

# Analytic and Algebraic Methods in Physics XXI

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## Some recent applications of the Generalized Uncertainty Principle



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Plan de Recuperación,  
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- 1 Introduction & Motivation
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Due to the importance of a **unified theory of quantum mechanics + gravity**, and the existence of a minimal length (**Planck scale**), we consider

- A **modified Schrödinger equation from a generalized uncertainty principle (GUP)**, with a quantum mechanically corrected gravitational interaction. The resulting equation cannot be solved by common exact approaches  $\Rightarrow$  a **Bethe-ansatz** approach.
  - **Dirac equation with a generalized gravitational interaction** which includes post-Newtonian (relativistic) and quantum corrections to the classical potential.  
The **Bethe-ansatz** approach is also proposed to attack this challenging problem.
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Obviously, those are not well studied problems in mathematical physics, where the differential equations that appear, whether relativistic or non-relativistic, are of the first or second order.

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- (d) **Ground and first excited states** are explicitly determined.

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$$[x_G, p_G] = i\hbar \left( 1 + \frac{\beta}{2\hbar^2} p^2 \right), \quad 0 \leq \beta \leq 1,$$

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$$\left( -\frac{d^2}{dx^2} + V_e(x) \right) \psi_n^G(x) = 0,$$

with an effective potential

$$V_e(x) := \frac{2m}{\hbar^2} (V(x) - E_n^G) + \left( \frac{2m}{\hbar^2} \right)^2 \beta (V(x) - E_n^G)^2,$$

with  $E_n$  the  $\beta = 0$  eigenvalues of the energy and  $E_n^G$  the eigenenergies for  $\beta \neq 0$ .

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$$\frac{2m}{\hbar^2} V(x) = \frac{\alpha_1}{x} + \frac{\alpha_2}{x^2} + \frac{\alpha_3}{x^3} + \frac{\alpha_4^2}{x^4}, \quad \alpha_4 > 0, \alpha_1 < 0,$$

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$$V_e(x) = \gamma_0 + \frac{\gamma_1}{x} + \frac{\gamma_2}{x^2} + \frac{\gamma_3}{x^3} + \frac{\gamma_4}{x^4} + \frac{\gamma_5}{x^5} + \frac{\gamma_6}{x^6} + \frac{\gamma_7}{x^7} + \frac{\gamma_8}{x^8},$$

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in which the  $\gamma_i$ ,  $i = 0, 1, \dots, 8$  are as follows

$$\begin{aligned} \gamma_0 &= \beta \epsilon_n^2 - \epsilon_n^G, & \gamma_1 &= \alpha_1(1 - 2\beta \epsilon_n), \\ \gamma_2 &= \alpha_2 + \beta(\alpha_1^2 - 2\alpha_2 \epsilon_n), & \gamma_3 &= \alpha_3 + 2\beta(\alpha_1 \alpha_2 - \alpha_3 \epsilon_n), \\ \gamma_4 &= \alpha_4^2 + \beta(2\alpha_1 \alpha_3 + \alpha_2^2 - 2\alpha_4^2 \epsilon_n), & \gamma_5 &= 2\beta(\alpha_1 \alpha_4^2 + \alpha_2 \alpha_3), \\ \gamma_6 &= \beta(2\alpha_2 \alpha_4^2 + \alpha_3^2), & \gamma_7 &= 2\beta \alpha_3 \alpha_4^2, & \gamma_8 &= \beta \alpha_4^4, \end{aligned}$$

where we have introduced the notation

$$\epsilon_n = \frac{2m}{\hbar^2} E_n, \quad \epsilon_n^G = \frac{2m}{\hbar^2} E_n^G.$$

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Let us start the analysis of the ordinary case ( $\beta = 0$ ), as the solutions of the model with GUP depend on them.

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**The ordinary case  $\beta = 0$  (not GUP)** has already been studied via the ansatz method. Here, we will obtain the general solutions of the model using the Lie-algebraic method within the framework of quasi-exact solvability.

