On differential operators with PT-symmetric coefficients having a purely real spectrum

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AAMP XXI, 28/08/2024

We consider the Bloch eigenvalues and spectrum of the non-self-adjoint differential operator L generated by the differential expression

$$l(y) = (-i)^n y^{(n)}(x) + \sum_{v=2}^{n} (-i)^{n-v} p_v(x) y^{(n-v)}(x), \tag{1}$$

of odd order n with the periodic PT-symmetric coefficients, where $n > 1$,

$$p_v(x+1) = p_v(x), \ \ \overline{p_v(-x)} = p_v(x), \ \ (p_v)^{(n-v)} \in L_2[0,1].$$

We study the structure of the spectrum. Moreover, we find conditions on the norm of the coefficients under which the spectrum of L is purely real and coincides with the real line.

It is well-known [Rofe-Beketov (1963), McGarvey (1965)] that the spectrum $\sigma(L)$ of the operator L is the union of the spectra of the operators L_t for $t \in (-1, 1]$ generated in $L_2[0, 1]$ by (1) and the boundary conditions

$$y^{(\nu)}(1) = e^{i\pi t} y^{(\nu)}(0) \tag{2}$$

for $\nu = 0, 1, ..., (n-1)$. The spectrum of L_t consists of the eigenvalues $\lambda_1(t)$, $\lambda_2(t)$, ... called the Bloch eigenvalues of L. Thus, the spectrum of L is the union of the Bloch eigenvalues for $t \in (-1, 1]$ and hence consists of the sets $\{\lambda_n(t) : t \in (-1, 1]\}$ for $n = 1, 2, ...$ which is called the nth band of the spectrum.

If (1) is a self-adjoint expression, then the Bloch eigenvalues are the real numbers and the bands are the intervals of the real axis. If the coefficients are arbitrary complex-valued function, then the Bloch eigenvalues are the complex numbers and the bands are the curves lying in the complex plane.

Note that the results and research methods for the odd case $(n = 2v + 1)$ differ significantly from the results and research methods for the even case $(n = 2v)$. Let us briefly explain the reason for the differences. The eigenvalues of

$$L_{t,\varepsilon}y = (-i)^n y^{(n)}(x) + \varepsilon \sum_{v=2}^{n} (-i)^{n-v} p_v(x) y^{(n-v)}(x)$$

are located in the small neighborhood of the eigenvalues of $(-i)^n y^{(n)}$. The operators L and L_t are denoted by $L(0)$ and $L_t(0)$ if $p_2, p_3, ..., p_n$, are the zero functions. It is clear that $(2\pi k + \pi t)^n$ and $e^{i\pi(2k+t)x}$ for $k \in \mathbb{Z}$ are, respectively, the eigenvalues and eigenfunctions of $L_t(0)$. The numbers $(2\pi k + \pi t)^n$ for $k \in \mathbb{Z}$ are the simple eigenvalues of $L_t(0)$ and the set of all Bloch eigenvalues of $L(0)$ covers the real axis if n is an odd number.

If n is an even number, then $(2\pi k + \pi t)^n = (-2\pi k + \pi t)^n$ for $t = 0$ and $(2\pi k + \pi t)^n = (2\pi(-k-1) + \pi t)^n$ for $t = 1$. Therefore, periodic and antiperiodic eigenvalues are double eigenvalue. If $t \neq 0, 1$, then these numbers are the simple eigenvalues. Thus, if n is an even number, the periodic and antiperiodic eigenvalues are exceptional points in the spectrum of L. However, if n is an odd number, the spectrum of L has no exceptional points. This situation and the following property of differential operators with PT-symmetric coefficients helps us prove that the spectrum of L, under certain conditions on the coefficient, is purely real if n is an odd number. If λ is an eigenvalue of L_t, then its conjugate $\overline{\lambda}$ is also an eigenvalue of L_t. On the other hand, according to general perturbation theory, if the operator A has only one eigenvalue λ inside the circle $\gamma = \{z \in \mathbb{C} : \ |\lambda - z| = \delta\}$, then for small value of ε the operator $A + \varepsilon B$ also has only one eigenvalue $\lambda(\varepsilon)$ inside γ. If $\lambda(\varepsilon)$ is not a real number, then $\lambda(\varepsilon)$ and its complex conjugate are eigenvalues of $A + \varepsilon B$ lying inside γ. This implies that $A + \varepsilon B$ has two eigenvalue inside γ, which leads to a contradiction. Thus, $\lambda(\varepsilon)$ must be a real number if n is an odd number.

The general even case is similar to the case of the Schrödinger operator ($n = 2$). For example, under perturbation by an optical potential, these double periodic eigenvalues (which can be thought of as two coinciding eigenvalues) separate, with one periodic eigenvalue shifting to the left and the other to the right. A different situation is observed with antiperiodic eigenvalues: one antiperiodic eigenvalue moves up and the other moves down.

