Some Novel Aspects of Singular Interactions

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The resolvent of the modified Hamiltonian by singular delta potentials supported by a point a in two or three dimensions has been studied extensively in the literature and given by the Krein's formula:

$$R(E) = R_0(E) + (\Phi(E))^{-1} R_0(E)|a\rangle\langle a|R_0(E)$$

where $R_0(E) = (H_0 - E)^{-1}$ is the resolvent of the initial Hamiltonian H_0, and $G_0(x, y)$ is integral kernel of the resolvent R_0 or the Green's function of H_0, and $\Phi(E)$ known explicitly.

- Discrete Spectrum Modified by a δ Interaction

♠ We consider the case in which H_0 is *formally* modified by a single δ function at $x = a$,

$$H = H_0 - \alpha \delta_a \ , \tag{1}$$

where α is to be renormalized eventually.

- H_0 is self-adjoint on some dense domain $D(H_0) \subset L^2(\mathcal{M})$, where $\mathcal{M}$ is a two or three dimensional compact Riemannian manifold.

• Often, it is essential to assume some regularity on the geometry, experience has shown that a lower bound on the Ricci curvature satisfies most of the technical requirements:

$$Ric_g(\cdot,\cdot) \geq (D-1)\kappa g(\cdot,\cdot)\ . \tag{2}$$

• For two dimensional compact manifolds, this does not impose any restriction, as Ricci curvature is exactly given by

$$Ric_g(\cdot,\cdot) = \frac{R}{2} g(\cdot,\cdot),$$

where R is the scalar curvature, and R has a minimum (and a maximum) value on a *compact* manifold.

♯ For three dimensional manifolds, this puts some restriction. If $\kappa > 0$, one has much better control for various bounds on heat kernels.

- Spectrum of H_0 is discrete $\sigma_d(H_0)$ (set of eigenvalues),

- The discrete spectrum has no accumulation point,

- For stability, we assume H_0 has spectrum bounded from below.

If we choose a proper $H_0 = -\frac{\hbar^2}{2m}\Delta + V$ on $D = 2, 3$ dimensional Euclidean space,
OR consider $H_0 = -\frac{\hbar^2}{2m}\Delta_g$ on a compact Riemannian manifold (again of dimension 2 or 3) with a metric g_{ij}, where Δ_g is the Laplace-Beltrami operator or Laplacian given by

$$(\Delta_g\psi)(x) = \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{D} \frac{\partial}{\partial x^i}\left(\sqrt{\det g}\, g^{ij}\frac{\partial\psi(x)}{\partial x^j}\right) , \quad (3)$$

in some local coordinates, with g^{ij} being the components of inverse of the metric g.

• For Δ_g, there exists a complete orthonormal system of C^∞ eigenfunctions $\{\phi_n\}_{n=0}^{\infty}$ in $L^2(\mathcal{M})$ and the spectrum $\sigma(H_0) = \{E_n\} = \{0 = E_0 \leq E_1 \leq E_2 \leq \ldots\}$, with E_n tending to infinity as $n \to \infty$; moreover each eigenvalue has finite multiplicity.

♣ Some eigenvalues are repeated according to their multiplicity. The multiplicity of the first eigenvalue $E_0 = 0$ is one and the corresponding eigenfunction is constant. From now on, we pretend that there is no degeneracy in the spectrum of the Laplacian for simplicity. General case can be done, but more technical.

The integral kernel of the resolvent $R_0(E)$ for H_0 or simply Green's function is given by

$$\left((H_0 - E)^{-1}\psi\right)(x) = \int_{\mathcal{M}} G_0(x, y|E)\psi(y)d\mu(y) \qquad (4)$$

where $d\mu(y)$ is the volume element in $\mathcal{M}$ (on a manifold, expressed in local coordinates, it has the usual $\sqrt{\det g}$ factor in it). Away from the diagonal $x = y$,

$$G_0(x, y) = \sum_{n=0}^{\infty} \frac{\phi_n(x)\overline{\phi_n(y)}}{E_n - E}\,, \qquad (5)$$

where $\{\phi_n\}$ is the complete set of eigenfunctions of H_0. The Green's function $G_0(x, y|E)$ is a square-integrable function of x for almost all values of y and vice versa [ReedSimon].

$\sharp$ The standard route is to construct this Green's function and establish that the Hamiltonian defined by this expression is self-adjoint.

♠ By the spectral theorem, there is a complete set of eigenfunctions!

• We prove directly by means of the explicit expression of the constructed Green's function that the corresponding Hamiltonian still has a complete set of eigenfunctions. For this we use the completeness property of the eigenfunctions of the initial Hamiltonian H_0, having only a discrete spectrum, and an interlacing theorem for the poles of the new Green's function.

• As a result, we thus establish the self-adjointness of the resulting Hamiltonian in a novel way (Remark 4.3).

• A great pedagogical value in establishing the existence of an orthonormal basis for a given Hamiltonian, as it demonstrates clearly the validity of one of the fundamental postulates of quantum mechanics.

