On the Ginzburg-Landau Energy of Corners

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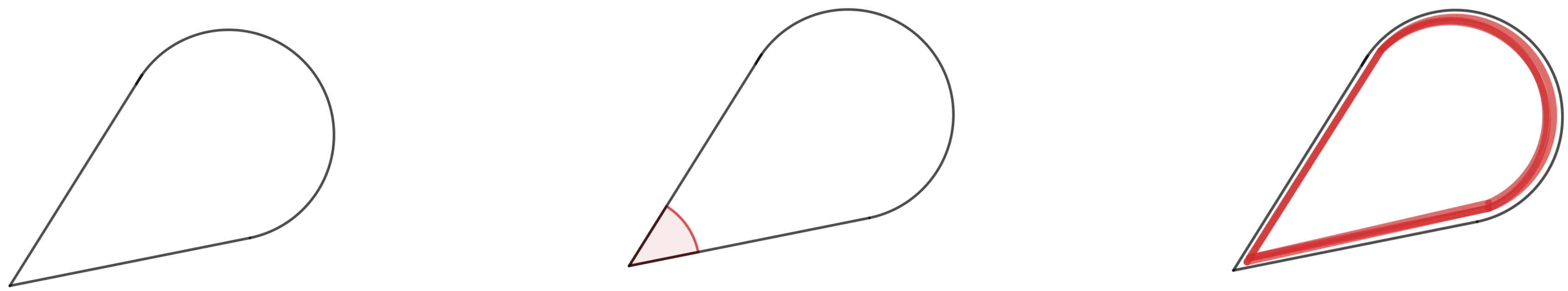
Onset of superconductivity in decreasing fields

A superconductor subject to an external magnetic field (of intensity H) undergoes phase transitions. The presence of corners in the sample influences the phase transitions.

[Fomin et al., Europhys. Lett. 42, (1998)] and [Moshchalkov et al., Nature, 373 (1995) 319]



Breakdown of superconductivity for $H > H_C$; onset of superconductivity at the corner for $H_C^{\text{corner}} < H < H_C$; onset of superconductivity along the surface for $H < H_C^{\text{corner}}$.

The Ginzburg—Landau order parameter

The phenomenological Ginzburg—Landau model detects the state of superconductivity via an order parameter, $\psi : \Omega \to \mathbb{C}$.

If $\psi(x) \neq 0,$ the sample is in a superconducting state locally at x and $|\psi(x)|^2$ is a measure of the density of superconductivity.

If $\psi(x) = 0,$ the sample is in a normal state (not superconducting) locally at x. In this case $|\psi(x)|^2 = 0.$

Effective model in a sector

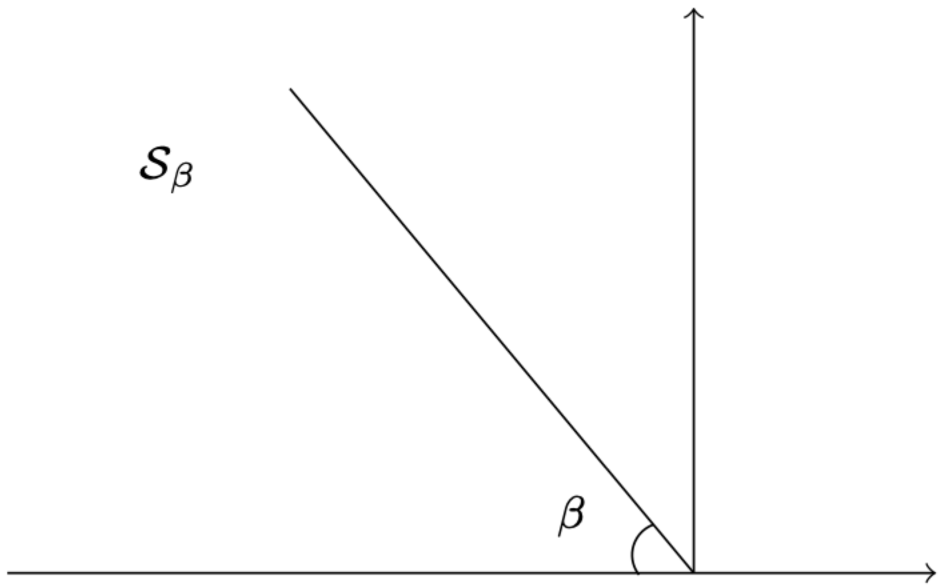
In 2007, Fournais and Bonnaillie-Noël introduced the following effective energy in the sector S_β :