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$$\left( -\frac{d^2}{dx^2} + \frac{\alpha_1}{x} + \frac{\alpha_2}{x^2} + \frac{\alpha_3}{x^3} + \frac{\alpha_4^2}{x^4} - \epsilon_n \right) \psi_n(x) = 0.$$

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Due to the asymptotic behavior of the wave function  $\psi_n(x)$ , we use the ansatz

$$\psi_n(x) = x^\delta \exp \left[ - \left( x \sqrt{-\epsilon_n} + \frac{\alpha_4}{x} \right) \right] \varphi_n(x), \quad \delta = 1 + \frac{\alpha_3}{2\alpha_4} > 0,$$



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which transforms Schrödinger equation (9) into the form

$$\left\{ -x^2 \frac{d^2}{dx^2} - 2 \left( \alpha_4 + \delta x - \sqrt{-\epsilon_n} x^2 \right) \frac{d}{dx} + (\lambda_1 x + \lambda_2) \right\} \varphi_n(x) = 0,$$

where

$$\lambda_1 = \alpha_1 + \left( 2 + \frac{\alpha_3}{\alpha_4} \right) \sqrt{-\epsilon_n}, \quad \lambda_2 = \alpha_2 - \frac{\alpha_3^2}{4\alpha_4^2} - \frac{\alpha_3}{2\alpha_4} + 2\alpha_4 \sqrt{-\epsilon_n}.$$

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$$\lambda_1 = -2n\sqrt{-\epsilon_n}$$

holds, the Schrödinger equation can then be expressed as a quasi-exactly solvable (QES) differential operator in the Lie-algebraic form

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with

$$H_{ges} = -\mathcal{J}_n^+ \mathcal{J}_n^- + 2\sqrt{-\epsilon_n} \mathcal{J}_n^+ - 2\alpha_4 \mathcal{J}_n^- - (2\delta + n) \mathcal{J}_n^0 - \frac{n^2}{2} - n\delta + \lambda_2.$$

Here,

$$\mathcal{J}_n^+ = x^2 \frac{d}{dx} - nx, \quad \mathcal{J}_n^0 = x \frac{d}{dx} - \frac{n}{2}, \quad \mathcal{J}_n^- = \frac{d}{dx},$$

are the generators of the  $sl(2)$  Lie algebra satisfying the commutation relations

$$[\mathcal{J}_n^+, \mathcal{J}_n^-] = -2\mathcal{J}_n^0, \quad [\mathcal{J}_n^\pm, \mathcal{J}_n^0] = \mp \mathcal{J}_n^\pm.$$

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These operators leave invariant the  $(n+1)$ -dimensional linear space of polynomials

$$\varphi_n(x) = \sum_{k=0}^n c_k x^k,$$

where the coefficients  $c_k$  satisfy the three-term recursion relation ( $c_{-1} = c_{n+1} = 0$ )

$$c_{k+1} = \frac{(\lambda_2 - 2k\delta - k(k-1))c_k - 4\sqrt{-\epsilon_n}c_{k-1}}{2(k+1)\alpha_4}, \quad k = 0, 1, \dots, n,$$

assuming  $c_{-1} = 0$  and  $c_{n+1} = 0$ .

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assuming  $c_{-1} = 0$  and  $c_{n+1} = 0$ . Equivalently, the recursion relation (11) can be rewritten as a tridiagonal matrix equation the nontrivial solutions of which are

$$\begin{vmatrix} \lambda_2 & -2\alpha_4 & & & \\ -2n\sqrt{-\epsilon_n} & (\lambda_2 - 2\delta) & -4\alpha_4 & & \\ & -2(n-1)\sqrt{-\epsilon_n} & \ddots & \ddots & \\ & & \ddots & \ddots & -2n\alpha_4 \\ & & & -2\sqrt{-\epsilon_n} & (n - n^2 + \lambda_2 - 2n\delta) \end{vmatrix} = 0.$$

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On the other hand, from the expressions of  $\lambda_1$  and  $\lambda_2$  the following expression of the energy in closed form can be obtained

$$\epsilon_n = -\frac{\alpha_1^2 \alpha_4^2}{(\alpha_3 + 2(n+1)\alpha_4)^2},$$

provided  $\alpha_1 < 0$ .

