Polynomial algebras in the universal enveloping algebras of simple (exceptional) Lie algebras

Rutwig Campoamor-Stursberg

Instituto de Matemática Interdisciplinar
Universidad Complutense de Madrid

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[1] RCS, Marquette I 2024 *J. Lie Theory* **34** 17; RCS, D Latini, I Marquette, Y-Z Zhang 2023 *J. Phys. A: Math. Theor.* **56** 045202; RCS, D Latini, I Marquette, J Zhang, Y-Z Zhang 2024 *arXiv:2406.01958*

Résumé

Motivation

1. Decompose universal enveloping algebras as $\mathfrak{s}$-modules
2. Commutants and types of labelling problems
3. Applications to BR and bases of representations
4. Purely algebraic description of (super)integrable systems

Outline

1. Commutants in $\mathcal{U}(\mathfrak{s})$
2. Algebraic vs. analytical approach
3. Computing with Berezin bracket
4. Commutants and missing label problem
5. Root systems and polynomial algebras
6. Polynomial algebra for G_2
7. Polynomial algebras and regular subalgebras
8. Open questions

- Generic references on the subject:[2]

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[2] Derome, J R, Sharp, W T 1965 *J. Math. Phys.* **6** 1584; Winternitz P, Smorodinsky Ya A, Uhlir M Friš 1966 *Yad. Fiz.* **4** 625; Calogero F 1969 *J. Math. Phys.* **10** 2191; Turbiner A V, Ushveridze A G 1987 *Phys. Lett.* **A126** 181; Shifman M A,Turbiner A V 1989 *Comm. Math. Phys.* **126** 347; Freidel L, Maillet J M 1991 *Phys. Lett.* **A 262** 278; González López A, Kamran N, Olver P J 1991 *J. Phys. A: Math. Gen.* **24** 3995; Tempesta P, Turbiner A V, Winternitz P 2001 *J. Math. Phys.* **42** 4248; Miller W, Post S, Winternitz P 2013 *J. Phys. A: Math. Theor.* **46** 423001; RCS 2004 *Phys. Lett. A* **327** 138; RCS 2004 *Acta Phys. Pol. B* **35** 2059; RCS 2005 *Acta Phys. Pol. B* **36** 2869; RCS 2006 *Acta Phys. Pol. B* **37** 2745; RCS 2007 *Physics Letters A* **362** 360; RCS 2007 *J. Phys. A: Math. Theor.* **40** 5355; RCS 2007 *J. Phys. A: Math. Theor.* **40** 14773; RCS 2008 *Acta Phys. Pol. B* **39** 755; RCS 2008 *J. Phys. A: Math. Theor.* **41** 365297; RCS 2008 *J. Phys. Conf. Ser.* **128** 012052; RCS and S. G. Low 2009 *J. Phys. A: Math. Theor.* **42** 065205; L. J. Boya and RCS 2009 *J. Phys. A: Math. Theor.* **42** 235203; RCS 2009 *J. Phys. Conf. Ser.* **175** 012008; RCS 2010 *Acta Phys. Pol. B* **41** 53 ; RCS 2011 *J. Phys. A: Math. Theor.* **44** 025204; RCS 2011 *Int. J. Theor. Phys.* **50** 2153; RCS 2011 *Acta Phys. Pol. B* **42** 1797; S. G. Low, P. D. Jarvis and RCS *Ann. Phys.* **327** (2012), 74; RCS 2013 *Lith. J. Phys.* **53** 71; RCS 2017 *J. Lie Theory* **27** 315; Liao Y, Marquette I, Zhang YZ 2018 *J. Phys. A. Math. Theor.* **51** 255201; Yates LA, Jarvis PD 2018 *J. Phys. A: Math. Theor.* **51** 145203; RCS 2019 *J. Phys. A: Math. Theor.* **52** 125201; F. Correa, Md F. Hoque, I. Marquette and Y.-Z. Zhang 2021 *J. Phys. A: Math. Theor.* **54** (2021), 395201; RCS, I. Marquette 2021 *Annals Phys.* **424** 168378; RCS, Marquette I 2022 *Annals Phys.* **437**, 168694; RCS 2022 *Acta Polytech.* **62** 16; RCS, D. Latini, I. Marquette and Y.-Z. Zhang 2023 *J. Phys. A: Math. Theor.* **56** (2023), 045202; RCS, D. Latini, I. Marquette and Y.-Z. Zhang 2023 em J. Phys.: Conf. Ser. **2667** 012037; RCS 2024 *J. Phys. Conf. Ser.* to appear, RCS, I. Marquette 2024 *J. Lie Theory* **34** 17; RCS, Latini D, Marquette I, Zhang J, Zhang Y-Z 2024 arXiv:2406.01958

Serre–Chevalley relations

- $\mathfrak{s}$ be a complex semisimple Lie algebra of rank ℓ
  1. $\{H_1, \ldots, H_\ell\}$ a basis of a Cartan subalgebra $\mathfrak{h}$
  2. $\mathcal{R}$ the corresponding root system
  3. $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ a basis of simple roots
- Cartan integers

$$n(i,j) = \langle \alpha_j, \alpha_i \rangle = \frac{2\,(\alpha_j, \alpha_i)}{(\alpha_i, \alpha_i)}, \quad (\alpha_i, \alpha_j) = \kappa\,(H_i, H_j)$$

- Presentation with generators H_i, X_i, Y_i and relations

$$\begin{aligned} &[H_i, H_j] = 0, \quad [H_i, X_j] = n(i,j)X_j, \quad [H_i, Y_j] = -n(i,j)Y_j, \\ &[X_i, Y_j] = \delta_i^j H_i, \quad \mathrm{ad}(X_i)^{1-n(i,j)} X_j = 0, \quad \mathrm{ad}(Y_i)^{1-n(i,j)} Y_j = 0 \ (i \neq j). \end{aligned} \tag{1}$$

- Weight space $\mathfrak{s}_\alpha$

$$\mathfrak{s}_\alpha = \{X \in \mathfrak{s} \mid [H, X] = \alpha(H)X, \quad H \in \mathfrak{h}\},$$

- $\mathfrak{h} = \mathfrak{s}^0$, $\alpha \in \mathfrak{h}^*$ s.t. $\mathfrak{s}^\alpha \neq 0$ correspond to the roots in $\mathcal{R}$

- Induces root-space decomposition

$$\mathfrak{s} = \mathfrak{h} \oplus \sum_{\alpha \in \mathcal{R}} \mathfrak{s}_\alpha.$$

- Grading: for any $\alpha_i, \alpha_j \in R$ the relations (1) imply that

$$[\mathfrak{s}^{\alpha_i}, \mathfrak{s}^{\alpha_j}] \subset \mathfrak{s}^{\alpha_i + \alpha_j}$$

Commutants in $\mathcal{U}(\mathfrak{s})$[4]

- $\mathcal{U}(\mathfrak{s})$ universal enveloping algebra of $\mathfrak{s}$.