· Green's function G_0 in terms of the heat the kernel $K_t(x, y)$ associated with the operator H_0 given by

$$G_0(x, y|E) = \int_0^\infty K_t(x, y) e^{tE} dt \;, \tag{6}$$

where $\text{Re}(E) < 0$ and

$$H_0 K_t(x, y) = \frac{\partial}{\partial t} K_t(x, y)$$

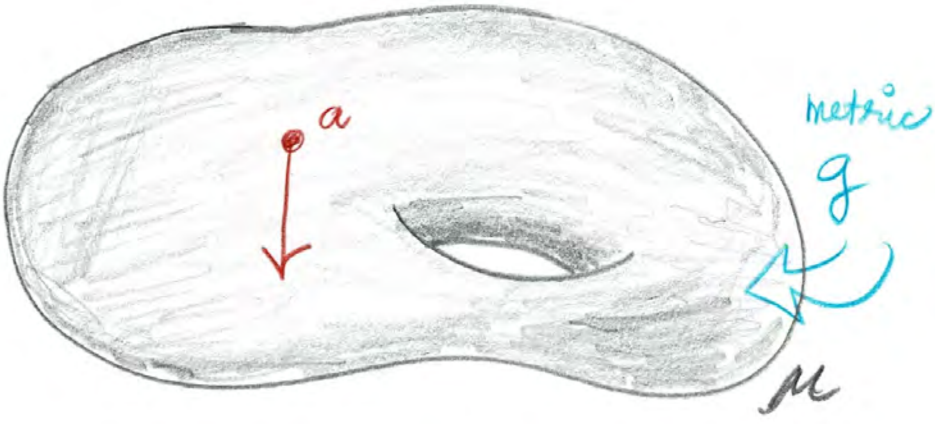
♠ Recall that the first term in the short-time asymptotic expansion of the diagonal heat kernel for any self-adjoint elliptic second-order differential operator in D dimensions, is given by

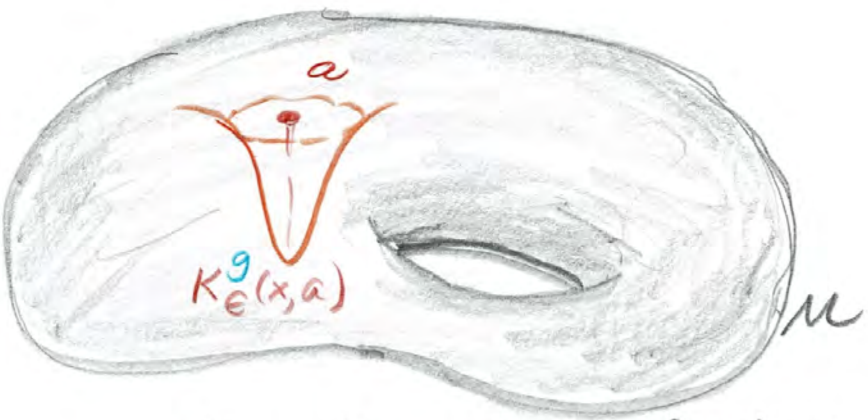
$$K_t(x,x) \sim t^{-D/2} \, . \tag{7}$$

This leads to the divergence around $t = 0$ in the diagonal part of the Green's function $G_0(x,x|E)$:

$$\int_0^\infty \frac{e^{-t|E|}}{t^{D/2}} \, dt \, , \tag{8}$$

• We thus use this idea to regularize the delta potential and then perform a renormalization prescription.





Regularized δ-function

♣ A natural choice for absorbing the divergent part in a redefinition of the coupling constant is given by

$$\frac{1}{\alpha(\epsilon)} = \frac{1}{\alpha_R(M)} + \int_{\epsilon}^{\infty} K_t(a,a) e^{tM} dt \ , \tag{9}$$

where M is the renormalization scale and could be eliminated in favor of a physical parameter by imposing a renormalization condition.

• Take the limit as $\epsilon \to 0$, we obtain the integral kernel

$$G(x,y|E) = G_0(x,y|E) + \frac{G_0(x,a|E)G_0(a,y|E)}{\Phi(E)}$$

where

$$\Phi(E) = \frac{1}{\alpha_R(M)} + \int_0^{\infty} K_t(a,a)\left(e^{tM} - e^{tE}\right) dt$$

we assume $\Re(E) < 0, M < 0$

• Since the bound state energy of the system can be found from the poles of the Green's function, or equivalently zeroes of the function Φ, there must be a relation among M, $\alpha_R(M)$, and the bound state energy of the particle say $-\mu^2$.