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For the sake of clarity, next we will find the explicit solutions of the ground state and the first excited state.

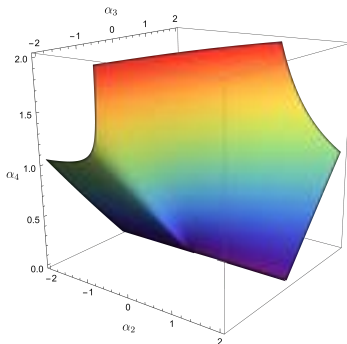
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### Ground state and the associated wave function

$$\epsilon_0 = -\frac{\alpha_1^2 \alpha_4^2}{(\alpha_3 + 2\alpha_4)^2}, \quad \psi_0(x) = c_0 x^{1+\alpha_3/(2\alpha_4)} \exp \left[ - \left( x\sqrt{-\epsilon_0} + \frac{\alpha_4}{x} \right) \right],$$

where the restriction on the parameters of the potential is determined by

$$4\alpha_2 \alpha_4^2 - \alpha_3^2 - 2\alpha_3 \alpha_4 + 8\alpha_4^3 \sqrt{-\epsilon_0} = 0.$$



$$\alpha_1 = -1/10$$

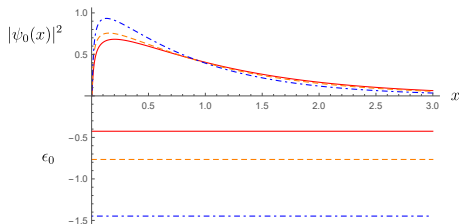
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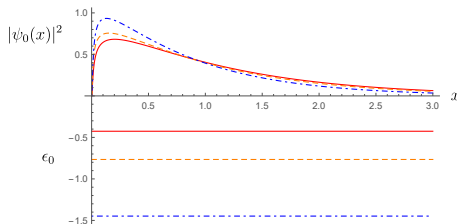
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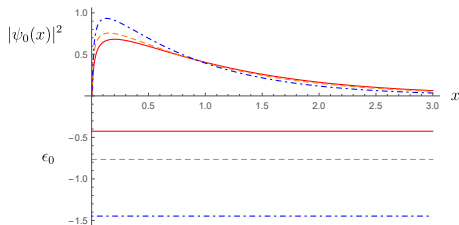


On the right, the form of the potential.

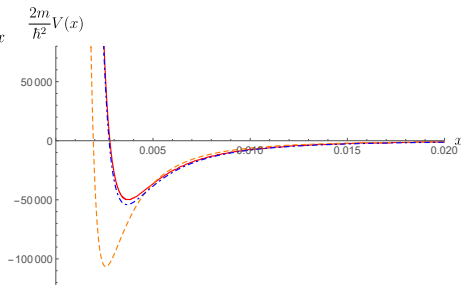
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## 2.– Quantum Correction to the Coulomb Potential in GUP

### First excited state ( $n = 1$ ) and the associated wave function

The energy and the corresponding wave function are given by

$$\epsilon_1 = -\frac{\alpha_1^2 \alpha_4^2}{(\alpha_3 + 4\alpha_4)^2}, \quad \psi_1(x) = (c_0 + c_1 x) x^{1+\alpha_3/(2\alpha_4)} \exp \left[ -\left( x\sqrt{-\epsilon_1} + \frac{\alpha_4}{x} \right) \right].$$