- For fixed positive integer p ($n = \dim \mathfrak{s}$)

$$\mathcal{U}_{(p)}(\mathfrak{g}) = \langle X_1^{a_1} \dots X_n^{a_n} \mid a_1 + a_2 + \dots + a_n \leq p \rangle$$

- $P \in \mathcal{U}(\mathfrak{s})$ of degree d if $d = \inf \left\{ k \mid P \in \mathcal{U}_{(k)}(\mathfrak{s}) \right\}$

- $\mathcal{U}(\mathfrak{s})$ naturally filtered $\implies$

$$\mathcal{U}_{(0)}(\mathfrak{s}) = \mathbb{C},\ \mathcal{U}_{(p)}(\mathfrak{s})\mathcal{U}_{(q)}(\mathfrak{s}) \subset \mathcal{U}_{(p+q)}(\mathfrak{s}), \quad p, q \geq 0.$$

- Consequence: $\mathcal{U}_{(p)}(\mathfrak{s})$ can be seen as f.d. representation of $\mathfrak{s}$.[3]

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[3] Flath D E 1990 *J. Math. Phys.* **31** 1076; Ovsienko V, Turbiner A V 1992 *Comptes Rendus Acad. Sci. A* **314** 13; Burdík Č, Navrátil O, Pošta s 2007 *Acta Polytech.* **47** 25; Burdík Č, Navrátil O 2009 *Generalized Lie Theory in Mathematics, Physics and beyond*, Springer Verlag, Berlin, pp. 297

[4] J. Dixmier 1974 *Algèbres enveloppantes*, Gauthier-Villars, Paris

Algebraic vs. analytical approach

- Adjoint action of $\mathfrak{s}$ on $\mathcal{U}(\mathfrak{s})$ (resp., the symmetric algebra $S(\mathfrak{s})$)

$$\begin{aligned} P \in \mathcal{U}(\mathfrak{s}) \mapsto & \quad P.X_i := [X_i, P] = X_i P - P X_i \in \mathcal{U}(\mathfrak{s}), \\ P\left(x_1, \ldots, x_n\right) \in S(\mathfrak{s}) \mapsto & \quad \widehat{X}_i(P) = C_{ij}^k x_k \frac{\partial}{\partial x_j} \in S(\mathfrak{s}), \end{aligned} ,$$

- Ad-comm. linear isomorphism $\Lambda : S(\mathfrak{s}) \to \mathcal{U}(\mathfrak{s})$ through symmetrization

$$\Lambda\left(x_{j_1} \ldots x_{j_p}\right) = \frac{1}{p!} \sum_{\sigma \in \Sigma_p} X_{j_{\sigma(1)}} \ldots X_{j_{\sigma(p)}},$$

- $\mathcal{U}^{(p)}(\mathfrak{s}) = \Lambda\left(S^{(p)}(\mathfrak{s})\right) \implies \mathcal{U}_{(p)}(\mathfrak{s}) = \sum_{k=0}^{p} \mathcal{U}^{(k)}(\mathfrak{s})$

- For any $P \in \mathcal{U}_{(p)}(\mathfrak{s}), Q \in \mathcal{U}_{(q)}(\mathfrak{s})$

$$[P, Q] \in \mathcal{U}_{(p+q-1)}(\mathfrak{s}).$$

- By Poincaré–Birkhoff–Witt theorem

$$\dim \mathcal{U}^{(p)}(\mathfrak{s}) = \dim \frac{\mathcal{U}_{(p)}(\mathfrak{s})}{\mathcal{U}_{(p-1)}(\mathfrak{s})} = \dim S^{(p)}(\mathfrak{s}) = \left(\begin{array}{c} \dim \mathfrak{s} + p - 1 \\ p \end{array} \right).$$

- Invariant polynomials of $\mathfrak{s}$ as centre of $\mathcal{U}(\mathfrak{s})$:

$$Z\left(\mathcal{U}(\mathfrak{s})\right) = \left\{ P \in \mathcal{U}(\mathfrak{s}) \mid [\mathfrak{s}, P] = 0 \right\}.$$

- Can be identified with (polynomial) solutions of diff. operators

- Commutant $C_{\mathcal{U}(\mathfrak{s})}(P)$ of $P \in \mathcal{U}(\mathfrak{s})$ as centralizer of P in $\mathcal{U}(\mathfrak{g})$:

$$C_{\mathcal{U}(\mathfrak{s})}(P) = \left\{ Q \in \mathcal{U}(\mathfrak{s}) \mid [P, Q] = 0 \right\}.$$

- $\mathfrak{s}$ semisimple, $C_{\mathcal{U}(\mathfrak{s})}(P)$ finitely generated (has integrity basis).

- $\widehat{X}_i = C_{ij}^k x_k \dfrac{\partial}{\partial x_j}$ inf. generator of 1-par. subgroup (wrt X_i)

- $S(\mathfrak{s})$ identified with $\mathbb{K}[x_1, \ldots, x_n]$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}$)

- Inherits structure of Poisson algebra through Berezin bracket[5]

$$\{P, Q\} = C_{ij}^k x_k \frac{\partial P}{\partial x_i} \frac{\partial Q}{\partial x_j}, \; P, Q \in S(\mathfrak{s}).$$

- $S^{(p)}(\mathfrak{s})$ hom. pol.:

$$\mathcal{U}^{(p)}(\mathfrak{s}) = \Lambda\left(S^{(p)}(\mathfrak{s})\right) \implies \mathcal{U}_{(p)}(\mathfrak{s}) = \sum_{k=0}^{p} \mathcal{U}^{(k)}(\mathfrak{s})$$

- Some caution:[6]

$$[\Lambda(P), \Lambda(Q)] = 0 \;\overset{\not\Leftarrow}{\Rightarrow}\; \{P, Q\} = 0$$

[5] Berezin F A 1967 *Funkt. Anal. Prilozh.* **1**(2) 1–14.

[6] Olshanski G 1997 *Transformation Groups* **2** 197; Boya L J, RCS 2009 *J. Phys. A: Math. Theor.* **42** 235203

- Commutant $C_{\mathcal{U}(\mathfrak{s})}(\mathfrak{a})$ of $\mathfrak{a} \subset \mathfrak{s}$

$$C_{\mathcal{U}(\mathfrak{s})}(\mathfrak{a}) = \{ Q \in \mathcal{U}(\mathfrak{s}) \mid [P, Q] = 0, \quad \forall P \in \mathfrak{a} \} . \tag{2}$$

- Special case: $\mathfrak{a} = \mathfrak{s}$

$$Z(\mathcal{U}(\mathfrak{s})) = \{ P \in \mathcal{U}(\mathfrak{s}) \mid [\mathfrak{s}, P] = 0 \} . \tag{3}$$

- If $C_{\mathcal{U}(\mathfrak{s})}(\mathfrak{a})$ is FG, $\exists \{P_1, \dots, P_s\}$ of lin. ind.,[7] s.t.

$$P_1^{a_1} P_2^{a_2} \dots P_s^{a_s}, \quad a_i \in \mathbb{N} \cup 0, \tag{4}$$

- Constants a_i satisfy certain algebraic relations.