• Note that α_R varies with respect to M in a precise way to keep the physics independent of this arbitrary choice. We set the renormalization scale at $M = -\mu^2$ (thinking of a bound state below E_0) for simplicity. Then,

$$\begin{aligned} \Phi(E) &= \frac{1}{\alpha_R} + \int_0^\infty K_t(a,a) \left(e^{-t\mu^2} - e^{tE} \right) \, dt \\ &= \frac{1}{\alpha_R} + \sum_{n=0}^{\infty} \left(\frac{|\phi_n(a)|^2}{(E_n + \mu^2)} - \frac{|\phi_n(a)|^2}{(E_n - E)} \right) \\ &= \frac{1}{\alpha_R} - \sum_{n=0}^{\infty} \frac{|\phi_n(a)|^2 (E + \mu^2)}{(E_n - E)(E_n + \mu^2)} \, . \end{aligned} \tag{10}$$

Here we employ the eigenfunction expansion of the heat kernel $K_t(x,y) = \sum_n \overline{\phi_n(x)} \phi_n(y) e^{-tE_n}$ of the Laplacian.

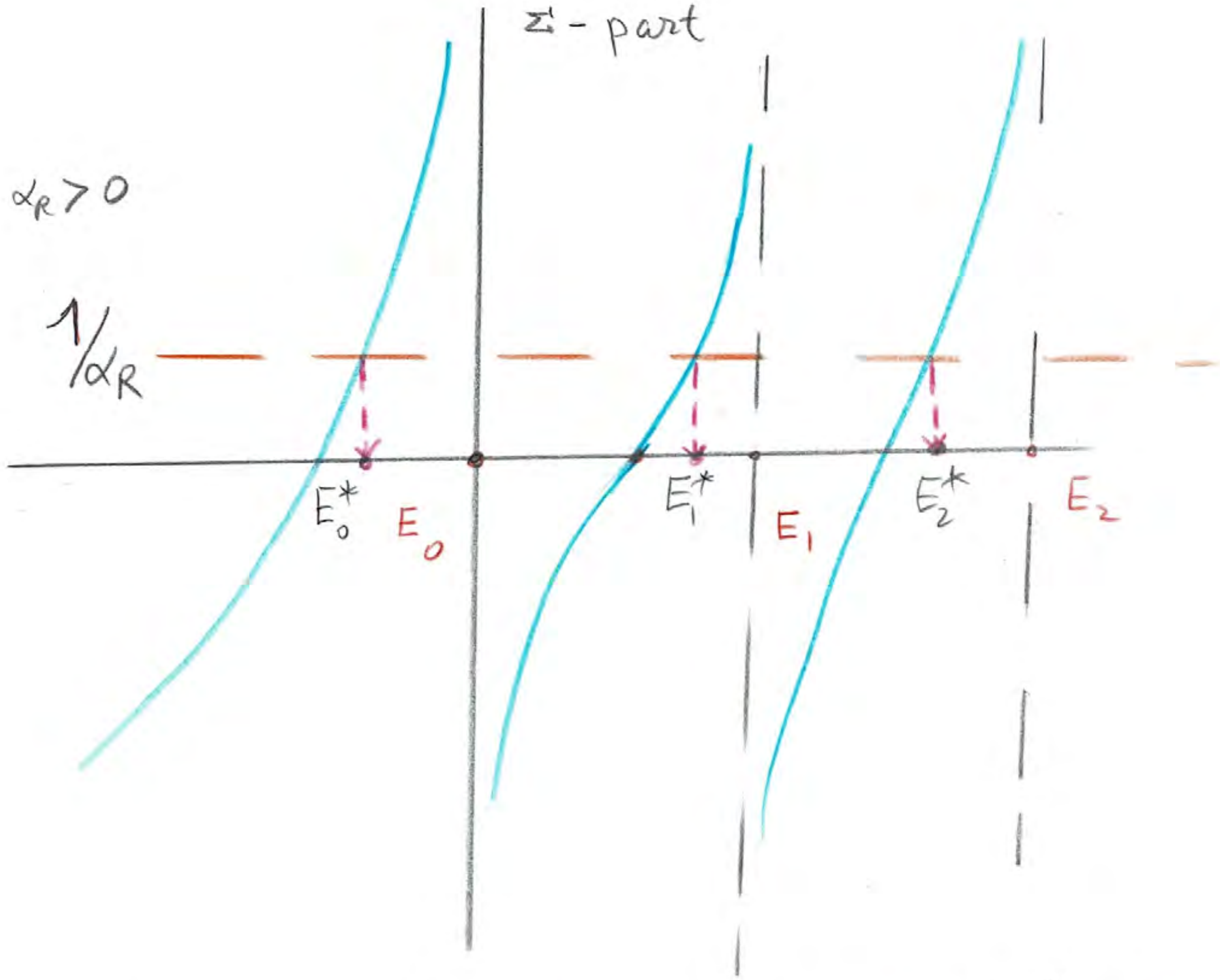
• The convergence of this sum can be shown by using the upper bounds of the heat kernel.