Note that there is a large number of papers for the Schrödinger operator. We only note that, in the first papers [Bender, etc. (1999)] about the PT-symmetric periodic potential, the disappearance of real energy bands for certain complex-valued PT-symmetric periodic potentials was reported. Shin (2004) showed that the disappearance of such real energy bands implies the existence of nonreal band spectra. In my papers (Veliev: Int. J. Geom. Methods Mod. Phys. (2017) and J. Math. Phys. (2020)), I investigated the spectrum of L in detail. I proved that the main part of the spectrum of L is real and covers the large portion of $[0, \infty)$. However, in general, the spectrum also contains infinitely many nonreal components.

Now let us discuss in detail the case when n is an odd number. First of all, I proved that if the coefficients of

$$(-i)^n y^{(n)}(x) + \sum_{v=2}^{n} (-i)^{n-v} P_v(x) y^{(n-v)}(x),$$

are the $m \times m$ matrices with PT-symmetric elements and m is an odd number, then $\mathbb{R} \subset \sigma(L)$. However, there are eigenvalues that are not real numbers for the following reasons. In this case, the Bloch eigenvalues $\lambda_{k,j}(t)$ for $k \in \mathbb{Z}$, $j = 1, 2, ..., m$ satisfy the asymptotic formulas

$$\lambda_{k,j}(t) = (2\pi k + t)^n + \mu_j (2\pi k + t)^{n-2} + O(k^{n-3} \ln |k|),$$

where $\mu_1, \mu_2, ..., \mu_m$ are the eigenvalues of the matrix $P = \int_0^1 P_2(x)\, dx$. The entries of the matrix P are the real numbers. Therefore, P has the real eigenvalues $\mu_1, \mu_2, ..., \mu_p$ and non-real eigenvalues $\mu_j = a_j \pm ib_j$ for $j = p+1, p+2, .., p+q$, where $p + 2q = m$. If μ_j is a nonreal eigenvalue, then $\lambda_{k,j}(t)$ is a nonreal Bloch eigenvalue.

In the case of $m = 1$, in order to prove that all Bloch eigenvalues are real numbers, under certain conditions on the coefficient, we consider in detail the localization of Bloch eigenvalues. Let's divide plane $\mathbb{C}$ into strips:

$$S(N,t) = \{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda \in [(-2\pi N + \pi + \pi t)^n, (2\pi N - \pi + \pi t)^n)\}$$

and

$$P(k,t) = \{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda \in [(2\pi k + \pi t - \pi)^n, (2\pi k + \pi t + \pi)^n)\}$$

for $|k| \geq N$, where N be the smallest integer satisfying $N \geq \pi^{-2} C + 1$ and

$$C = \sum_{v=2}^{n} \sum_{s=0}^{n-v} \frac{(n-v)! \left\| (p_v)^{(s)} \right\|}{s!(n-v-s)!\pi^{v+s-2}}.$$

First we prove that the eigenvalues of L_t lying in $P(k,t)$, lie in the disk

$$U(k,t) = \{\lambda \in \mathbb{C} : |\lambda - (2\pi k + \pi t)^n| < \delta_k(t)\}$$

and this disk contains only one eigenvalue of L_t, where

$$\delta_k(t) := \frac{3}{2}\pi^{n-2} C \left|(2k+t)\right|^{n-2}.$$

Therefore, this eigenvalue is a real number. This means that all large eigenvalues are real numbers.

It remains to consider the eigenvalues lying in $S(N,t)$. We prove that the eigenvalues lying in $S(N,t)$ are contained in the rectangle

$$\left\{\lambda \in \mathbb{C} : |\mathrm{Re}\,\lambda| \le (2\pi N)^n, \ \ |\mathrm{Im}\,\lambda| < \frac{\sqrt{10}}{3}(2N+1)^{n-3/2}\,\pi^{n-2}C\right\}. \tag{3}$$

Moreover, if $C \le \pi^2 2^{-n+1/2}$, then all eigenvalues of L_t are contained in the disks

$$U(0,t) = \left\{\lambda \in \mathbb{C} : |\lambda - (\pi t)^n| < \frac{1}{5}\pi^n\right\},$$

$$U(1,t) = \left\{\lambda \in \mathbb{C} : |\lambda - (2\pi + \pi t)^n| < \frac{3}{10}\,|2+t|^{n-2}\,\pi^n\right\},$$

$$U(-1,t) = \left\{\lambda \in \mathbb{C} : |\lambda - (\pi t - 2\pi)^n| < \frac{3}{10}\,|t-2|^{n-2}\,\pi^n\right\}$$

and $U(k,t)$ for $|k| > 1$. The closures of these disks are pairwise disjoint sets, each of which contains only one eigenvalue. Therefore, these eigenvalues are real numbers.

Now we are ready to formulate the following main results.

Theorem

(a) Each of the disks $U(k,t)$ for $|k| \geq N$ contains only one eigenvalues of L_t. This eigenvalue is a real number.
(b) The real part $\sigma(L) \cap \mathbb{R}$ of the spectrum $\sigma(L)$ of L is $\mathbb{R}$ and the nonreal part $\sigma(L) \backslash \mathbb{R}$ of $\sigma(L)$ consists of the curves lying in the rectangle (3).
(c) If $C \leq \pi^2 2^{-n+1/2}$, then (a) is valid for all $k \in \mathbb{Z}$ and $\sigma(L) = \mathbb{R}$.

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