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The restriction on the parameters of the potential is given by

$$64\alpha_1^2\alpha_4^8 + 16\alpha_4^4(\alpha_3 + 4\alpha_4)(\alpha_3^2 + 4\alpha_3\alpha_4 + 8\alpha_4^2 - 4\alpha_2\alpha_4^2)\alpha_1 \\ + (\alpha_3 + 4\alpha_4)^2(\alpha_3^2 + 2\alpha_3\alpha_4 - 4\alpha_2\alpha_4^2)(\alpha_3^2 + 6\alpha_3\alpha_4 + 8\alpha_4^2 - 4\alpha_2\alpha_4^2) = 0.$$

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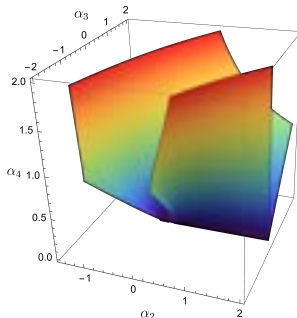
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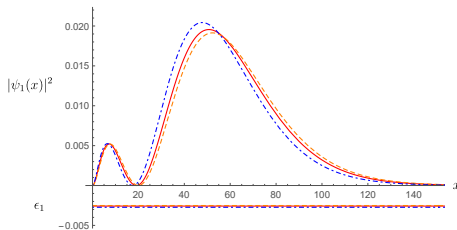
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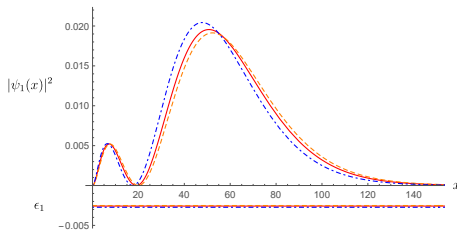
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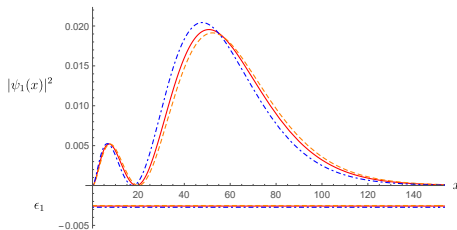


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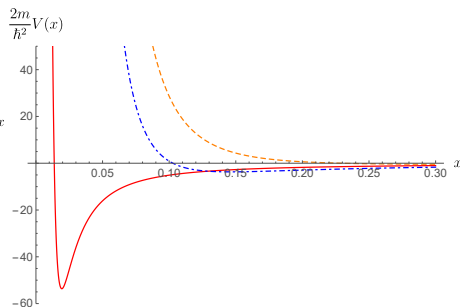
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## 2.– Quantum Correction to the Coulomb Potential in GUP

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**The ordinary case as a double-confluent Heun equation:**



## 2.– Quantum Correction to the Coulomb Potential in GUP

**The ordinary case as a double-confluent Heun equation:** By changing the independent variable  $y = 2\sqrt{-\epsilon_n} x$ , the Schrödinger differential equation

$$\left\{ -x^2 \frac{d^2}{dx^2} - 2(\alpha_4 + \delta x - \sqrt{-\epsilon_n} x^2) \frac{d}{dx} + (\lambda_1 x + \lambda_2) \right\} \varphi_n(x) = 0,$$

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is transformed into the double-confluent Heun equation

$$\left\{ y^2 \frac{d^2}{dy^2} + (-y^2 + \rho y + \eta) \frac{d}{dy} - (\omega y + \lambda_2) \right\} \varphi_n(y) = 0,$$

in which we denote  $\rho = 2 + \frac{\alpha_3}{\alpha_4}$ ,  $\eta = 4\alpha_4\sqrt{-\epsilon_n}$ ,  $\omega = 1 + \frac{\alpha_1}{2\sqrt{-\epsilon_n}} + \frac{\alpha_3}{2\alpha_4}$ .