The spectral properties of the full Hamiltonian is given by the following proposition:

♣ Let $\phi_k(x)$ be the wave function of H_0 associated with the eigenvalue E_k. Then, the (new) energy eigenvalue E_k^* of H, is found from the unique solution of the equation

$$\Phi(E) = \frac{1}{\alpha_R} - \sum_{n=0}^{\infty} \frac{|\phi_n(a)|^2(E + \mu^2)}{(E_n - E)(E_n + \mu^2)} = 0$$

which lies in between E_{k-1} and E_k, if $\phi_k(a) \neq 0$ for this particular k. If we have $\phi_k(a) = 0$, the corresponding energy eigenvalue does not change, i.e., $E_k^* = E_k$. For the ground state ($k = 0$), we always have $E_0^* < E_0$.



We get $E_0^* < E_0 < E_1^* < E_1 < E_2^* < E_2$

♠ Using a contour integral of the resolvent $R(E) = (H - E)^{-1}$ around each simple eigenvalue E_k^*, we obtain the projection operator onto the eigen-subspace corresponding to the eigenvalue E_k^*,

$$\mathbb{P}_k = \frac{1}{2\pi i} \oint_{\Gamma_k} R(E)\, dE \ , \tag{11}$$

where Γ_k is the closed contour anticlockwise oriented around each simple pole E_k^*, or equivalently

$$\psi_k(x)\overline{\psi_k(y)} = \frac{1}{2\pi i} \oint_{\Gamma_k} G(x, y|E)\, dE \ . \tag{12}$$

• From the explicit expression of the Green's function and the residue theorem, we obtain

$$\psi_k(x) = \frac{G_0(x, a|E_k^*)}{\left(\left. \frac{d\Phi(E)}{dE} \right|_{E=E_k^*} \right)^{1/2}}. \tag{13}$$

♠ Remark: These functions are not in the domain of H_0, if you compute the kinetic energy it is divergent, yet they are square integrable. They become singular when $x \to y$.

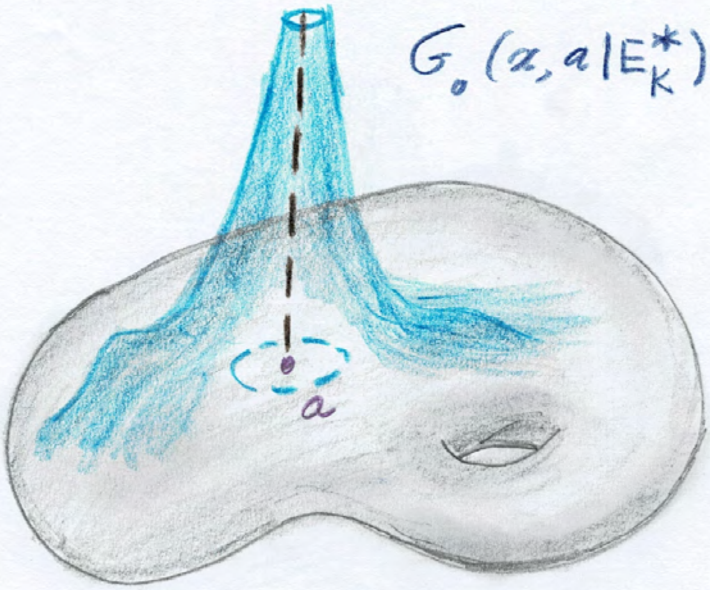
♯ Note that we have

$$\left.\frac{d\Phi(E)}{dE}\right|_{E_k^*} = \sum_{n=0}^{\infty} \frac{|\phi_n(a)|^2}{(E_n - E_k^*)^2} > 0. \tag{14}$$

♠ If $E_k^* = E_k$, then $\phi_k(a) = 0$, thus this term is skipped in the sum ensuring the expression being well-defined in all these cases. Moreover, in these special cases, the corresponding eigenfunction becomes,

$$\psi_k(x) = \phi_k(x). \tag{15}$$

Remark: The eigenfunctions $\phi_n(x)$ of Laplacian on a compact manifold are smooth functions, new eigenfunctions are singular at $x = a$, they are square-integrable and smooth functions outside of an open set around point a.



- G_0 is the Green's function of the Laplacian

♠ Perturbative approximation: Assume $\alpha_R << 1$. Let $E = E_k^* = E_k + \delta E_k$, where E_k is any specific eigenstate of H_0. The desired pole of the Green's function $G(x, y|E)$ is given by the solution of

$$\begin{aligned} & \frac{1}{\alpha_R} - \sum_{n=0}^{\infty} \frac{|\phi_n(a)|^2 (E_k + \delta E_k + \mu^2)}{(E_n + \mu^2)(E_n - (E_k + \delta E_k))} \\ = & \frac{1}{\alpha_R} - \sum_{n \neq k} \frac{|\phi_n(a)|^2 (E_k + \delta E_k + \mu^2)}{(E_n + \mu^2)(E_n - (E_k + \delta E_k))} \\ + & \frac{|\phi_k(a)|^2 (E_k + \delta E_k + \mu^2)}{(E_k + \mu^2)(\delta E_k)} \end{aligned} \tag{16}$$