$$\mathscr{E}_\mu(\psi) = \int_{S_\beta} \left(|(\nabla + i\mathbf{F})\psi|^2 - \mu|\psi|^2 + \frac{\mu}{2}|\psi|^4 \right) dx$$

where

– $\mathbf{F}(x_1, x_2) = \frac{1}{2}(-x_2, x_1)$

– $\mu > 0$ corresponds to the intensity of the applied magnetic field



An eigenvalue problem

Consider
$$\mu(\beta) = \inf_{\substack{\psi \in W^{1,2}_{\mathbf{F}}(S_\beta) \\ \psi \neq 0}} \frac{\int_{S_\beta} |(\nabla + i\mathbf{F})\psi|^2 \, dx}{\int_{S_\beta} |\psi|^2 \, dx},$$

where $W^{1,2}_{\mathbf{F}}(S_\beta) = \{\psi \in L^2(S_\beta) : (\nabla + i\mathbf{F})\psi \in L^2(S_\beta; \mathbb{C}^2)\}$.

In terms of $\mathscr{L}_\beta = -(\nabla + i\mathbf{F})^2$, the magnetic Laplacian on S_β, $\mu(\beta)$ is $\inf \sigma(\mathscr{L}_\beta)$, and $\sigma_{\mathrm{ess}}(\mathscr{L}_\beta) = [\Theta_0, +\infty)$, where $\Theta_0 = \mu(\pi) \approx 0.59$.

[Bonnaillie, Asympt. Anal., Vol. 41, 2005]

Back to the effective model

$$\mathscr{E}_\mu(\psi) = \int_{S_\beta} \left(|(\nabla + i\mathbf{F})\psi|^2 - \mu |\psi|^2 + \frac{\mu}{2} |\psi|^4 \right) dx$$

We have:

– $\mathscr{E}_\mu(0) = 0$

– $\mathscr{E}_\mu(\psi) \geq (\mu(\beta) - \mu) \int_{S_\beta} |\psi|^2 dx + \frac{\mu}{2} \int_{S_\beta} |\psi|^4 dx\,.$

– If ψ^{gs} is a ground state of $\mu(\beta)$, we have for any $t > 0$,

$$\mathscr{E}_\mu(t\psi^{\mathrm{gs}}) \leq t^2(\mu(\beta) - \mu) \int_{S_\beta} |\psi^{\mathrm{gs}}|^2 dx + t^4 \int_{S_\beta} |\psi^{\mathrm{gs}}|^4 dx\,.$$

Consider $E(\mu) = \inf_{\psi \in W^{1,2}_{\mathbf{F}}(S_\beta)} \mathscr{E}_\mu(\psi)$, where

$$\mathscr{E}_\mu(\psi) = \int_{S_\beta} \left(|(\nabla + i\mathbf{F})\psi|^2 - \mu|\psi|^2 + \frac{\mu}{2}|\psi|^4 \right) dx\,.$$

Proposition 1. (Fournais—Bonnaillie-Noël, 2007)

1. If $\mu \leq \mu(\beta)$, then $E(\mu) = 0$ and $\psi = 0$ is a minimizer.
2. If $\mu > \mu(\beta)$ and $\mu(\beta) < \Theta_0$, then $E(\mu) < 0$.
3. If $\mu(\beta) < \mu < \Theta_0$, then $-\infty < E(\mu) < 0$ and there is a minizimizer $\psi_* \neq 0$.