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Its regular solutions at origin are given by ( $h_0 = 1$ )

$$\varphi := \varphi_n(y; \rho, \eta, \omega, \lambda_2) = \sum_{n=0}^{\infty} h_n x^n,$$

$$h_{n+2} = \frac{(\lambda_2 - n(n + \rho + 1) - \rho) h_{n+1} + (n + \omega) h_n}{(n + 2)\eta} \quad \text{and} \quad h_1 = \frac{\lambda_2}{\eta} h_0,$$

Consequently,  $\varphi$  can admit polynomial solution of degree  $m$  if  $(m + \omega)$  and  $h_{m+1}$  vanish simultaneously. In this way, general solutions to the problem can be obtained in terms of the associated Heun functions.

## 2.– Quantum Correction to the Coulomb Potential in GUP

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Having obtained the solutions of the Coulomb–4 model without GUP in the previous section, let us now return to the modified Schrödinger equation in a formalism with GUP, that is, to the equations

$$\left[ -\frac{d^2}{dx^2} + V_e(x) \right] \psi_n^G(x) = 0, \quad V_e(x) := \frac{2m}{\hbar^2} (V(x) - E_n^G) + \left( \frac{2m}{\hbar^2} \right)^2 \beta (V(x) - E_n)^2,$$

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To ensure **proper asymptotic behavior** of the wave function  $\psi_n^G(x)$ , after inspecting the differential equation we propose the following ansatz

$$\psi_n^G(x) = x^f e^{g(x)} \varphi_n^G(x), \quad g(x) = -a x - \frac{b}{x} - \frac{c}{x^2} - \frac{d}{x^3},$$

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It can be seen that the previous parameters are determined by:

## 2.– Quantum Correction to the Coulomb Potential in GUP

$$a = \sqrt{\gamma_0} = \sqrt{\beta \epsilon_n^2 - \epsilon_n^G} > 0,$$

$$b = -\frac{\gamma_7^2 - 4\gamma_6 \gamma_8}{8(\gamma_8)^{3/2}} = \alpha_2 \sqrt{\beta},$$

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$$d = \frac{\sqrt{\gamma_8}}{3} = \frac{1}{3} \alpha_4^2 \sqrt{\beta},$$

$$f = 2 + \frac{8\gamma_5 \gamma_8^2 - 4\gamma_6 \gamma_7 \gamma_8 + \gamma_7^3}{16(\gamma_8)^{5/2}} = 2 + \alpha_1 \sqrt{\beta} > 0.$$

## 2.– Quantum Correction to the Coulomb Potential in GUP

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Then, the differential equation for  $\varphi_n^G(x)$  simplifies to

$$\left\{ P_4(x) \frac{d^2}{dx^2} + Q_4(x) \frac{d}{dx} + W_3(x) \right\} \varphi_n^G(x) = 0,$$

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where

$$\begin{aligned}
 P_4(x) &= x^4, \\
 Q_4(x) &= 6d + 4cx + 2bx^2 + 2fx^3 - 2ax^4, \\
 W_3(x) &= (b^2 - 6ad + 4cf - 6c - \gamma_4) + (2(bf - b - 2ac) - \gamma_3)x \\
 &\quad + (f(f - 1) - 2ab - \gamma_2)x^2 - (\gamma_1 + 2af)x^3.
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## 2.– Quantum Correction to the Coulomb Potential in GUP

To find the solution  $\varphi_n^G(x)$  we use the general **Bethe ansatz method**, looking for polynomial solutions of the form

$$\varphi_n^G(x) = \begin{cases} 1, & n = 0, \\ \prod_{i=1}^n (x - x_i), & n \in \mathbb{N}, \end{cases}$$

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After some calculations we get the energy relation: for a given  $n$ , the energy  $\epsilon_n^G$  is

$$\epsilon_n^G(\epsilon_n; \alpha_1, \beta) = -\frac{\alpha_1^2}{4} \left( \frac{2\beta \epsilon_n - 1}{\alpha_1 \sqrt{\beta} + n + 2} \right)^2 + \beta \epsilon_n^2.$$

The denominator of the fraction,  $\alpha_1 \sqrt{\beta} + n + 2$ , is always positive.