near E_k. Expand in powers of δE_k, (for a specific k),

$$\begin{aligned} & 0 = \delta E_k + \alpha_R |\phi_k(a)|^2 + \frac{\alpha_R |\phi_k(a)|^2}{E_k + \mu^2} \delta E_k \\ - & \alpha_R \delta E_k \sum_{n \neq k} \frac{|\phi_n(a)|^2 (E_k + \mu^2)}{(E_n - E_k)(E_n + \mu^2)} \left(1 + \frac{\delta E_k}{E_n - E_k} + \ldots \right) \\ - & \alpha_R \delta E_k \sum_{n \neq k} \frac{|\phi_n(a)|^2 \delta E_k}{(E_n - E_k)(E_n + \mu^2)} \left(1 + \frac{\delta E_k}{E_n - E_k} + \ldots \right) \end{aligned}$$

♯ We assume that δE_k has an expansion in powers of α_R,

$$\delta E_k = E_k^{(1)} + E_k^{(2)} + \cdots ,$$

where $E_k^{(n)} \sim \alpha_R^n$. Solve $E_k^{(1)}$ and $E_k^{(2)}$ term by term,

$$
\begin{aligned}
E_k^{(1)} &= -\alpha_R |\phi_k(a)|^2 \\
E_k^{(2)} &= \alpha_R^2 |\phi_k(a)|^2 \left(\frac{|\phi_k(a)|^2}{E_k + \mu^2} - \sum_{n \neq k} \frac{|\phi_n(a)|^2 (E_k + \mu^2)}{(E_n - E_k)(E_n + \mu^2)} \right)
\end{aligned}
$$

Note that the first order result is the same as the regular perturbation theory, except that α is replaced by the renormalized coupling constant α_R. However, the second order result is completely different from the regular potential case.

- Digression: A cut-off version.

We introduce a cut-off in the eigenvalues of H_0, N and define $\delta_N(x,a)$ as our regularized delta-interaction on a manifold. Here

$$\delta_N(x,a) = \sum_n e^{-n/N} \overline{\phi_n(x)} \phi_n(a) \; . \tag{17}$$

Clearly as $N \to \infty$ we get $\delta_N(x,a) \to \delta(x,a)$ (in the weak sense). Let us define

$$\frac{1}{\alpha} = \frac{1}{\alpha_R} + G_0(a,a| - \mu^2; N) \; , \tag{18}$$

where

$$G_0(a,a| - \mu^2; N) = \sum_n \frac{e^{-n/N} |\phi_n(a)|^2}{E_n + \mu^2} \; , \tag{19}$$

or equivalently,

$$\alpha = \frac{\alpha_R}{1 + \alpha_R G_0(a, a| - \mu^2; N)} \; . \tag{20}$$

Now, within the usual perturbative approach, we assume α_R is a *formal parameter, that can be made arbitrarily small so as to organize our expansions accordingly*. This means we should order everything according to powers of α_R and formally expand, which gives

$$\alpha = \alpha_R - \alpha_R^2 G_0(a, a| - \mu^2; N) + O(\alpha_R^3) \; . \tag{21}$$

The formal δ-interaction term in the Hamiltonian now becomes,

$$-\left[\alpha_R - \alpha_R^2 G_0(a, a| - \mu^2; N) + O(\alpha_R^3)\right] \delta_N(x, a) \; . \tag{22}$$

Hence, a standard perturbative expansion, organized according to the powers of α_R will have mixed terms as *the interaction term now is a power series in* α_R. First order perturbation now keeps only the first term of the interaction,

$$E^{(1)}_{N;k} = -\alpha_R \langle \phi_k | \delta_N(\cdot, a) | \phi_k \rangle, \tag{23}$$

where $d = 2$ or $d = 3$. The limit $N \to \infty$ reproduces our previous answer. The second order term has two parts, one from the interaction, treated at first order as it has α_R^2, then another term, treated at second order:

$$E_k^{(2)} = \alpha_R^2 \sum_{m \neq k} \frac{\langle \phi_k | \delta_N(\cdot, a) | \phi_m \rangle \langle \phi_m | \delta_N(\cdot, a) | \phi_k \rangle}{E_k - E_m} + \alpha_R^2 \sum_m \frac{e^{-m/N} |\phi_m(a)|^2}{E_m + \mu^2} \langle \phi_k | \delta_N(\cdot, a) | \phi_k \rangle.$$

In the limit $N \to \infty$ (leaving aside delicate convergence issues),

$$E_k^{(2)} = \alpha_R^2 |\phi_k(a)|^2 \Big(\frac{|\phi_k(a)|^2}{E_k + \mu^2} - \sum_{m \neq k} \frac{|\phi_m(a)|^2 (E_k + \mu^2)}{(E_m - E_k)(E_m + \mu^2)} \Big),$$

where the last term comes from isolating the $m = k$ term from the $G_N(a, a| -\mu^2)$ part.