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Question 1. When $\mu(\beta) < \Theta_0$?

Question 2. Does $-\infty < E(\mu) < 0$ for $\mu = \Theta_0$?

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3. If $\mu(\beta) < \mu < \Theta_0$, then $-\infty < E(\mu) < 0$ and there is a minizimizer $\psi_* \neq 0$. Moreover, $E(\mu) = -\infty$ for $\mu > \Theta_0$.

Question 1. When $\mu(\beta) < \Theta_0$?

Question 2. Does $-\infty < E(\mu) < 0$ for $\mu = \Theta_0$?

Question 3. Effective energy for $\mu > \Theta_0$?

On Question 1:

Conjecture. (Bonnaillie 2005). $\mu(\beta) < \Theta_0$ for $0 < \beta < \pi$.

Progress:

i) [Jadallah 2001, Pan 2002, Bonnaillie 2005] For $\beta \leq \dfrac{\pi}{2} + \epsilon$

and $\epsilon \approx 0.011$ small.

ii) [Exner—Lotoreichik—Pérez-Obiol 2018] For $\beta \leq 0.595\pi$.

iii) [Bonnaillie-Noël—Fournais—K.—Raymond, 2024] For

$\pi - \delta \leq \beta < \pi$ and δ small.

On Question 2:

Theorem 1. (Correggi-Giacomelli-K. 2023)

Suppose that $\mu(\beta) < \Theta_0$.

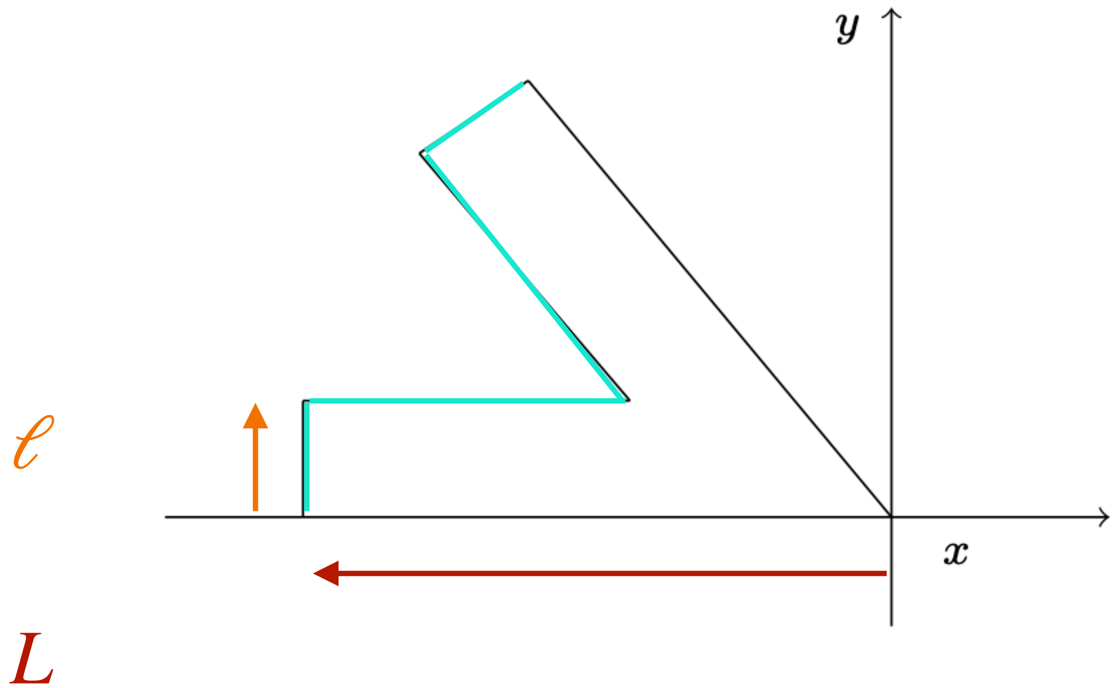
1. $\lim_{\mu \searrow \Theta_0} E(\mu) = E(\Theta_0)$.