- Linear dimension $\dim_L C_{\mathcal{U}(\mathfrak{s})}(\mathfrak{a}) = s$.

- $\mathcal{C}_{\mathcal{U}(\mathfrak{s})}(\mathfrak{a})$ contains invariants of $\mathfrak{a}$ and $C_1, \dots, C_\ell$ of $\mathfrak{s}$.

- $\mathcal{C}_{\mathcal{U}(\mathfrak{s})}(\mathfrak{a})$ is a free module over $\mathbb{C}[C_1, \dots, C_\ell]$.

[7] Functional dependence relations are defined in the field of fractions, and not in $\mathcal{U}(\mathfrak{s})$.

Computing with Berezin bracket

- $\mathfrak{a}^* \subset \mathfrak{s}^*$ [commuting coordinates]

$$C_{S(\mathfrak{s})}(\mathfrak{a}) = \{Q \in S(\mathfrak{s}) \mid \{P, Q\} = 0,\ P \in \mathfrak{a}\}$$

- Solutions of linear first-order system of PDEs

$$\widehat{X}_i(Q) := \{x_i, Q\} = C_{ij}^k x_k \frac{\partial Q}{\partial x_j} = 0,\ 1 \leq i \leq m = \dim \mathfrak{a}, \qquad (5)$$

- $\mathfrak{J}_d$ linearly independent elements. Filtration property

$$\mathfrak{J}_2 \subset \mathfrak{J}_3 \subset \cdots \subset \mathfrak{J}_d \subset \dots$$

- Berezin bracket

$$\{\mathfrak{J}_p, \mathfrak{J}_q\} \subset \mathfrak{J}_{p+q-1}, \quad p, q, \geq 2 \qquad (6)$$

- Commutants in $\mathcal{U}(\mathfrak{s})$: special case of "internal labelling problem"

- Chain of subalgebras

$$\mathfrak{s}_1 \subset \mathfrak{s}_2 \subset \cdots \subset \mathfrak{s}.$$

- Embedding chain of commutants

$$C_{S(\mathfrak{s})}(\mathfrak{s}_1) \supset C_{S(\mathfrak{s})}(\mathfrak{s}_2) \supset \cdots \supset C_{S(\mathfrak{s})}(\mathfrak{s}) . \tag{7}$$

- Applications: Algebraic Hamiltonian (w.t.r $\mathfrak{a}$)

$$\mathcal{H}_a = \sum_i \alpha_i X_i X_j + \sum_k \beta_k X_k + \sum_\ell \gamma_\ell \mathcal{C}_\ell, \quad X_i, X_j, X_k \in \mathfrak{a}; \quad \alpha_{ij}, \beta_k, \gamma_\ell \in \mathbb{K},$$

- Elements in commutant: constants of the motion of $\mathcal{H}_a$

- Two cases:
  1. Trivial Berezin bracket in $C_{S(\mathfrak{s})}(\mathcal{H}_a)$ all have trivial Berezin bracket: Abelian symmetry algebra.
  2. $\left\{ C_{S(\mathfrak{s})}(\mathcal{H}_a), C_{S(\mathfrak{s})}(\mathcal{H}_a) \right\} \neq 0$: non-commutative polynomial symmetry algebra.

Commutants and missing label problem (MLP)

- Classification schemes related to embedding chains

$$\left| \begin{array}{ccccccccc} \mathfrak{s} & \supset & \mathfrak{s}' & \supset & \mathfrak{s}'' & \supset & \ldots & \supset & \mathfrak{s}^{n\prime} \\ \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\ [\lambda] & & [\lambda'] & & [\lambda''] & & \ldots & & [\lambda^{n\prime}] \end{array} \right\rangle$$

- Labelling of $\mathfrak{s}$ IR by means of "internal symmetries".

- Total number of labels needed to describe states :

$$i_0 + \mathcal{N}(\mathfrak{s}) = \frac{1}{2}(\dim \mathfrak{s} - \mathcal{N}(\mathfrak{s})) + \mathcal{N}(\mathfrak{s}), \tag{8}$$

with i_0 number of internal labels.

- Covered in multiplicity free cases [Gel'fand-Tsetlin patterns]

- Non-multiplicity free case: requires additional operators

$$n_0 = \frac{1}{2}\left(\dim \mathfrak{s} - \mathcal{N}(\mathfrak{s}) - \dim \mathfrak{s}' - \mathcal{N}(\mathfrak{s}')\right) + \ell' \tag{9}$$

- $\mathfrak{s}'$ is CSA of $\mathfrak{s}$ [Racah operators].

$$n_0 = \frac{1}{2}\left(\dim \mathfrak{s} - 3\ell\right)$$

- Chain $\mathfrak{s} \subset \mathfrak{s} \oplus \mathfrak{s}$ [external MLP, related to CG series]

$$n_0 = \frac{1}{2}\left(\dim \mathfrak{s} - \ell\right)$$

- $\mathfrak{n} \subset \mathfrak{s}$ with $\mathfrak{n}$ NR of Borel subalgebra [Decomposition of $\mathcal{U}(\mathfrak{s})$][8]

$$n_0 = \frac{1}{2}\left(\dim \mathfrak{s} - \ell - 2\mathcal{N}(\mathfrak{n})\right)$$

- MLP corresponds to $\mathcal{A} \subset \mathcal{C}_{\mathcal{U}(\mathfrak{s})}(\mathfrak{s}')$ s.t.

$$[\mathcal{A}, \mathcal{A}] = 0.$$

[8] RCS, Marquette I 2024 *J. Lie Theory* **34** 17

Casimirs as sums of labelling operators

- The chain $\mathfrak{s} \supset \mathfrak{s}'$ implies $\mathrm{ad}(\mathfrak{s}) = \mathrm{ad}(\mathfrak{s}') \oplus \Gamma$.[9]
- $\{X_1, .., X_{n_0}\}$ – $\mathfrak{s}'$ basis; $\{X_1, .., X_{n_0}, .., X_n\}$ – $\mathfrak{s}$ basis
- Consider the scaling transformations

$$\Phi(\varepsilon)(X_i) = \begin{cases} X_i, & 1 \leq i \leq n_0 \\ \varepsilon X_i, & n_0 + 1 \leq i \leq n \end{cases}, \qquad (10)$$

- $\mathfrak{s}'$ and Γ remain invariant.
- $C_p = \kappa^{i_1 \dots i_p} X_{i_1} \dots X_{i_p}$ – hom. p-Casimir operator of $\mathfrak{s}$.

$$C_p(\Phi^{-1}(\varepsilon)(X_{i_1}), .., \Phi^{-1}(\varepsilon)(X_{i_p})) = \varepsilon^{-(n_{i_1} + \dots + n_{i_p})} \kappa^{i_1 \dots i_p} X_{i_1} \dots X_{i_p}. \qquad (11)$$

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[9] J. Patera and G. Sankoff *Tables of Branching Rules for Representations of Simple Lie Algebras*, Presses de L'Université de Montréal, Montréal, 1973 ; W. G. McKay, J. Patera *Tables of Dimensions, Indices and Branching Rules for Representations of Simple Lie algebras*, M. Dekker, New York, 1981; W. G. McKay, J. Patera and D. R. Rand. *Tables of Representations of Simple Lie algebras*, CRM, Montréal 1990.

