Nonexistence of local Hamiltonian structures and complete characterization of local conservation laws for generalized Infeld–Rowlands equation

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Introduction

Let $u_{ij} = \partial^{i+j} u / \partial x^i \partial y^j$, $i, j = 0, 1, 2, \ldots$, with $u_{00} \equiv u$.

A smooth function of x, y, t and finitely many u_{ij} is called *local*.

Consider an evolutionary partial differential equation of the form

$$u_t = F \tag{1}$$

in one unknown function $u = u(t, x, y)$ of three independent variables x, y, t, where F is a local function.

Define the operators of total derivatives adapted to equation (1)

$$\begin{gathered} D_x = \frac{\partial}{\partial x} + \sum_{i,j=0}^{\infty} u_{i+1,j} \frac{\partial}{\partial u_{ij}}, \quad D_y = \frac{\partial}{\partial y} + \sum_{i,j=0}^{\infty} u_{i,j+1} \frac{\partial}{\partial u_{ij}}, \\ D_t = \frac{\partial}{\partial t} + \sum_{i,j=0}^{\infty} D_x^i D_y^j(F) \frac{\partial}{\partial u_{ij}}. \end{gathered} \tag{2}$$

Conservation laws I

A *local conservation law* for (1) is an identity of the form

$$D_t(\rho) + D_x(\sigma) + D_y(\zeta) = 0 \tag{3}$$

that holds modulo (1) and its differential consequences; here ρ, σ, ζ are local functions, not all of which are zero. Then ρ is called the *density* of the conservation law under study, and σ and ζ are known as x- and y-*flux components*.

Let $\delta/\delta u$ denote the operator of variational derivative on local functions

$$\delta/\delta u = \sum_{i,j=0}^{\infty} (-1)^{i+j} D_x^i D_y^j \circ \partial/\partial u_{ij}.$$

Note that for any local function g the expression $\delta g/\delta u$ contains only finitely many terms, so there are no convergence issues.

For a local conservation law (3) its *characteristic* is defined as $\delta\rho/\delta u$ (in our setting this definition is readily seen to be equivalent to the more standard one).

Conservation laws II

Consider two local conservation laws for (1)

$$D_t(\rho) + D_x(\sigma) + D_y(\zeta) = 0 \quad \text{and} \quad D_t(\tilde{\rho}) + D_x(\tilde{\sigma}) + D_y(\tilde{\zeta}) = 0$$

A linear combination thereof

$$D_t(c_1\rho + c_2\tilde{\rho}) + D_x(c_1\sigma + c_2\tilde{\sigma}) + D_y(c_1\zeta + c_2\tilde{\zeta}) = 0$$

where c_1 and c_2 are constants, obviously is again a local conservation law for (1), i.e., local conservation laws for (1) form a vector space.

A local conservation law for (1) is *trivial*, if there exist local functions α, β, γ such that

$$\rho = D_x(\alpha) - D_y(\beta), \quad \sigma = D_y(\gamma) - D_t(\alpha), \quad \zeta = D_t(\beta) - D_x(\gamma)$$

Two local conservation laws for (1) are *equivalent* if their difference is a trivial local conservation law. In what follows we shall tacitly consider local conservation laws modulo trivial ones.

A complete description of all nontrivial local conservation laws of all orders for a given PDE is a difficult task and was done successfully only for a very small number of PDEs, especially in the case of three or more independent variables.

Characteristics of symmetries and cosymmetries

For any local function h define its linearization, also known as the formal Frechet derivative, as

$$D_h = \sum_{i,j=0}^{\infty} \partial h / \partial u_{ij} D_x^i D_y^j$$

For example for $K = -D_x(u_{xxx} + a u_y + f(u, u_x))$ we have

$$D_K = -D_x^4 - a D_y D_x - D_x \circ (f_u + f_{u_x} D_x),$$

here $\circ$ denotes composition of operators in total derivatives.

By definition G is a *characteristic of local generalized symmetry* for (1) if G is a local function that satisfies

$$D_t(G) - D_F(G) = 0 \tag{4}$$

and γ is a *local cosymmetry* for (1) if it is a local function that satisfies

$$D_t(\gamma) + D_F^*(\gamma) = 0, \tag{5}$$

where D_F^* is a formal adjoint of D_F, cf. next slide.

