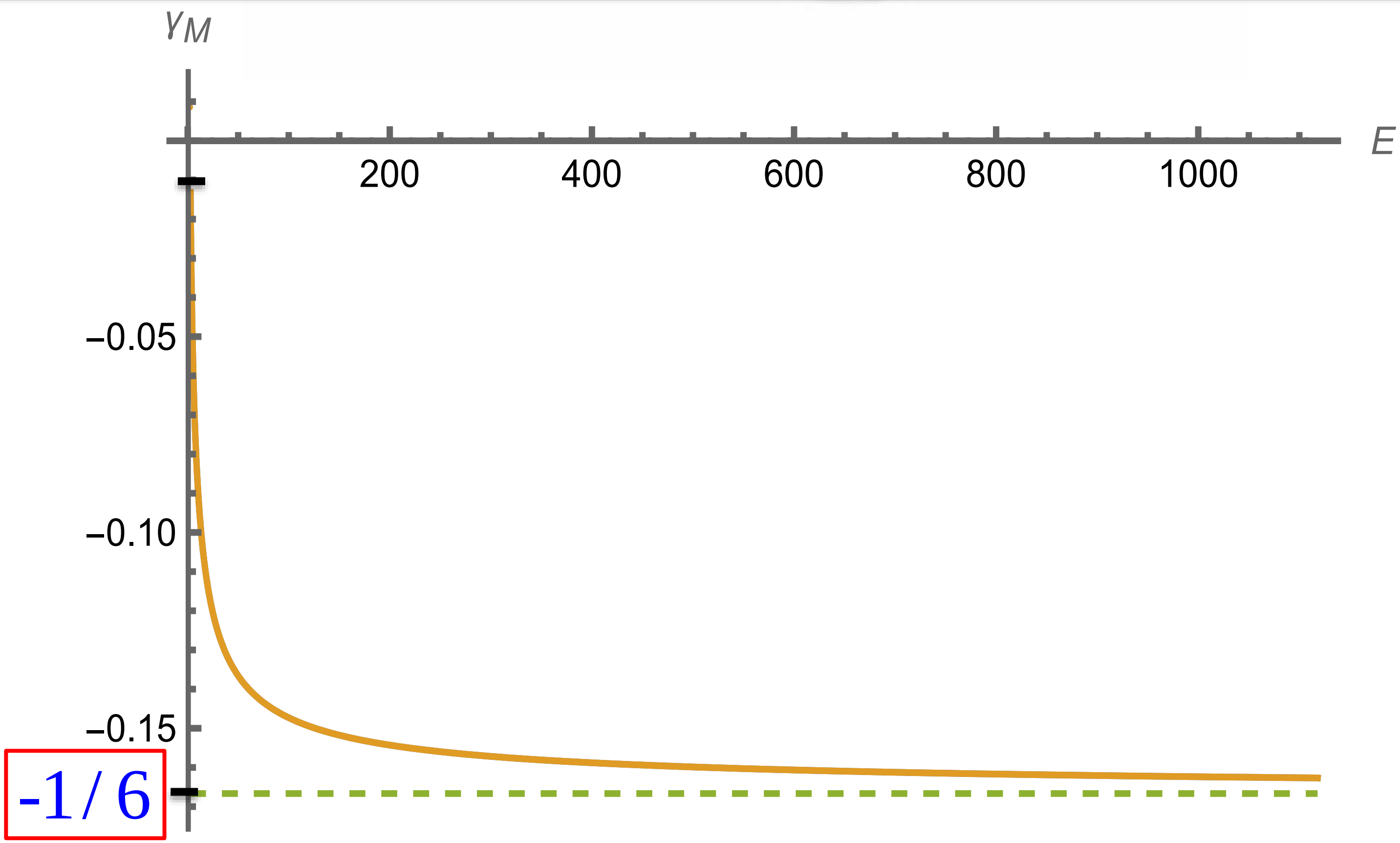
$$s = \frac{\sqrt{3}W_1}{\sqrt{2c\hbar W_2}}$$

Approximate spectrum with the potential term $W_1 x^{-1/3}$

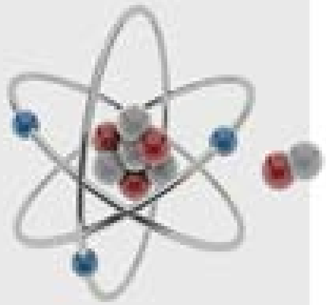


$$E_n \approx \pm\sqrt{m^2c^4 + 2W_1W_2 + 2W_2\sqrt{W_1^2 + \frac{4c\hbar W_2}{3}\left(n-\frac{1}{6}\right)}}$$