- $M_p = \max\left\{ n_{i_1} + \ldots + n_{i_p} \mid \kappa^{i_1 \ldots i_p} \neq 0 \right\} \leq p$.

$$C_p = \sum_{\alpha=0}^{M_p} \varepsilon^{\alpha} \Theta^{[p-\alpha,\alpha]},$$

- $\Theta^{[p-\alpha,\alpha]}$ hom. pol. of bi-degree $(p-\alpha, \alpha)$ in $(\mathfrak{s}', \Gamma)$.
- $\forall X_i \in \mathfrak{s}' \Rightarrow \left[X_i, \Theta^{[p-\alpha,\alpha]}\right] = 0 \Rightarrow \Theta^{[p-\alpha,\alpha]} \in \mathcal{C}_{\mathcal{U}(\mathfrak{s})}(\mathfrak{s}')$.
- $\Theta^{[p-\alpha,\alpha]}$ obtainable as trace operators.
- Not any commuting polynomial $P \in \mathcal{C}_{\mathcal{U}(\mathfrak{s})}(\mathfrak{s}')$ arises in this form.
- When does P appear as a trace operator? [Open question]

Root systems and polynomial algebras

- $\mathfrak{h} \subset \mathfrak{s}$ Cartan subalgebra: $\{h_1, \dots, h_\ell\}$
- $C_{\mathcal{U}(\mathfrak{s})}(\mathfrak{h})$ of $\mathfrak{h} \in \mathcal{U}(\mathfrak{s})$ determined by root system $\mathcal{R}$
- Simple roots $\Delta = \{\alpha_1, \dots, \alpha_\ell\} \leftrightarrow \{X_{\pm\alpha_1}, \dots, X_{\pm\alpha_\ell}\}$ generators
- Positive non-simple roots $\{\beta_1, \dots, \beta_{r_0}\}, \quad r_0 = |\mathcal{R}^+|$
- Basis of $\mathfrak{s}$

$$\mathcal{B} = \left\{\mathfrak{h};\ X_{\pm\alpha_1}, \dots, X_{\pm\alpha_\ell}, X_{\pm\beta_1}, \dots, X_{\pm\beta_{r_0}}\right\}$$

- By SCH relations

$$[h_i, X_\beta] = \lambda_i(\beta) X_\beta, \quad \beta \in \mathcal{R}, \quad \lambda_i(\beta) \in \mathbb{Z}$$

- Differential operators

$$\widehat{H}_i = \sum_{k=1}^{2r_0} \lambda_i(\beta_k) x_{\beta_k} \frac{\partial}{\partial x_{\beta_k}}$$

- Commutant in analytical version

$$C_{S(\mathfrak{s})}(\mathfrak{h}) = \left\{ P \mid \{P, H_i\} = \widehat{H}_i(P) = 0 \right\} \supset \mathfrak{h}$$

- For monomials $Q = x_{\beta_1}^{v_1} \cdots x_{\beta_{2r_0}}^{\nu_{2r_0}}$

$$\begin{aligned} \widehat{H}_i \left(x_{\beta_1}^{v_1} \cdots x_{\beta_{2r_0}}^{\nu_{2r_0}} \right) &= \left(\sum_{k=1}^{2r_0} \lambda_i(\beta_k) \nu_k \right) Q = 0 \\ &\Longrightarrow \left(\sum_{k=1}^{2r_0} \lambda_i(\beta_k) \nu_k \right) = 0, \quad 1 \leq i \leq \ell \end{aligned} \tag{12}$$

- Elements in $C_{S(\mathfrak{s})}(\mathfrak{h})$ have thus 'zero weight' w.r.t. $\mathfrak{h}$
- We proceed recursively on $d = \deg Q$
  - $d = 1$: $\mathfrak{h}$
  - $d = 2$: $x_{\beta_1} x_{\beta_2} \in C_{S(\mathfrak{s})}(\mathfrak{h}) \implies \beta_1 + \beta_2 = 0$
  - $\vdots$
  - $d = p$: $x_{\beta_1} \cdots x_{\beta_p} \in C_{S(\mathfrak{s})}(\mathfrak{h}) \implies \beta_1 + \cdots + \beta_p = 0$
- Basis of lin. ind. generators of $C_{S(\mathfrak{s})}(\mathfrak{h})$ obtained in steps
- Highest degree of ind. monomial $d_0 = 1 + \max\{ht(\beta) \mid \beta \in \mathcal{R}\}$
- For $d > d_0$ a monomial in $C_{S(\mathfrak{s})}(\mathfrak{h})$ is decomposable

- Maximal heights of roots for classical SLA:[10]
  - A_ℓ:

$$\sum_{k=1}^{\ell} \alpha_k \implies d_0 = \ell + 1$$

  - B_ℓ:

$$\alpha_1 + 2\sum_{k=2}^{\ell} \alpha_k \implies d_0 = 2\ell$$

  - C_ℓ:

$$\alpha_\ell + 2\sum_{k=1}^{\ell-1} \alpha_k \implies d_0 = 2\ell$$

  - D_ℓ:

$$\alpha_1 + \sum_{k=2}^{\ell-2} + \alpha_{\ell-1} + \alpha_\ell \implies d_0 = 2\ell - 2$$

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[10] RCS, Latini D, Marquette I, Zhang Y-Z 2023 *J. Phys.: Conf. Ser.* **2667** 012037; RCS, Latini D, Marquette I, Zhang Y-Z 2023 *J. Phys. A: Math. Theor.* **56** 045202; RCS, Latini D, Marquette I, Zhang J, Zhang Y-Z 2024 arXiv:2406.01958

- Maximal heights of roots for ESLA:
  - G_2: $3\alpha_1 + 2\alpha_2 \implies d_0 = 6$
  - F_4: $2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 \implies d_0 = 12$
  - E_6: $\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6 \implies d_0 = 12$
  - E_7: $2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + 2\alpha_7 \implies d_0 = 18$
  - E_8:
    $2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + 3\alpha_8 \implies d_0 = 30$
- d_0 coincides with highest degree of primitive Casimir operator
- Exact dimension still to be determined generically
- Degree of polynomial algebra $\mathcal{A}$: $\deg \leq d_0$

- Number of monomials for A_ℓ series:

Table: Number of monomials for A_ℓ series

ℓ	2	3	4	5	6	7	8	9	10	11
2	3	2								
3	6	8	6							
4	10	20	30	24						
5	15	40	90	144	120					
6	21	70	210	504	340	720				
7	28	112	420	1344	3360	5760	5040			
8	36	168	756	3024	10080	25920	45360	8!		
9	45	240	1260	6048	25200	86400	226800	403200	9!	
10	55	330	1980	11088	55440	237600	831600	2217600	3991680	10!