- Our direct approach provides a sound basis for a regularized perturbation theory.

Let ϕ_n be orthonormal set of eigenfunctions of H_0, i.e.,

$$\begin{aligned} H_0 \phi_n &= E_n \phi_n \\ \int_{\mathcal{M}} \overline{\phi_n(x)} \phi_m(x) \, d\mu(x) &= \delta_{nm}. \end{aligned} \tag{24}$$

Then, the eigenfunctions ψ_n for H_0 modified by a delta interaction supported at $x = a$ are orthonormal, that is,

$$\int_{\mathcal{M}} \overline{\psi_n(x)} \psi_m(x) \, d\mu(x) = \delta_{nm} \, , \tag{25}$$

where $D = 2, 3$ (so renormalization is needed).

Proof:

• If $n = m$, then it is easy to show that the eigenfunctions ψ_n's are normalized.

• For $n \neq m$, we first formally decompose the expression in the summation with a cut-off N as a sum of two partial fractions

$$\sum_{k=0}^{N} \frac{|\phi_k(a)|^2}{(E_k - E_n^*)(E_k - E_m^*)} = \sum_{k=0}^{N} \frac{|\phi_k(a)|^2}{(E_n^* - E_m^*)} \Big(\frac{1}{E_k - E_n^*} - \frac{1}{E_k - E_m^*} \Big).$$

Each term is divergent as $N \to \infty$. Motivated by this, we add and subtract

$$\frac{1}{\alpha_R} + \sum_{k=0}^{N} \frac{|\phi_k(a)|^2}{E_k + \mu^2}$$

to the above expression and obtain in the limit $N \to \infty$

$$\int_{\mathcal{M}} \overline{\psi_n(x)} \psi_m(x) \, d\mu(x) = \frac{(\Phi(E_n^*) - \Phi(E_m^*))}{(E_n^* - E_m^*)} \times \frac{1}{\left(\frac{d\Phi(E)}{dE}\Big|_{E_n^*} \right)^{1/2} \left(\frac{d\Phi(E)}{dE}\Big|_{E_m^*} \right)^{1/2}}.$$

The zeroes of the function Φ are the eigenvalues of the modified system: $\Phi(E_n^*) = 0$ and $\Phi(E_m^*) = 0$ for all n, m (when $n \neq m$) we get 0, this completes our proof of the orthogonality of eigen-functions for δ-modified Hamiltonian.

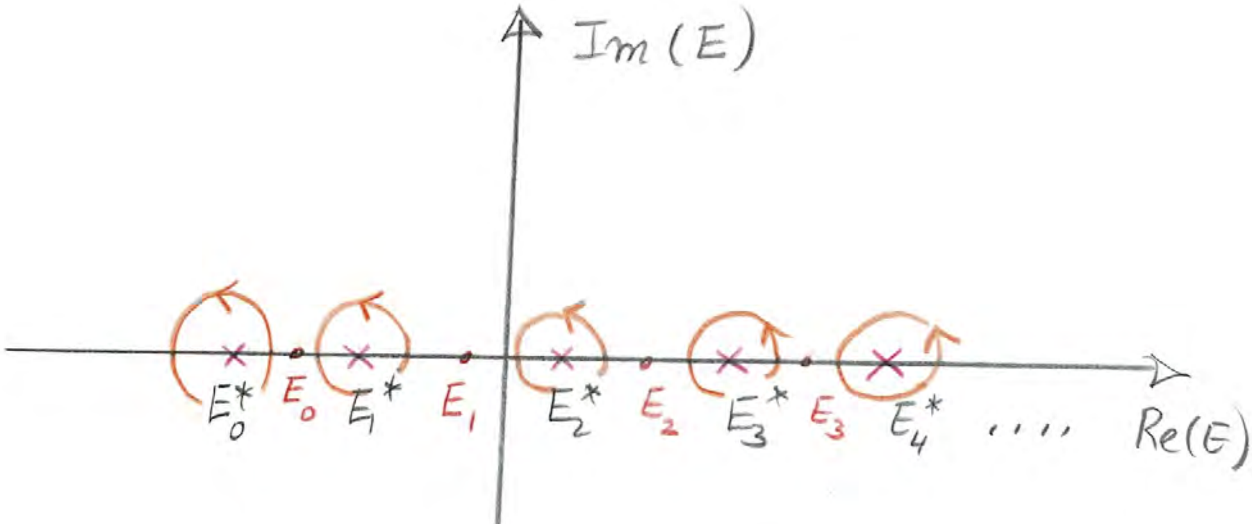
♠ Let ϕ_n be a complete set of eigen-functions of H_0, i.e.,

$$\begin{aligned} H_0 \phi_n &= E_n \phi_n \\ \sum_{n=0}^{\infty} \overline{\phi_n(x)} \phi_n(y) &= \delta(x-y) \; . \end{aligned} \tag{26}$$