The dependence of γ_M on energy

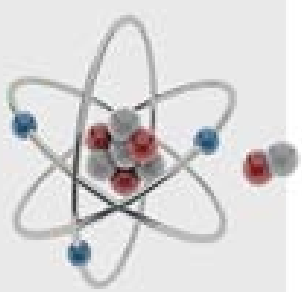


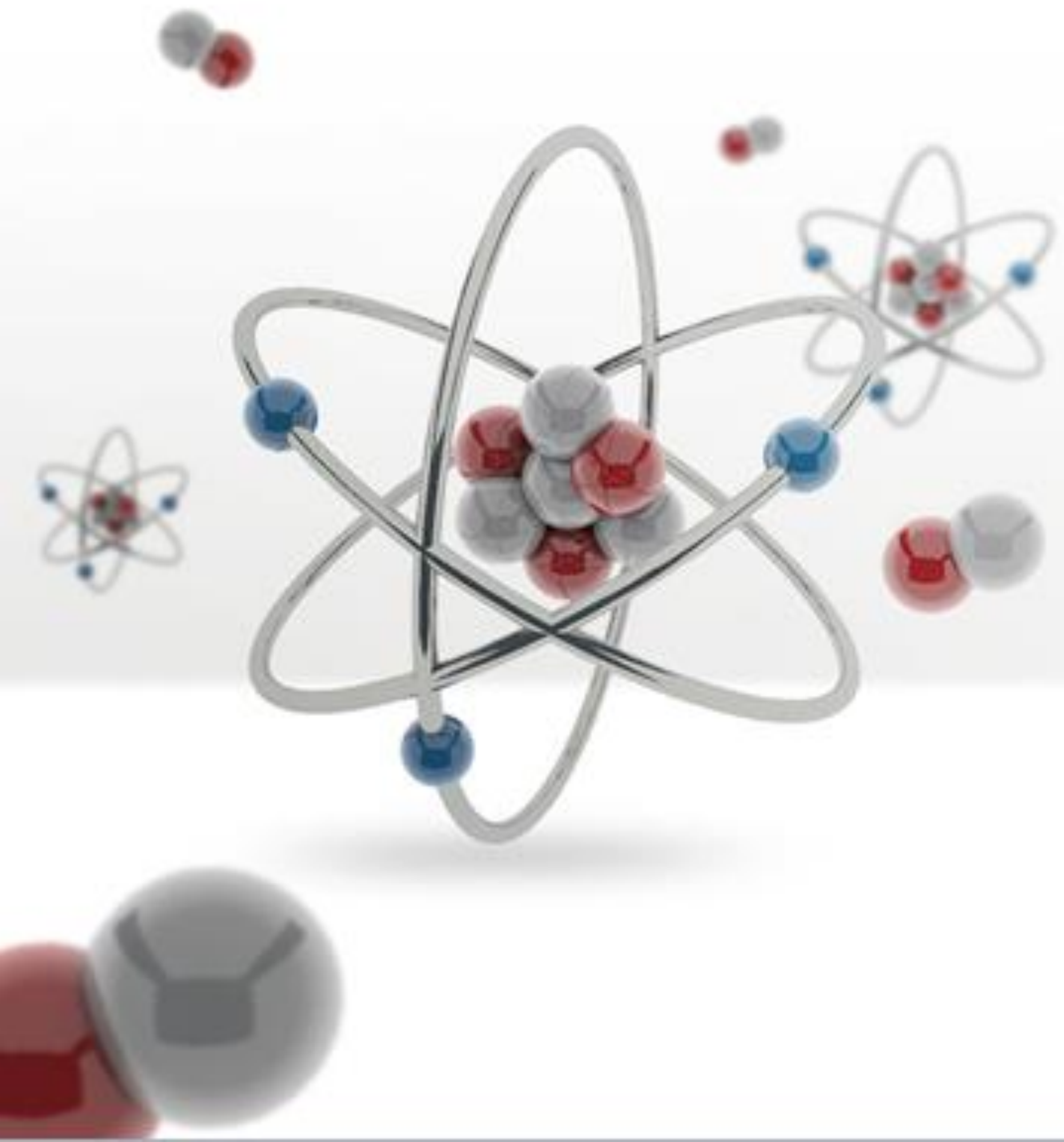
The black mark is not zero, it depends on the value of
$W_1, \; W_2$



Discussion

- We have examined an analytically solvable pseudoscalar interaction potential for the one-dimensional stationary Dirac equation
- The general solution to the Dirac equation is written in terms of non-integer index Hermite functions and confluent hypergeometric functions
- We have derived the exact equation for the energy spectrum, developed an approximation for the spectrum
- Our results demonstrate that the inclusion of the $x^{-1/3}$ term has a significant impact on the energy spectrum and eigenfunctions, particularly, on the low-lying energy levels.
- These results indicate that the pseudoscalar interaction potential with an $x^{-1/3}$ term can be used to model a variety of physical systems, particularly low-dimensional systems such as graphene, semiconductor nanostructures, or topological insulators, where relativistic effects and confinement are significant factors





Future Directions

- ✓ Addition of Vector Potential

$$V(x) = \frac{V_0}{x^{2/3}} + V_1 x^{2/3}$$

- ✓ Scalar Potential
- ✓ The case $x < 0$
- ✓ …
- ✓ Applications

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THANK YOU