- Generating function can be found

$$\xi(\ell) = \sum_{k=1}^{\ell+1} \frac{(\ell+1)!}{(\ell+1-d)!d} - 1$$

Polynomial algebra for G_2

- Using regular embedding $A_2 \subset G_2$
- Branching rule

$$\Gamma[1,0] = \Lambda[1,1] + \Lambda[1,0] + \Lambda[0,1], \tag{13}$$

- Basis $\left\{E_{ij}, a_1, a_2, a_3, b^1, b^2, b^3\right\}$ $(E_{11} + E_{22} + E_{33} = 0)$,

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj}, \quad [E_{ij}, a_k] = \delta_{jk} a_i, \quad \left[E_{ij}, b^k\right] = -\delta_{ik} b^j,$$
$$[a_i, a_j] = -2\varepsilon_{ijk} b^k, \quad \left[b^i, b^j\right] = 2\varepsilon_{ijk} a_k, \quad \left[a_i, b^j\right] = 3E_{ij}.$$

- Cartan subalgebra $h_1 = E_{11} - 2E_{22} + E_{33}$, $h_2 = E_{22} - E_{33}$

Table: Eigenvalues of $\mathfrak{H}$ over the basis (23)

	3	5	7	4	6	8	9	10	11	12	13	14
X	E_{12}	E_{23}	E_{13}	E_{21}	E_{32}	E_{31}	a_1	a_2	a_3	b^1	b^2	b^3
$\lambda_1(X)$	3	3	0	-3	-3	0	1	-2	1	-1	2	-1
$\lambda_2(X)$	-1	-2	1	1	2	-1	0	1	-1	0	-1	1

- Differential operators

$$\begin{aligned}\widehat{X}_1(F) =& 3x_3\partial_{x_3}F - 3x_4\partial_{x_4}F + 3x_5\partial_{x_5}F - 3x_6\partial_{x_6}F + x_9\partial_{x_9}F - 2x_{10}\partial_{x_{10}}F + x_{11}\partial_{x_{11}}F \\ & - x_{12}\partial_{x_{12}}F + 2x_{13}\partial_{x_{13}}F - x_{14}\partial_{x_{14}}F = 0, \\ \widehat{X}_2(F) =& - x_3\partial_{x_3}F + x_4\partial_{x_4}F - 2x_5\partial_{x_5}F + 2x_6\partial_{x_6}F + x_7\partial_{x_7}F - x_8\partial_{x_8}F + x_{10}\partial_{x_{10}}F \\ & - x_{11}\partial_{x_{11}}F - x_{13}\partial_{x_{13}}F + x_{14}\partial_{x_{14}}F = 0.\end{aligned}$$

- Constraint

$$\sum_{k=1}^{14}\nu_k\lambda_1(x_k) = 0, \quad \sum_{k=1}^{14}\nu_k\lambda_2(x_k) = 0.$$

$$\mathcal{A}_1 = \{Q_1 = x_1, Q_2 = x_2\}$$

$$\mathcal{A}_2 = \{Q_3 = x_3x_4, Q_4 = x_5x_6, Q_5 = x_7x_8, Q_6 = x_9x_{12}, Q_7 = x_{10}x_{13}, Q_8 = x_{11}x_{14}\}$$

$$\begin{aligned}\mathcal{A}_3 = \{&Q_9 = x_3x_6x_8, Q_{10} = x_3x_{10}x_{12}, Q_{11} = x_4x_5x_7, Q_{12} = x_4x_9x_{13}, Q_{13} = x_5x_{10}x_{14},\\ &Q_{14} = x_6x_{11}x_{13}, Q_{15} = x_7x_{11}x_{12}, Q_{16} = x_8x_9x_{14}, Q_{17} = x_9x_{10}x_{11}, Q_{18} = x_{12}x_{13}x_{14}\}\end{aligned}$$

$$\begin{aligned}\mathcal{A}_4 = \Big\{&Q_{19} = x_3x_6x_{11}x_{12}, Q_{20} = x_3x_8x_{10}x_{14}, Q_{21} = x_3x_{10}^2x_{11}, Q_{22} = x_3x_{12}^2x_{14}, Q_{23} = x_4x_5x_9x_{14},\\ &Q_{24} = x_4x_7x_{11}x_{13}, Q_{25} = x_4x_9^2x_{11}, Q_{26} = x_4x_{13}^2x_{14}, Q_{27} = x_5x_7x_{10}x_{12}, Q_{28} = x_5x_9x_{10}^2,\\ &Q_{29} = x_5x_{12}x_{14}^2, Q_{30} = x_6x_8x_9x_{13}, Q_{31} = x_6x_9x_{11}^2, Q_{32} = x_6x_{12}x_{13}^2, Q_{33} = x_7x_{10}x_{11}^2,\\ &Q_{34} = x_7x_{12}x_{13}^2, Q_{35} = x_8x_{13}x_{14}^2, Q_{36} = x_8x_9^2x_{10}\Big\}\end{aligned}$$

$$\begin{aligned}\mathcal{A}_5 = \Big\{&Q_{37} = x_3x_5x_{10}^3, Q_{38} = x_3x_6x_{10}x_{11}^2, Q_{39} = x_3x_6x_{12}^2x_{13}, Q_{40} = x_3x_7x_{13}^3, Q_{41} = x_3x_8x_9x_{10}^2,\\ &Q_{42} = x_3x_8x_{12}x_{14}^2, Q_{43} = x_4x_5x_9^2x_{10}, Q_{44} = x_4x_5x_{13}x_{14}^2, Q_{45} = x_4x_6x_{13}^3, Q_{46} = x_4x_7x_9x_{11}^2,\\ &Q_{47} = x_4x_7x_{12}x_{13}^2, Q_{48} = x_4x_8x_9^3, Q_{49} = x_5x_7x_{10}^2x_{11}, Q_{50} = x_6x_7x_{11}^3, Q_{51} = x_5x_7x_{12}^2x_{14},\\ &Q_{52} = x_5x_8x_{14}^3, Q_{53} = x_6x_8x_9^2x_{11}, Q_{54} = x_6x_8x_{13}^2x_{14}\Big\}\end{aligned}$$

$$\begin{aligned}\mathcal{A}_6 = \Big\{&Q_{55} = x_3^2x_6x_{12}^3, Q_{56} = x_3x_6^2x_{11}^3, Q_{57} = x_3x_8^2x_{14}^3, Q_{58} = x_3^2x_8x_{10}^3, Q_{59} = x_4x_5^2x_{14}^3,\\ &Q_{60} = x_4^2x_5x_9^3, Q_{61} = x_4^2x_7x_{13}^3, Q_{62} = x_4x_7^2x_{11}^3, Q_{63} = x_5^2x_7x_{10}^3, Q_{64} = x_5x_7^2x_{12}^3,\\ &Q_{65} = x_6^2x_8x_{13}^3, Q_{66} = x_6^2x_8^2x_9^2\Big\}\end{aligned}$$