2. $-\infty < E(\Theta_0) < 0$ and there is $\psi_* \neq 0$ such that

   $E(\Theta_0) = \mathscr{E}_{\Theta_0}(\psi_*)$.

Remark. $E(\Theta_0) < 0$ by Proposition 1.

To prove Theorem 1, we work in the corner region $\Gamma_\beta(L, \ell)$, with the "inner boundary" highlighted.



Outline of Proof and an intermediate energy

i) We show first that $E(\Theta_0) = \lim_{\mu \searrow \Theta_0} E(\mu) = E^D_{\text{corner}}(\Theta_0)$.

ii) $E^D_{\text{corner}}(\Theta_0)$ is defined as $E^D_{\text{corner}}(\Theta_0) = \inf_{(L,\ell)\in\mathscr{A}} E_{L,\ell}(\Theta_0)$, where

$$\mathscr{A} = \{(L,\ell)\colon 1 \le \ell < L\tan(\beta/2) \le \ell^2\},$$

$$E_{L,\ell}(\mu) = \inf_{\psi\in\mathscr{D}_0} \int_{\Gamma_\beta(L,\ell)} \left(|(\nabla + i\mathbf{F})\psi|^2 - \mu|\psi|^2 + \frac{\mu}{2}|\psi|^4\right)dx$$

iii) $\Gamma_\beta(L,\ell)$ is a layer of thickness ℓ of a finite corner and $\psi \in \mathscr{D}_0$ means that $\psi = 0$ on the inner boundary of $\Gamma_\beta(L,\ell)$.

iv) A minimizer $\psi_{L,\ell}$ on $\Gamma_\beta(L,\ell)$ converges to ψ_* as $(L,\ell) \to \infty$.

On Question 3.

Suppose that $\Theta_0 < \mu < 1$. Choose (f, α) such that

$$\mathcal{F}_\alpha(f) = \inf_{(g,\xi)} \mathcal{F}_\xi(g), \quad \mathcal{F}_\xi(g) := \int_0^\ell \Big(|g'(t)|^2 + (t+\xi)^2 |g(t)|^2 - \mu |g(t)|^2 + \frac{\mu}{2} |g(t)|^4 \Big) dt\,.$$

Let $\mathcal{D} = \{\psi : \psi = \psi_0 \text{ on the inner boundary of } \Gamma_\beta(L, \ell)\}$, where $\psi_0(s,t) = f(t) e^{i\alpha s - \frac{i}{2} st}$, and (s,t) are coordinates defined by the tangential and normal distances to the boundary.

Let $\mathcal{G}_\mu(\psi) = \displaystyle\int_{\Gamma_\beta(L,\ell)} \Big(|(\nabla + i\mathbf{F})\psi|^2 - \mu |\psi|^2 + \frac{\mu}{2} |\psi|^4 \Big) dx.$

Let $G_{L,\ell}(\mu) = \inf_{\psi \in \mathscr{D}} \mathscr{G}_\mu(\psi)$.

Theorem 2. (Correggi—Giacomelli 2021)

The following limit exists (and depends on the angle β)

$$G(\mu) = \lim_{\substack{(L,\ell) \to \infty \\ (L,\ell) \in \mathscr{A}}} \Big(G_{L,\ell}(\mu) - 2L\mathscr{F}_\alpha(f) \Big).$$

Moreover, as $\beta \to \pi$, we have $G(\mu) = -(\pi - \beta)C(\mu) + o(1)$,

with $C(\mu) > 0$ and $C(\Theta_0) = 0$.

Conjecture. For $\beta < \pi$, we have $G(\mu) = -(\pi - \beta)C(\mu)$.