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As in the ordinary case (without a GUP,  $\beta = 0$ ), we will now look for explicit solutions for the ground state and the first excited state.

## 2.– Quantum Correction to the Coulomb Potential in GUP

### Ground state and the associated wave function

For  $n = 0$ , the energy of ground state,  $\epsilon_0^G$ , is determined in closed form:

$$\epsilon_0^G(\epsilon_0; \alpha_1, \beta) = -\frac{\alpha_1^2}{4} \left( \frac{2\beta \epsilon_0 - 1}{\alpha_1 \sqrt{\beta} + 2} \right)^2 + \beta \epsilon_0^2.$$



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The explicit form of the associated wave function is given by

$$\psi_0^G(x) = C_0 x^{2+\alpha_1\sqrt{\beta}} \exp \left[ -x \sqrt{\beta \epsilon_0^2 - \epsilon_0^G} - \frac{\sqrt{\beta}}{2} \left( \frac{2\alpha_4^2}{3x^3} + \frac{\alpha_3}{x^2} + \frac{2\alpha_2}{x} \right) \right],$$

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## 2.– Quantum Correction to the Coulomb Potential in GUP

### First excited state and the associated wave function

For  $n = 1$ , the first excited state energy,  $\epsilon_1^G$ , is given by

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The wave function is explicitly given by

$$\psi_1^G(x) = C_1 (x - x_1) x^{2+\alpha_1 \sqrt{\beta}} \exp \left[ -x \sqrt{\beta \epsilon_1^2 - \epsilon_1^G} - \frac{\sqrt{\beta}}{2} \left( \frac{2\alpha_4^2}{3x^3} + \frac{\alpha_3}{x^2} + \frac{2\alpha_2}{x} \right) \right],$$

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In the last part of the talk, the **Dirac equation** is considered with the recently proposed **generalized gravitational interaction** (Kepler or Coulomb), which includes post-Newtonian (relativistic) and quantum corrections to the classical potential.

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The general idea in choosing the metric is that the spacetime contributions are contained in an external potential or in an electromagnetic potential which can be considered as a good basis for future studies on space quantum communication.

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We also discuss several known generalizations of the Coulomb potential within this formulation in terms of certain Heun functions.

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#### **Roadmap:**

- (A) **A review** of the essential formulae of the Dirac equation in the desired metric.
- (B) Using the **Bethe-ansatz approach**, we report the general solution for arbitrary  $n$ , in particular solutions for the ground and the first excited states.

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In curved spacetime  $\nabla_\mu = \partial_\mu + iA_\mu/c + \Omega_\mu$ , with  $\partial_\mu$  the covariant derivative on flat spacetime and  $\Omega_\mu$  the spin connection. Choosing  $A_\mu = (V(r), cA_r(r), 0, 0)$ , the spinor wave function is

$$\Psi_c(r, \theta, \phi) = N \begin{pmatrix} R_1(r) \mathcal{Y}_{j+1/2}^{|m|j}(\theta, \phi) \\ iR_2(r) \mathcal{Y}_{j-1/2}^{|m|j}(\theta, \phi) \end{pmatrix}.$$

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#### (A) Curved spacetime Dirac equation with a generalized metric

The Dirac equation  $(i\gamma^\mu \nabla_\mu - mc) \Psi = 0$  will be considered for the metric

$$g_{\mu\nu} = \text{diag} \left( e^{2f(r)}, -e^{2g(r)}, -r^2, -r^2 \sin^2 \theta \right),$$

where  $f(r)$  and  $g(r)$  are arbitrary functions of the radial coordinate, being the angular parts the same as in (3+1)–Minkowski spacetime.