Formal series and their adjoints

Consider a formal seriers of the form

$$L = \sum_{i=-\infty}^{k} \sum_{j=-\infty}^{l} h_{ij} D_x^i D_y^j,$$

where h_{ij} are local functions.
Assuming that $h_{kl} \neq 0$ introduce the obvious notation $k = \deg_x L$ and $l = \deg_y L$, with the standard convention that $\deg_x 0 = \deg_y 0 = -\infty$.

For the above L we define

$$D_t(L) = \sum_{i=-\infty}^{k} \sum_{j=-\infty}^{l} D_t(h_{ij}) D_x^i D_y^j,$$

while the formal adjoint L^* for the above L is defined as

$$L^* = \sum_{i=-\infty}^{k} \sum_{j=-\infty}^{l} (-D_x)^i (-D_y)^j \circ h_{ij},$$

where $\circ$ is defined below.

Multiplication of formal series

The usual associative multiplication of differential operators can be extended to formal series while preserving associativity.

The multiplication in question is extended in an obvious manner by linearity from its definition on monomials, and in fact it is easily seen that it suffices to define a very special case of multiplication in question:

$$D_x^k D_y^l \circ h = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ijkl} D_x^i D_y^j(h) D_x^{k-i} D_y^{l-j},$$

where h is a local function and

$$c_{ijkl} = \frac{k(k-1)\cdots(k-i+1)}{i!} \frac{l(l-1)\cdots(l-j+1)}{j!}.$$

It is also easily seen that the commutator $[L, M] = L \circ M - M \circ L$ turns the vector space of formal series of the general form

$$\sum_{i=-\infty}^{k} \sum_{j=-\infty}^{l} h_{ij} D_x^i D_y^j,$$

with local coefficients h_{ij} into a Lie algebra.

Noether operators

An operator of the form

$$N = \sum_{i=-\infty}^{r} \sum_{j=-\infty}^{s} h_{ij} D_x^i D_y^j \qquad (6)$$

where h_{ij} are local functions, is called a *Noether operator* for (1), resp. a *inverse Noether operator* for (1), if

$$D_t(N) - D_F \circ N - N \circ D_F^* = 0,$$

resp. if

$$D_t(N) + D_F \circ N + N \circ D_F = 0.$$

We shall say that a formal series of the form (6) is a *local operator* if $b_{ij} = 0$ for $i < 0$ or $j < 0$.

Hamiltonian and symplectic structures

An operator of the form $P = \sum_{i=-\infty}^{r} \sum_{j=-\infty}^{s} h_{ij} D_x^i D_y^j$, where h_{ij} are local functions, defines a *Hamiltonian structure*, or is said to be a *Hamiltonian operator*, or an *implectic operator*, if it is formally anti-self-adjoint ($P^* = -P$) and the associated bracket on functionals

$$\{\mathscr{H}, \mathscr{K}\} = \int dxdy\, \delta\mathscr{H} P \delta\mathscr{K}$$

satisfies the Jacobi identity. Here $\mathscr{H} = \int H dxdy$, $\mathscr{K} = \int K dxdy$ where H, K are local functions, and δ is the operator of variational derivative: $\delta\mathscr{H} = \sum_{i,j=0}^{\infty} (-D_x)^i (-D_y)^j \partial H / \partial u_{ij}$.

Equation $u_t = F$ is *Hamiltonian* w.r.t. HO P if $F = P\delta\mathscr{H}$ for some $\mathscr{H}$.

An operator of the form $J = \sum_{i=-\infty}^{r'} \sum_{j=-\infty}^{s'} b_{ij} D_x^i D_y^j$, where b_{ij} are local functions, defines a *symplectic structure*, or is said to be a *symplectic operator*, if it is formally anti-self-adjoint ($J^* = -J$) and its formal inverse P (s.t. $P \circ J = 1$) is a HO.

Noether and inverse Noether vs. symplectic and Hamiltonian operators

If N is a local Noether operator for (1), then for any local cosymmetry γ of (1) we have that $N(\gamma)$ is characteristic of a local generalized symmetry for (1).

Likewise, if J is a local inverse Noether operator for (1), then for any characteristic G of a local generalized symmetry for (1) the quantity $J(G)$ is a local cosymmetry for (1).

While any symplectic operator for (1) that can be represented as a formal series like (6) is automatically is an inverse Noether operator, the converse, generally speaking, is not true.

Likewise, while any Hamiltonian operator for (1) that can be represented as a formal series like (6) automatically is a Noether operator for (1), the converse in general does not hold.

Generalized Infeld–Rowlands equation

Consider the following equation from the class (1)

$$u_t = -D_x(u_{xxx} + au_y + f(u, u_x)) \tag{7}$$

where a is a nonzero constant and f is smooth.