- Number of generating monomials

$$(2) + 6 + 10 + 18 + 18 + 12 = 66$$

- Casimir operators C_2', C_3' of A_2 and C_2, C_6 of G_2[11]

$$C_2' = Q_1^2 + 3(Q_1 Q_2 + Q_2^2) + 3(Q_4 + Q_7 + Q_8), \quad C_2 = C_2' + Q_3 + Q_5 + Q_6,$$
$$C_3' = 2Q_1^3 + 9(Q_1^2 Q_2 + Q_1 Q_2^2) + 9Q_1(Q_7 - 2Q_4 + Q_8) - 27Q_2(Q_4 - Q_7) + 27(Q_{11} + Q_{12}).$$

- Algebraic dependence relations

$$P = \sum_{s=1}^{r_0} \mu_s \prod_{k=1}^{66} Q_k^{a_{k,s}} = 0,$$

- r_0 number of non-negative integer solutions of

$$w_0 = \sum_{k=1}^{66} a_{k,s} \mathrm{deg}(Q_k).$$

[11] C_6 is skipped due to its length

- Some algebraic relations in degree six:

$$
\begin{gathered}
Q_3Q_{20}-Q_{14}Q_{18}=0, \quad Q_3Q_{22}-Q_{15}Q_{17}=0, \quad Q_3Q_{24}-Q_{13}Q_{18}=0,\\
Q_3Q_{25}-Q_{13}Q_{17}=0, \quad Q_3Q_{27}-Q_{15}Q_{16}=0, \quad Q_3Q_{28}-Q_{14}Q_{16}=0,\\
Q_4Q_{32}-Q_{11}Q_{18}=0, \quad Q_4Q_{36}-Q_{12}Q_{15}=0, \quad Q_5Q_{19}-Q_{14}Q_{15}=0,\\
Q_5Q_{21}-Q_{17}Q_{18}=0, \quad Q_5Q_{30}-Q_9Q_{15}=0, \quad Q_5Q_{31}-Q_9Q_{17}=0,\\
Q_5Q_{34}-Q_{10}Q_{18}=0, \quad Q_5Q_{35}-Q_{10}Q_{14}=0, \quad Q_6Q_{23}-Q_{13}Q_{14}=0,\\
Q_6Q_{26}-Q_{16}Q_{17}=0, \quad Q_6Q_{29}-Q_9Q_{14}=0, \quad Q_6Q_{32}-Q_9Q_{16}=0,\\
Q_6Q_{33}-Q_{10}Q_{17}=0, \quad Q_6Q_{36}-Q_{10}Q_{13}=0, \quad Q_7Q_{30}-Q_{11}Q_{13}=0,\\
Q_7Q_{34}-Q_{12}Q_{16}=0, \quad Q_8Q_{24}-Q_9Q_{12}=0, \quad Q_8Q_{27}-Q_{10}Q_{11}=0,\\
Q_3Q_4Q_5-Q_{15}Q_{18}=0, \quad Q_3Q_5Q_6-Q_{14}Q_{17}=0, \quad Q_3Q_6Q_7-Q_{13}Q_{16}=0,\\
Q_4Q_7Q_8-Q_{11}Q_{12}=0, \quad Q_5Q_6Q_8-Q_9Q_{10}=0.
\end{gathered}
$$