Theorem 3. (Correggi—Giacomelli—K. 2023)

We have $E(\Theta_0) = \lim_{\mu \nearrow \Theta_0} G(\mu)\,.$

Remarks.

i) The proof consists in showing that $\lim_{\mu \nearrow \Theta_0} G(\mu) = E^D_{\mathrm{corner}}(\Theta_0)$, where $E^D_{\mathrm{corner}}(\Theta_0)$ is the intermediate energy used in the proof of Theorem 1.

ii) Since $E(\Theta_0) < 0$, the Correggi-Giacomelli conjecture is false, at least for μ close to Θ_0.

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ii) Since $E(\Theta_0) < 0$, the Correggi-Giacomelli conjecture is false, at least for μ close to Θ_0.

iii) Will it hold for μ **not** close to Θ_0?

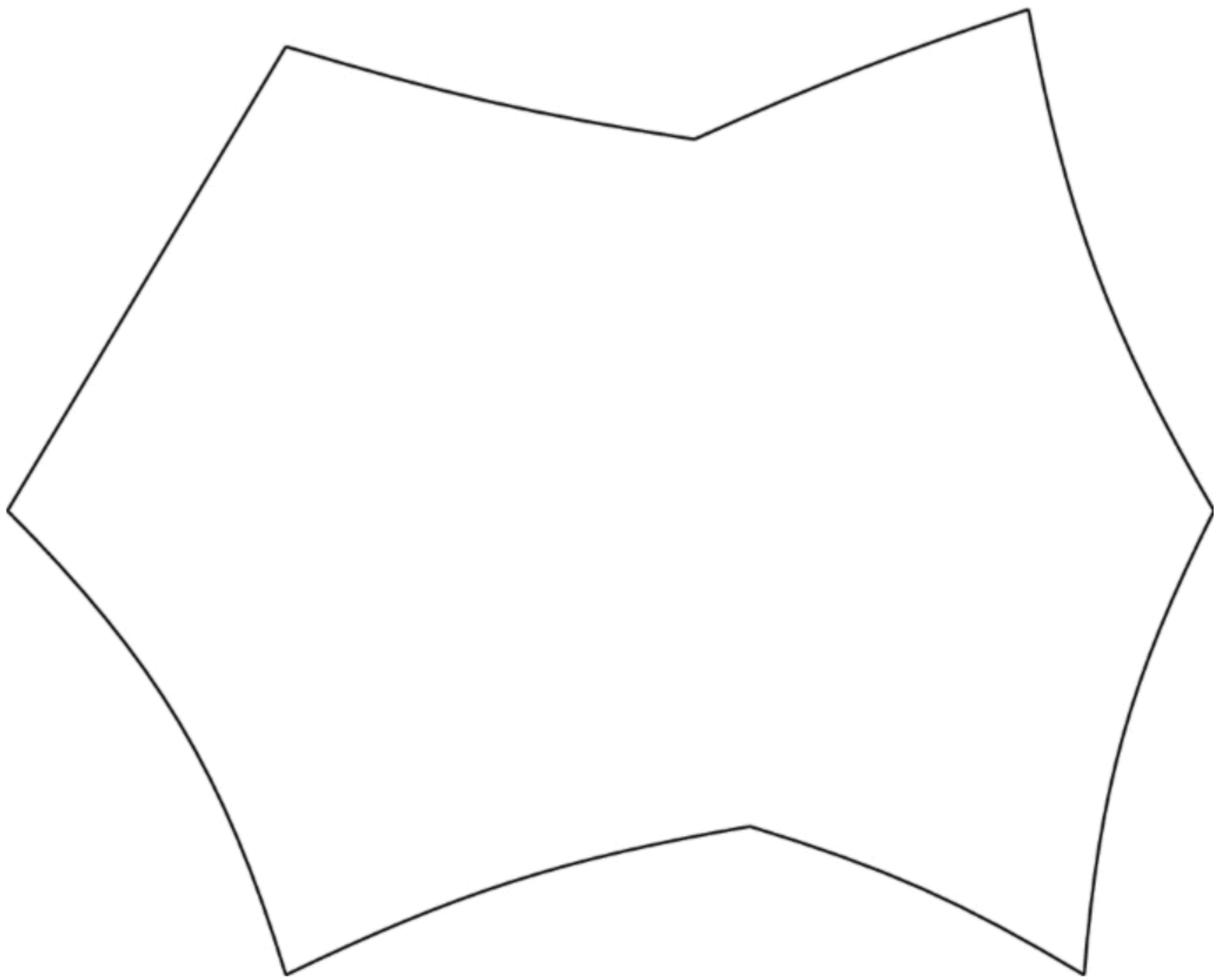
Geometric intuition behind the Correggi-Giacomelli conjecture