We shall refer to (7) as to the generalized IR equation, as for $a = 1$ and $f = u_x^2$ we recover the original IR equation which has applications for example in the study of the soliton stability of the Ginzburg–Landau equation.

The point symmetries for the IR equation were found by Faucher and Winternitz in their Phys Rev E 1993 paper.

Conservation laws I

Proposition 1

For any smooth $f(u, u_x)$ eqn (7)*, i.e.,* $u_t = -D_x(u_{xxx} + au_y + f(u, u_x))$*, has infinitely many nontrivial conservation laws of the form*

$$D_t(Mu) + D_x((u_{xxx} + au_y + f)M) = 0 \tag{8}$$

where M is an arbitrary smooth function of y.

Proposition 2 (Part I)

Equation (7) *with $a \neq 0$ and nonlinear smooth $f = f(u, u_x)$ further admits nontrivial local conservation laws other than (8) if and only if f is linear in u_x and one of the following holds:*

i) there exist a smooth nonlinear function $g = g(u)$ of u and constants k_0 and k_1 such that $f = g(u)u_x + k_1 u + k_0$;

ii) there exist a smooth nonlinear function $h = h(u)$ of u and constants c_0 and c_1 such that $c_1 \neq 0$ and $f = (c_1 \partial h(u)/\partial u + c_0)u_x + h(u)$

Conservation laws II

Proposition 2 (Part II)

The additional nontrivial local conservation laws in both cases i) and ii) have the form

$$D_t(\zeta u) + D_x\left(-(u_{xx} - K_1 u_x + q)\zeta_x + (u_{xxx} + a u_y + f - K_2)\zeta\right) + \\ + D_y(-a u \zeta_x) = 0, \tag{9}$$

where for the case i) we have that $K_1 = 0$, $K_2 = -k_0$, $q = q(u)$ *is a smooth function of* u *such that* $\partial q(u)/\partial u = g(u)$, *and*

$$\zeta = t(a\partial L/\partial y - k_1 L) + xL, \tag{10}$$

where L *is an arbitrary smooth function of* y,
while for the case ii) we have $q(u) = c_1 h(u) + (c_0 + 1/c_1^2)u$, $K_1 = 1/c_1$, $K_2 = 0$, *and*

$$\zeta = \exp(x/c_1 + t(c_0/c_1^2 + 1/c_1^4))F(at + c_1 y), \tag{11}$$

where F *is an arbitrary smooth function of its argument.*

Conservation laws and cosymmetries

Propositions 1 and 2 together give a complete description of nontrivial local conservation laws for (7), i.e., $u_t = -D_x(u_{xxx} + au_y + f(u, u_x))$, with $a \neq 0$ and nonlinear smooth f: in particular, if f is nonlinear, smooth and does *not* satisfy the conditions of Proposition 2, the only nontrivial local conservation laws admitted by (7) are those from Proposition 1.

Corollary 1

The only nontrivial local conservation laws admitted by the original Infeld–Rowlands equation, obtained from (7) *upon setting* $a = 1$ *and* $f = u_x^2$, *are those from Proposition 1.*

Proposition 3

All local cosymmetries of (7) *with* $a \neq 0$ *can depend at most on* x, y *and* t.

Proposition 4

The only local cosymmetries admitted by (7) *with* $a \neq 0$ *and nonlinear smooth* f *are characteristics of local conservation laws listed in Propositions 1 and 2.*

Proof of Proposition 3

Let γ be a local cosymmetry for (7); $k := \deg_x D_\gamma$ & $l := \deg_y D_\gamma$.

Clearly, γ depends at most on x, y, t iff $D_\gamma = 0$.

Seeking a contradiction, assume that $D_\gamma \neq 0$. Then obviously $k \geqslant 0$, and upon repeated use of (2) we find that taking the partial derivative of the determining equation for cosymmetries $D_t(\gamma) + D_F^*(\gamma) = 0$, i.e., (5) with F being the r.h.s. of (7) w.r.t. $u_{k+4,l}$ yields

$$2\partial\gamma/\partial u_{kl} = 0,$$

Taking this into account and acting by $\partial/\partial u_{k+4,l-1}$ on (5) now yields

$$2\partial\gamma/\partial u_{k,l-1} = 0,$$

and continuing in the same fashion we find that for all $j = 0, 1, \ldots, l$

$$\partial\gamma/\partial u_{kj} = 0,$$

so in fact $\deg_x D_\gamma$ is at most $k - 1$, which contradicts our initial assumption $\deg_x D_\gamma = k$.