- No dependence relations in degrees $p \leq 5$

- Polynomial algebra of degree 6

$$
\begin{aligned}
&\{\mathcal{A}_2,\mathcal{A}_2\}\subset \mathcal{A}_1\mathcal{A}_2+\mathcal{A}_3,\\
&\{\mathcal{A}_2,\mathcal{A}_3\}\subset \mathcal{A}_1\mathcal{A}_3+\mathcal{A}_2^2+\mathcal{A}_4,\quad \{\mathcal{A}_2,\mathcal{A}_4\}\subset \mathcal{A}_1\mathcal{A}_4+\mathcal{A}_1\mathcal{A}_2^2+\mathcal{A}_2\mathcal{A}_3+\mathcal{A}_5,\\
&\{\mathcal{A}_2,\mathcal{A}_5\}\subset \mathcal{A}_1\mathcal{A}_5+\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3+\mathcal{A}_2\mathcal{A}_4+\mathcal{A}_2^3+\mathcal{A}_3^2+\mathcal{A}_6,\\
&\{\mathcal{A}_2,\mathcal{A}_6\}\subset \mathcal{A}_1\mathcal{A}_6+\mathcal{A}_1\mathcal{A}_2^3+\mathcal{A}_1\mathcal{A}_3^2+\mathcal{A}_1\mathcal{A}_2\mathcal{A}_4+\mathcal{A}_2^2\mathcal{A}_3+\mathcal{A}_3\mathcal{A}_4+\mathcal{A}_2\mathcal{A}_5+\mathcal{A}_6,\\
&\{\mathcal{A}_3,\mathcal{A}_3\}\subset \mathcal{A}_1\mathcal{A}_4+\mathcal{A}_1\mathcal{A}_2^2+\mathcal{A}_2\mathcal{A}_3+\mathcal{A}_5,\\
&\{\mathcal{A}_3,\mathcal{A}_4\}\subset \mathcal{A}_1\mathcal{A}_5+\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3+\mathcal{A}_2\mathcal{A}_4+\mathcal{A}_2^3+\mathcal{A}_3^2+\mathcal{A}_6,\\
&\{\mathcal{A}_3,\mathcal{A}_5\}\subset \mathcal{A}_1\mathcal{A}_6+\mathcal{A}_1\mathcal{A}_2^3+\mathcal{A}_1\mathcal{A}_3^2+\mathcal{A}_1\mathcal{A}_2\mathcal{A}_4+\mathcal{A}_2^2\mathcal{A}_3+\mathcal{A}_3\mathcal{A}_4+\mathcal{A}_2\mathcal{A}_5+\mathcal{A}_6,\\
&\{\mathcal{A}_3,\mathcal{A}_6\}\subset \mathcal{A}_1\mathcal{A}_2^2\mathcal{A}_3+\mathcal{A}_1\mathcal{A}_2\mathcal{A}_5+\mathcal{A}_1\mathcal{A}_3\mathcal{A}_4+\mathcal{A}_2^4+\mathcal{A}_2\mathcal{A}_3^2+\mathcal{A}_2^2\mathcal{A}_4+\mathcal{A}_2\mathcal{A}_6+\mathcal{A}_3\mathcal{A}_5+\mathcal{A}_4^2,\\
&\{\mathcal{A}_4,\mathcal{A}_4\}\subset \mathcal{A}_1\mathcal{A}_6+\mathcal{A}_1\mathcal{A}_2^3+\mathcal{A}_1\mathcal{A}_3^2+\mathcal{A}_1\mathcal{A}_2\mathcal{A}_4+\mathcal{A}_2^2\mathcal{A}_3+\mathcal{A}_3\mathcal{A}_4+\mathcal{A}_2\mathcal{A}_5+\mathcal{A}_6,\\
&\{\mathcal{A}_4,\mathcal{A}_5\}\subset \mathcal{A}_1\mathcal{A}_2^2\mathcal{A}_3+\mathcal{A}_1\mathcal{A}_2\mathcal{A}_5+\mathcal{A}_1\mathcal{A}_3\mathcal{A}_4+\mathcal{A}_2^4+\mathcal{A}_2\mathcal{A}_3^2+\mathcal{A}_2^2\mathcal{A}_4+\mathcal{A}_2\mathcal{A}_6+\mathcal{A}_3\mathcal{A}_5+\mathcal{A}_4^2, \qquad (14)\\
&\{\mathcal{A}_4,\mathcal{A}_6\}\subset \mathcal{A}_1\mathcal{A}_2^4+\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3^2+\mathcal{A}_1\mathcal{A}_2^2\mathcal{A}_4+\mathcal{A}_1\mathcal{A}_2\mathcal{A}_6+\mathcal{A}_1\mathcal{A}_3\mathcal{A}_5+\mathcal{A}_1\mathcal{A}_4^2+\mathcal{A}_2^3\mathcal{A}_3+\mathcal{A}_2^2\mathcal{A}_5\\
&\qquad\qquad +\mathcal{A}_2\mathcal{A}_3\mathcal{A}_4+\mathcal{A}_3^3+\mathcal{A}_3\mathcal{A}_6\mathcal{A}_4\mathcal{A}_5,\\
&\{\mathcal{A}_5,\mathcal{A}_5\}\subset \mathcal{A}_1\mathcal{A}_2^4+\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3^2+\mathcal{A}_1\mathcal{A}_2^2\mathcal{A}_4+\mathcal{A}_1\mathcal{A}_2\mathcal{A}_6+\mathcal{A}_1\mathcal{A}_3\mathcal{A}_5+\mathcal{A}_1\mathcal{A}_4^2+\mathcal{A}_2^3\mathcal{A}_3+\mathcal{A}_2^2\mathcal{A}_5\\
&\qquad\qquad +\mathcal{A}_2\mathcal{A}_3\mathcal{A}_4+\mathcal{A}_3^3+\mathcal{A}_3\mathcal{A}_6\mathcal{A}_4\mathcal{A}_5,\\
&\{\mathcal{A}_5,\mathcal{A}_6\}\subset \mathcal{A}_1\mathcal{A}_2^3\mathcal{A}_3+\mathcal{A}_1\mathcal{A}_3^3+\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3\mathcal{A}_4+\mathcal{A}_1\mathcal{A}_2^2\mathcal{A}_5+\mathcal{A}_1\mathcal{A}_4\mathcal{A}_5+\mathcal{A}_1\mathcal{A}_3\mathcal{A}_6+\mathcal{A}_2^3\mathcal{A}_4\\
&\qquad\qquad +\mathcal{A}_2^5+\mathcal{A}_2^2\mathcal{A}_3^2+\mathcal{A}_2\mathcal{A}_4^2+\mathcal{A}_2\mathcal{A}_3\mathcal{A}_5+\mathcal{A}_2\mathcal{A}_6^2+\mathcal{A}_3^2\mathcal{A}_4+\mathcal{A}_4\mathcal{A}_6+\mathcal{A}_5^2,\\
&\{\mathcal{A}_6,\mathcal{A}_6\}\subset \mathcal{A}_1\mathcal{A}_2^5+\mathcal{A}_1\mathcal{A}_2^2\mathcal{A}_3^2+\mathcal{A}_1\mathcal{A}_2^3\mathcal{A}_4+\mathcal{A}_1\mathcal{A}_3^2\mathcal{A}_4+\mathcal{A}_1\mathcal{A}_2\mathcal{A}_4^2+\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3\mathcal{A}_5+\mathcal{A}_1\mathcal{A}_2^2\mathcal{A}_6\\
&\qquad\qquad +\mathcal{A}_1\mathcal{A}_5^2+\mathcal{A}_1\mathcal{A}_4\mathcal{A}_6+\mathcal{A}_2^4\mathcal{A}_3+\mathcal{A}_2^3\mathcal{A}_5+\mathcal{A}_2^2\mathcal{A}_3\mathcal{A}_4+\mathcal{A}_2\mathcal{A}_3\mathcal{A}_6+\mathcal{A}_2\mathcal{A}_4\mathcal{A}_5+\mathcal{A}_2\mathcal{A}_3^3\\
&\qquad\qquad +\mathcal{A}_3\mathcal{A}_4^2+\mathcal{A}_3^3\mathcal{A}_5+\mathcal{A}_5\mathcal{A}_6.
\end{aligned}
$$

- Degree as predicted

$$\{Q_{61}, Q_{65}\} = 4Q_1Q_4Q_6^3Q_8 - 18Q_1Q_4Q_6^2Q_8^2 + 8Q_2Q_4Q_6^3Q_8 - 17Q_2Q_4Q_6^2Q_8^2 - Q_2Q_6^3Q_8^2.$$

- Recovery of the polynomial algebra associated to A_2
- Set of A_i reduces to elements $Q_1, Q_2, Q_4, Q_7, Q_8, Q_{11}, Q_{12}$
- Cubic polynomial algebra[12]

$$\begin{aligned}
\{Q_4, Q_7\} =& Q_{12} - Q_{11}, \quad \{Q_4, Q_8\} = Q_{11} - Q_{12}, \quad \{Q_4, Q_{11}\} = Q_4(Q_8 - Q_7) + Q_2Q_{11},\\
\{Q_4, Q_{12}\} =& Q_4(Q_8 - Q_8) - Q_2Q_{11}, \quad \{Q_7, Q_{11}\} = (Q_1 + Q_2)Q_{11} + (Q_4 - Q_7)Q_{11},\\
\{Q_7, Q_8\} =& Q_{12} - Q_{11}, \quad \{Q_7, Q_{12}\} = -(Q_1 + Q_2)Q_{12} + (Q_8 - Q_4)Q_7,\\
\{Q_8, Q_{11}\} =& (Q_4 + Q_7)Q_8 - (Q_1 + 2Q_2)Q_{11}, \quad \{Q_8, Q_{12}\} = (Q_1 + 2Q_2)Q_{12} + (Q_4 - Q_7)Q_8,\\
\{Q_{11}, Q_{12}\} =& Q_1Q_4(Q_7 - Q_8) + Q_2Q_4(2Q_7 - Q_8) + Q_2Q_7Q_8.
\end{aligned}$$

[12] Obtained differently in RCS, Latini D, Marquette I, Zhang Y-Z 2023 *Annals Phys.* **459** 169496.