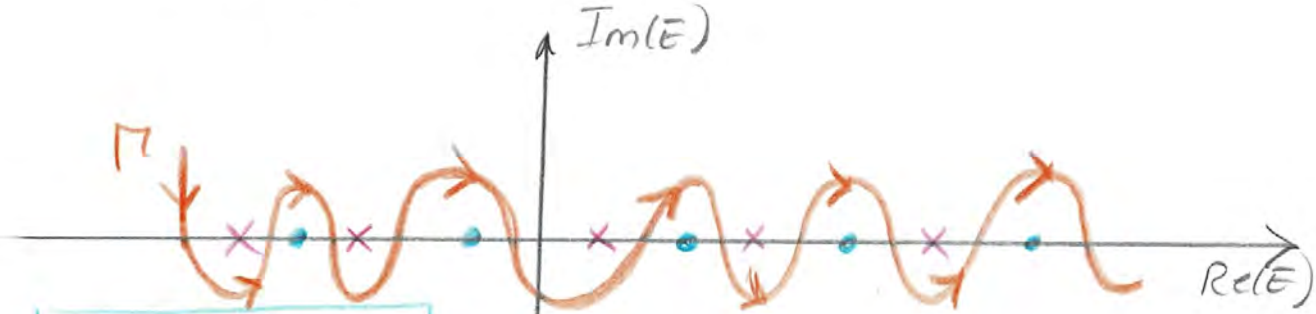
♣ Then, the eigen-functions ψ_n of H_0 modified by a delta-interaction supported at $x = a$, form a complete set, that is,

$$\sum_{n=0}^{\infty} \overline{\psi_n(x)} \psi_n(y) = \delta(x-y) \; . \tag{27}$$

Proof: Let Γ_n be the counter-clockwise oriented closed contours around each simple pole E_n^* and $\Gamma_n \cap \Gamma_m = \emptyset$ for $n \neq m$, as shown in the figure below.

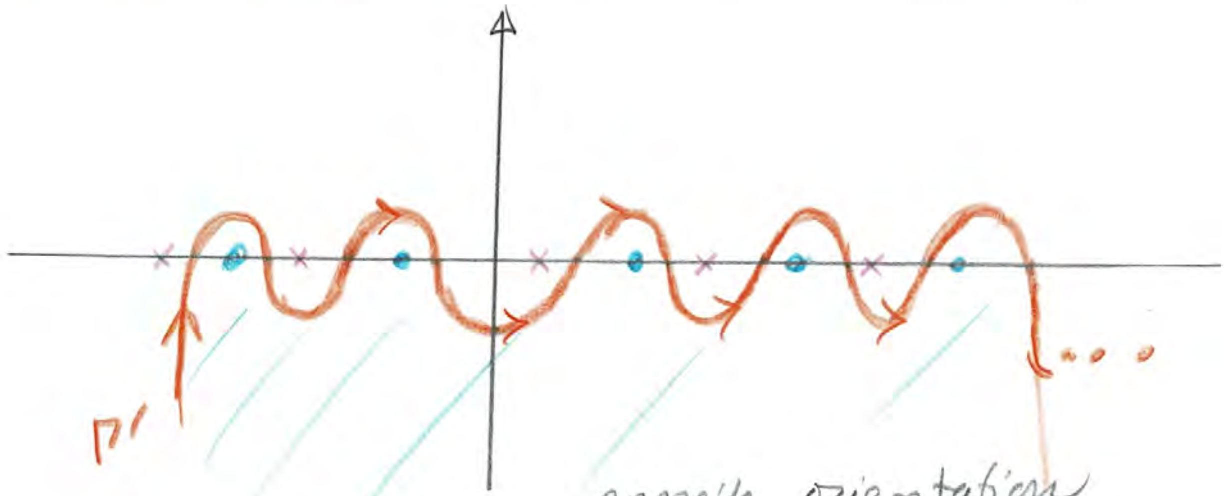


$$\sum_{n=0}^{\infty} \oint_{\Gamma_n} dE \; G(x,y|E) = \sum_{n=0}^{\infty} \psi_n(x)\psi_n^*(y)$$