In curved spacetime  $\nabla_\mu = \partial_\mu + iA_\mu/c + \Omega_\mu$ , with  $\partial_\mu$  the covariant derivative on flat spacetime and  $\Omega_\mu$  the spin connection. Choosing  $A_\mu = (V(r), cA_r(r), 0, 0)$ , the spinor wave function is

$$\Psi_c(r, \theta, \phi) = N \begin{pmatrix} R_1(r) \mathcal{Y}_{j+1/2}^{|m|j}(\theta, \phi) \\ iR_2(r) \mathcal{Y}_{j-1/2}^{|m|j}(\theta, \phi) \end{pmatrix}.$$

$$\mathcal{Y}_l^{j=l\pm 1/2 m}(\theta, \phi) = \frac{1}{\sqrt{2l+1}} \begin{pmatrix} \pm \sqrt{l \pm m + \frac{1}{2}} Y_l^{m-1/2}(\theta, \phi) \\ \sqrt{l \mp m + \frac{1}{2}} Y_l^{m+1/2}(\theta, \phi) \end{pmatrix}$$

are the **spinor spherical harmonics**, being  $Y_l^m(\theta, \phi)$  the usual spherical harmonics.

### 3.– Dirac eqn. with generalized gravitational interaction

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- $m = \hbar = 1$ ,  $c = 1/\alpha$ , with  $\alpha$  being the fine structure constant.
- $f(r) = g(r)$ , with  $e^{f(r)} = 1 + \alpha^2 U(r)$ ,
- $V(r) = a \mathbf{z}(r)$  and  $U(r) = b \mathbf{z}(r)$ ,
- $R_1(r) = \frac{u(r)}{r} e^{-f(r)/2}$  and  $R_2(r) = \frac{v(r)}{r} e^{-f(r)/2}$ ,
- $\begin{pmatrix} \rho_1(r) \\ \rho_2(r) \end{pmatrix} = \mathbf{U}' \begin{pmatrix} u(r) \\ v(r) \end{pmatrix}$ ,  $\mathbf{U}' = \exp(i\eta\sigma_2)$ ,
- $A_r(r) = \frac{\alpha C}{S} \left[ \frac{V(r)}{C} - U(r) \right] - \frac{\lambda}{r} [1 + \alpha^2 U(r)]$ ,  $C = \cos 2\eta$ ,  $S = \sin 2\eta$ ,

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and we obtain the final equation for the component  $\rho_1(r)$

$$\left[ \frac{d^2}{dr^2} + \frac{\alpha}{S} (aC - b) z'(r) - 2(b + \epsilon a) z(r) - \frac{\alpha^2}{S^2} (aC - b)^2 z^2(r) + \frac{\epsilon^2 - 1}{\alpha^2} \right] \rho_1(r) = 0$$

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### (B) Solutions by the Bethe-ansatz approach

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Let us consider the *potential*  $z(r)$  as in the previous Schrödinger-like equation

$$z(r) = \frac{u}{r} + \frac{v}{r^2} + \frac{w}{r^3}, \quad u, v, w < 0,$$

that has been investigated before in the GUP framework.

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$$\left[ \frac{d^2}{dr^2} + \frac{\epsilon_n^2 - 1}{\alpha^2} - \frac{2u(a\epsilon_n + b)}{r} + \frac{1}{S^2} \left( \frac{\Lambda_2}{r^2} + \frac{\Lambda_3}{r^3} + \frac{\Lambda_4}{r^4} \right) - \frac{\alpha^2(b - aC)^2}{S^2} \left( \frac{w^2}{r^6} + \frac{2vw}{r^5} \right) \right] \rho_{1,n}(r) = 0,$$

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where  $\Lambda_2$ ,  $\Lambda_3$  and  $\Lambda_4$  are some functions of  $u, v, w, a, b, \alpha, S, C, \epsilon_n$ . We have added the index  $n$  to  $\epsilon$  and  $\rho_1(r)$  to distinguish some states from others.

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Now, we propose

$$\rho_{1,n}(r) = e^{\Delta(r)} \mathcal{R}_{1,n}(r), \quad \Delta(r) = \delta \ln r + \frac{\gamma}{r^2} + \frac{\beta}{r} + \lambda r,$$

where  $\mathcal{R}_{1,n}(r)$  is a polynomial, and  $\lambda, \beta, \gamma < 0$ ,  $\delta > 0$  are parameters to be find.