Polynomial algebras and regular subalgebras

- For regular subalgebras (RSA) $\mathfrak{s}' \subset \mathfrak{s}$ we have

$$C_{S(\mathfrak{s}')}(\mathfrak{h}') \cap C_{S(\mathfrak{s})}(\mathfrak{h}) \subset C_{S(\mathfrak{s}')}(\mathfrak{h}) \subset C_{S(\mathfrak{s})}(\mathfrak{h})$$

- Chains provide (weak) lower bound for linear dimension
  - $E_6 \supset D_5 \implies \xi(E_6) > 209625$
  - $E_7 \supset E_6 \implies \xi(E_7) > \xi(E_6)$
  - $E_8 \supset E_7 \implies \xi(E_8) > \xi(E_7)$
  - $F_4 \supset B_4 \implies \xi(F_4) > 35884$
  - $G_2 \supset A_2 \implies \xi(G_2) > 7$ [Compare with dim $= 66$]

- Best approximation not necessarily related to maximal rank RSA

- Scheme of RSA of exceptional SLA[13]

G_2

$\underline{A_2}$	A_1
$A_1 + \widetilde{A}_1$	$\widetilde{A}_1$

F_4

$\underline{B_4}$	$2A_1 + \widetilde{A}_1$
$A_3 + \widetilde{A}_1$	$A_1 + \widetilde{A}_2$
$A_2 + \widetilde{A}_2$	C_3
$C_3 + A_1$	$3A_1$
D_4	A_2
$B_2 + 2A_1$	B_2
$4A_1$	$A_1 + \widetilde{A}_1$
B_3	$2A_1$
$B_2 + A_1$	$\widetilde{A}_2$
$A_2 + \widetilde{A}_1$	$\widetilde{A}_1$
A_3	A_1

E_6

$A_5 + A_1$	$A_2 + 2A_1$
$3A_2$	$4A_1$
A_5	$\underline{D_4}$
$2A_2 + A_1$	A_3
$A_4 + A_1$	$A_2 + A_1$
D_5	$3A_1$
$A_3 + 2A_1$	A_2
A_4	$2A_1$
$A_3 + A_1$	A_1
$2A_2$	

E_7

$D_6 + A_1$	$3A_2$	D_4
$A_5 + A_2$	$2A_3$	A_4
$2A_3 + A_1$	A_6	$[A_3 + A_1]'$
A_7	$6A_1$	$[A_3 + A_1]''$
$D_4 + 3A_1$	$\underline{D_5}$	$2A_2$
$7A_1$	$A_4 + A_1$	$A_2 + 2A_1$
$\underline{E_6}$	$2A_2 + A_1$	$[4A_1]'$
$D_5 + A_1$	$[A_5]'$	$[4A_1]''$
$A_4 + A_2$	$[A_5]''$	A_3
$A_3 + A_2 + A_1$	$D_4 + A_1$	$A_2 + A_1$
$[A_5 + A_1]'$	$A_3 + A_2$	$[3A_1]'$
$[A_5 + A_1]''$	$5A_1$	$[3A_1]''$
D_6	$A_2 + 3A_1$	A_2
$D_4 + 2A_1$	$[A_3 + 2A_1]'$	$2A_1$
$A_3 + 3A_1$	$[A_3 + 2A_1]''$	A_1

E_8

$\underline{A_8}$	$A_4 + A_2 + A_1$	$A_4 + A_3$	$A_4 + 2A_1$	$A_2 + 3A_1$
$\underline{D_8}$	$A_5 + A_2$	$A_5 + 2A_1$	A_6	$2A_2 + A_1$
$A_7 + A_1$	$3A_2 + A_1$	$[A_7]'$	$A_3 + A_2 + A_1$	D_4
$A_5 + A_2 + A_1$	$E_6 + A_1$	$[A_7]''$	$[A_5 + A_1]'$	$[4A_1]'$
$2A_4$	$\underline{E_7}$	$3A_2$	$[A_5 + A_1]''$	$[4A_1]''$
$4A_2$	D_7	E_6	$A_4 + A_2$	$A_2 + 2A_1$
$A_6 + A_2$	$D_5 + 2A_1$	D_6	$2A_2 + 2A_1$	$2A_2$
$A_7 + A_1$	$D_4 + 3A_1$	$D_4 + 2A_1$	D_5	$A_3 + A_1$
$D_6 + 2A_1$	$2A_3 + A_1$	$[2A_3]'$	$[A_3 + 2A_1]'$	A_4
$D_5 + A_3$	$7A_1$	$[2A_3]''$	$[A_3 + 2A_1]''$	A_3
$2D_4$	$D_6 + A_1$	$D_5 + A_1$	$A_3 + A_2$	$A_2 + A_1$
$D_4 + 4A_1$	$D_5 + A_2$	$A_3 + 3A_1$	A_5	$3A_1$
$2A_3 + 2A_1$	$A_3 + A_2 + 2A_1$	$D_4 + A_2$	$5A_1$	A_2
$8A_1$	$D_4 + A_3$	$6A_1$	$A_4 + A_1$	$2A_1$
$A_6 + A_1$	$A_3 + 4A_1$	$A_2 + 4A_1$	$D_4 + A_1$	A_1

[13] Extracted from Dynkin E B 1952 *Mat. Sbornik* (NS) **30** (**72**) 349

Open questions

- Obtain generating functions for $\xi(\mathfrak{s})$
- Verify that $\deg(\mathcal{A}) = d_0$ for classical and exceptional algebras
- Determine commutant decomposition w.r.t BR

$$\mathfrak{s} \subset \mathfrak{s}' : \mathrm{ad}(\mathfrak{s}') \downarrow \mathrm{ad}(\mathfrak{s}) \oplus \Gamma_{\mathrm{ch}}$$

- Characterize elements related to characteristic representation Γ_{ch}
- What elements can be identified with trace operators?
- Systematization for the case of S-subalgebras
- Construction of new systems with exceptional SGA[14]
- Dynamical/geometrical meaning for elements in $C_{S(\mathfrak{s})}(\mathcal{H}_a)$

[14] Joseph A 1974 *Comm. Math. Phys* **36** 325; Sokolov V V, Turbiner A V 2015 *J. Phys. A. Math. Theor.* **48**

Děkuji za pozornost

Thank you for your attention