(Correggi—Rougerie) For a smooth simply connected domain, the total curvature on the boundary is 2π, and the contribution of surface superconductivity is $-C(\mu)\int_{\partial\Omega} k(s)ds = -2\pi C(\mu)$.

(Correggi—Giacomelli) For a domain with N corners, the contribution is

$$-C(\mu)\int_{\partial\Omega} k(s)ds + \sum_{i=1}^{N} G_{\beta_j}(\mu),$$

and by the Gauss-Bonnet Theorem, $\displaystyle\int_{\partial\Omega} k(s)ds + \sum_{j=1}^{N}(\pi - \beta_j) = 2\pi$.

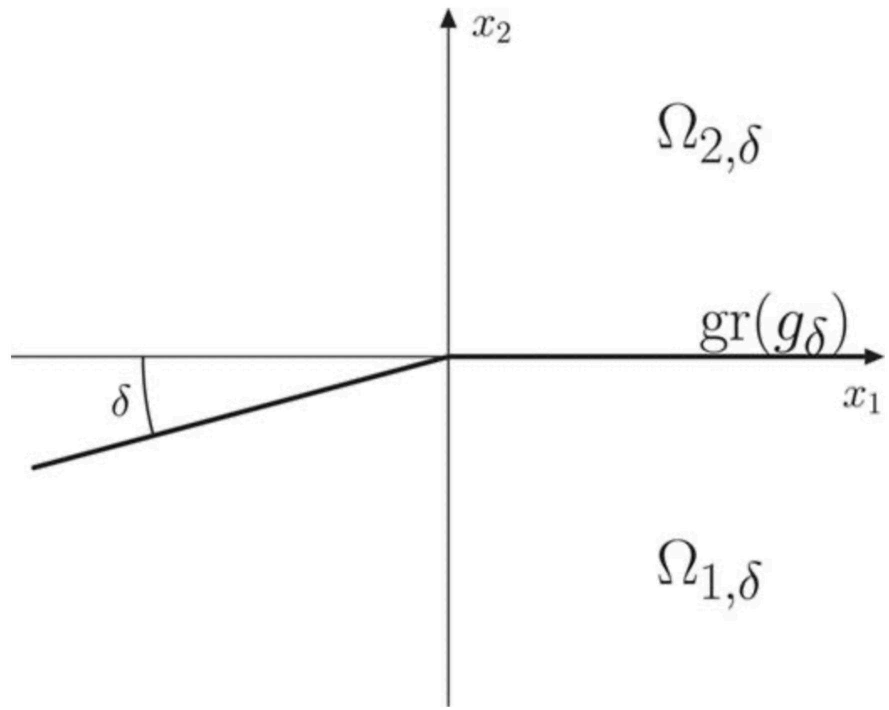


A possible extension: Magnetic steps

A similar problem occurs when the magnetic field B is discontinuous along a broken line:

$B = 1$ on $\Omega_{1,\delta}$

$B = a$ on $\Omega_{2,\delta}$



[Miranda, *J. Math. Phys.* 65, 072101 (2024)]