### 3.– Dirac eqn. with generalized gravitational interaction

Consequently, the differential equation for  $\mathcal{R}_{1,n}(r)$  simplifies to

$$\left\{ r^3 \frac{d^2}{dr^2} + (2\lambda r^3 + 2\delta r^2 - 2\beta r - 4\gamma) \frac{d}{dr} + \left( \xi_2 r^2 + \frac{\xi_1}{S^2} r + \frac{\xi_0}{S^2} \right) \right\} \mathcal{R}_{1,n}(r) = 0,$$

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where  $\xi_0, \xi_1, \xi_2$  are known functions of  $u, v, w, a, b, \alpha, S, C, \epsilon_n$ .

To solve this equation, we assume  $\mathcal{R}_{1,n}(r)$  to be a polynomial of the form

$$\mathcal{R}_{1,n}(r) = \begin{cases} 1, & n = 0, \\ \prod_{i=1}^n (r - r_i), & n \in \mathbb{N}, \end{cases}$$

where  $r_i$  are distinct roots to be determined. The general solutions are given by

$$\xi_2 + 2n\lambda = 0,$$

$$\xi_1 + 2\lambda \sum_{i=1}^n r_i + n(n-1) + 2n\delta = 0,$$

$$\xi_0 + 2\lambda \sum_{i=1}^n r_i^2 + 2(\delta + n - 1) \sum_{i=1}^n r_i - 2n\beta = 0,$$

the  $r_i$  given by the Bethe ansatz eqns 
$$\sum_{j=1, j \neq i}^n \frac{1}{r_i - r_j} + \frac{\lambda r_i^3 + \delta r_i^2 - \beta r_i - 2\gamma}{r_i^3} = 0.$$

### 3.– Dirac eqn. with generalized gravitational interaction

#### Ground state and the associated wave function

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For  $n = 0$  it follows that the ground state energy  $\epsilon_0$  is given by

$$\epsilon_0 = \frac{-4a\alpha^2\mathcal{A}^2bu^2 \pm \sqrt{4\alpha^2\mathcal{A}^2u^2(a^2 - b^2)(\mathcal{B} - 3\mathcal{A})^2 + (\mathcal{B} - 3\mathcal{A})^4}}{4a^2\alpha^2\mathcal{A}^2u^2 + (\mathcal{B} - 3\mathcal{A})^2},$$

with

$$\mathcal{A} := \sqrt{\alpha^2 S^2 (b - aC)^2}, \quad \mathcal{B} := \alpha(b - aC)(2\alpha u(b - aC) - 3S).$$

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with

$$\mathcal{A} := \sqrt{\alpha^2 S^2 (b - aC)^2}, \quad \mathcal{B} := \alpha(b - aC)(2\alpha u(b - aC) - 3S).$$

The associated wave function is  $\rho_{1,0}(r) \propto e^{\Delta(r)}$ , with

$$\Delta(r) = \left( \frac{3}{2} + \frac{\alpha w(b - aC)(2\alpha u(b - aC) - 3S)}{2\sigma S^2} \right) \ln r - \frac{\sigma}{r^2} \left( \frac{rv}{w} + \frac{1}{2} \right) - \sqrt{\frac{1 - \epsilon_0^2}{\alpha^2}} r,$$

The potential parameters  $v$  and  $w$  are given in terms of  $u, a, b, C, S, \alpha, \epsilon_0$ .

### 3.– Dirac eqn. with generalized gravitational interaction

#### First excited state and the associated wave function

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For  $n = 1$ , the first excited-state energy,  $\epsilon_1$ , is given by

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The associated wave function is  $\rho_{1,1}(r) \propto (r - r_1) e^{\Delta(r)}$ , with

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The special case of the Coulomb interaction  $v = w = 0$  can be obtained and compared the existing literature.

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**You are welcome to participate!**

**See you next January in Valladolid!**