The only way to resolve this contradiction is to assume that $D_\gamma = 0$, so γ can depend at most on x, y, t $\square$

Remarks on proving other results

With Proposition 3 proved, we know that the cosymmetries of (7) can depend at most on x, y, t. With this in mind, a careful analysis of the determining equation for cosymmetries, which boils down to

$$\frac{\partial\gamma}{\partial t} - \frac{\partial^4\gamma}{\partial x^4} - a\frac{\partial^2\gamma}{\partial x\partial y} - f_{u_x}\frac{\partial^2\gamma}{\partial x^2} + f_u\frac{\partial\gamma}{\partial x} - f_{u_x u_x}u_{xx}\frac{\partial\gamma}{\partial x} - f_{uu_x}u_x\frac{\partial\gamma}{\partial x} = 0,$$

proves Proposition 4, and finding conservation laws with given characteristics yields Propositions 1 and 2.

Note that while in our setting a characteristic of any local conservation law for (1) (and hence in particular for (7)) is necessarily a cosymmetry, the converse is in generally not true.

Noether, Hamiltonian and symplectic operators

Proposition 5

Equation (7), i.e., $u_t = -D_x(u_{xxx} + au_y + f(u, u_x))$, admits no nontrivial Noether and inverse Noether operators that can be represented as formal series of the form

$$\sum_{i=-\infty}^{r} \sum_{j=-\infty}^{s} b_{ij} D_x^i D_y^j \tag{12}$$

where r and s are any integers and b_{ij} are local functions.

Note that any symplectic operator for (7) is necessarily a inverse Noether operator and a Hamiltonian operator for (7) is necessarily a Noether operator.

Corollary 2

Equation (7) admits no nontrivial Hamiltonian and symplectic operators that can be represented as formal series of the form (12) with local coefficients.

Proof of nonexistence of Noether operators

Let P of the form

$$P = \sum_{i=-\infty}^{r} \sum_{j=-\infty}^{s} p_{ij} D_x^i D_y^j \tag{13}$$

where p_{ij} are local functions, be a Noether operator for (7), i.e., $\tilde{P} = 0$, where

$$\tilde{P} = D_t(P) - D_F^* \circ P - P \circ D_F,$$

where F now denotes the right-hand side of (7).

We readily see that the leading term of $\tilde{P}$ is $-2p_{rs} D_x^{r+4} D_y^s$ and since we require that $\tilde{P} = 0$, this leading term must vanish, i.e., $p_{rs} = 0$, and moreover $p_{rj} = 0$ for all $j = s-1, s-2, \dots$.

Continuing by replacing r by $r-1$ in the above considerations and so on establishes that $p_{ij} = 0$ for all i and j, so $P = 0$, i.e.(7) admits no nontrivial Noether operators of the form (13), which completes the part of the proof concerning the Noether operators.

Proof of nonexistence of inverse Noether operators

Likewise, assume that

$$B = \sum_{i=-\infty}^{k} \sum_{j=-\infty}^{l} b_{ij} D_x^i D_y^j,$$

where b_{ij} are local functions, is a inverse Noether operator for (7), i.e., it satisfies $\tilde{B} = 0$ where

$$\tilde{B} = D_t(B) + D_F^* \circ B + B \circ D_F;$$

here F again denotes the right-hand side of (7).

In a similar fashion as for the Noether operator case we see that $b_{kj} = 0$ for all $j = l, l-1, \dots$ and then that $b_{ij} = 0$ for all i and j, so $B = 0$ and the result follows.

$\square$

Conclusions

- The generalized Infeld–Rowlands equation

$$u_t + D_x(u_{xxx} + au_y + f(u, u_x)) = 0$$

where a is a nonzero constant and f is smooth, admits no Noether or inverse Noether operators that can be represented in the form

$$\sum_{i=-\infty}^{k} \sum_{j=-\infty}^{l} b_{ij} D_x^i D_y^j$$

with local coefficients b_{ij} and hence has no Hamiltonian or symplectic operators of the same kind.

- The method of proof of this result can be applied to many other PDEs
- Complete classification for all nontrivial local conservation laws for the generalized Infeld–Rowlands equation is obtained.

Further details can be found in

J. Vašíček: *Conservation laws and nonexistence of local Hamiltonian structures for generalized Infeld–Rowlands equation*. Rep. Math. Phys. 93 (2024), 287-300.

Thank you for your attention