Eigenvalues of operator functions: A commutativity result with an application to 1D Dirac operators

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Joint work with J. Behrndt and P. Siegl

Analytic and algebraic methods in physics XXII
Prague, August 26-29, 2025

Content

Motivation: Spectrum of perturbed 1D Dirac

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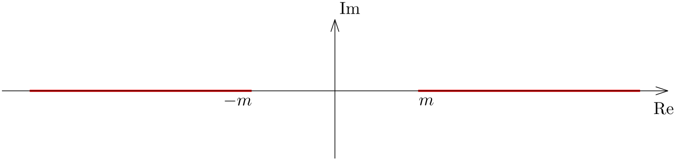

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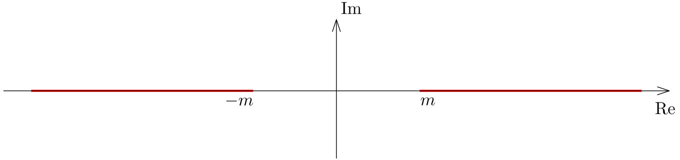

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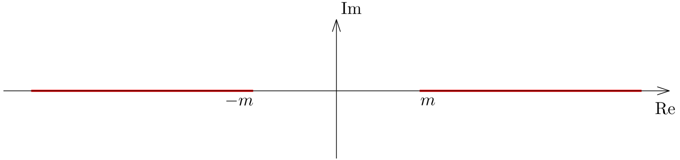

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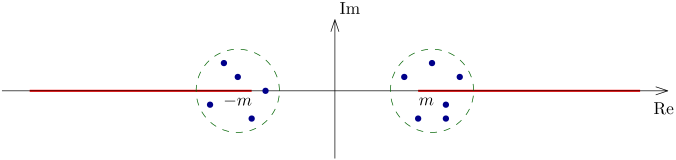

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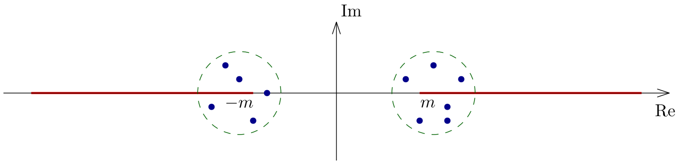

Does $H_0 + V$ have eigenvalues? $\rightsquigarrow H_\varepsilon = H_0 + \varepsilon V$.

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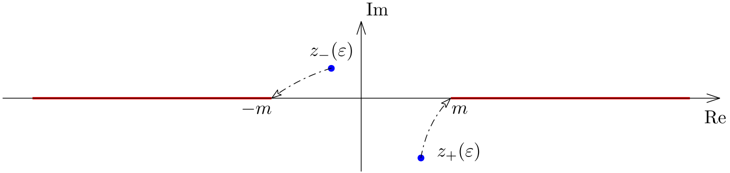

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Question: $m_a(z; I - TS) = m_a(z; I - ST)$?

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Outlines of the proofs

Theorem

Suppose

- $\Omega \subset \mathbb{C}$ *open.*
- $T : \Omega \to \mathcal{L}(\mathcal{H}, \mathcal{G})$, $S : \Omega \to \mathcal{L}(\mathcal{G}, \mathcal{H})$ *analytic.*

Then:

$$\forall z \in \Omega : \quad z \in \sigma_{\mathbf{ft}}(I - TS) \iff z \in \sigma_{\mathbf{ft}}(I - ST),$$

in which case

$$m_a(z_0; I - TS) = m_a(z_0; I - ST).$$

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$$\stackrel{def}{=} m_a(z_0; W_{ST}) \qquad \square$$

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Thank you for your attention!