$\boxed{\text{Drop } G_0(x,y)}$

$\Downarrow$ Further deform



opposite orientation

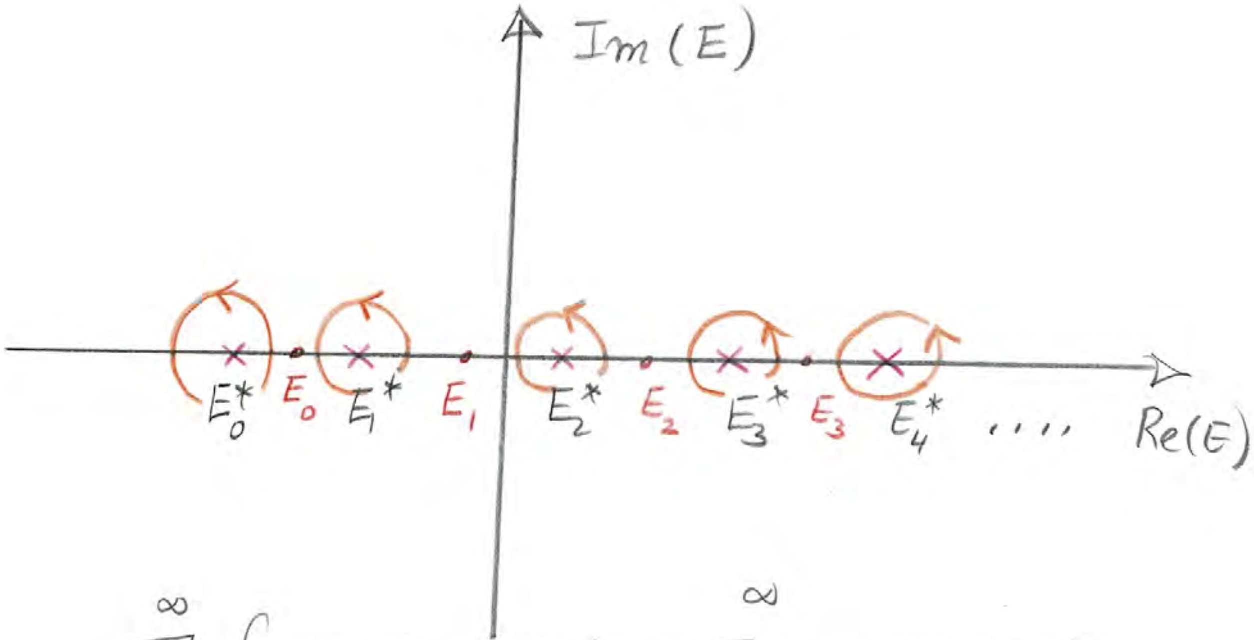
$$\oint_{\Gamma'} G_0 \Phi^{-1} G_0 \, dE$$

Then, the projection onto the associated eigenspace is given by the formula (12), and thanks to Krein's formula for the Green's function of the modified Hamiltonian, we have

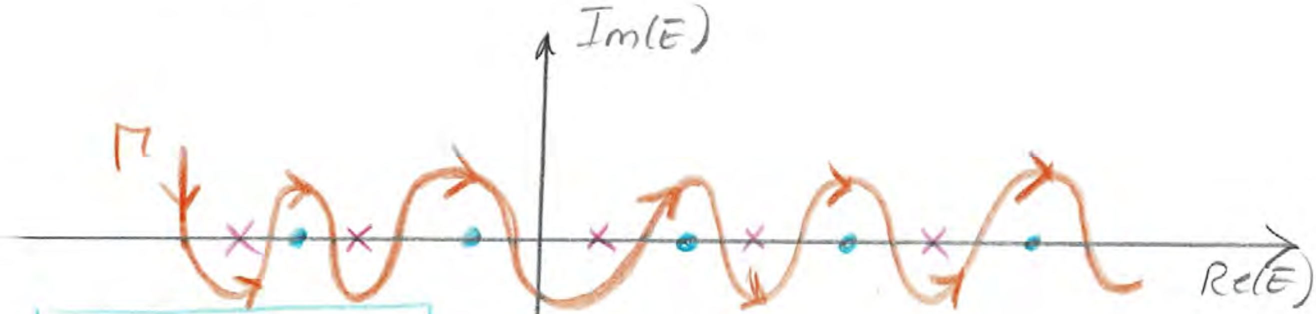
$$
\begin{aligned}
\sum_{n=0}^{\infty} \overline{\psi_n(x)}\psi_n(y) &= \frac{1}{2\pi i}\sum_{n=0}^{\infty} \oint_{\Gamma_n \supset E_n^*} \Big(G_0(x,y|E) \\
&+ \frac{G_0(x,a|E)G_0(a,y|E)}{\Phi(E)}\Big) dE.
\end{aligned}
$$

Note that the total expression in the Krein's formula has only poles at E_n^*'s; however, when we think of it as the sum of two separate expressions, we have the original eigenvalues, E_n, reappearing as poles.

• The contribution coming from the Green's function of the initial Hamiltonian H_0, which is the first term of Krein's formula, in the above contour integral *vanishes since the poles E_n of G_0 are all located outside* of each Γ_n. For simplicity, we assume that all $E_k^* \neq E_k$ from now on. Note that thanks to the denominators we elongate the contours to ellipses that extend to infinity along the imaginary direction (on the complex E-plane). We now continuously deform this contour.



$$\sum_{n=0}^{\infty} \oint_{\Gamma_n} dE \; G(x,y|E) = \sum_{n=0}^{\infty} \psi_n(x)\psi_n^*(y)$$



$\boxed{\text{Drop } G_0(x,y)}$

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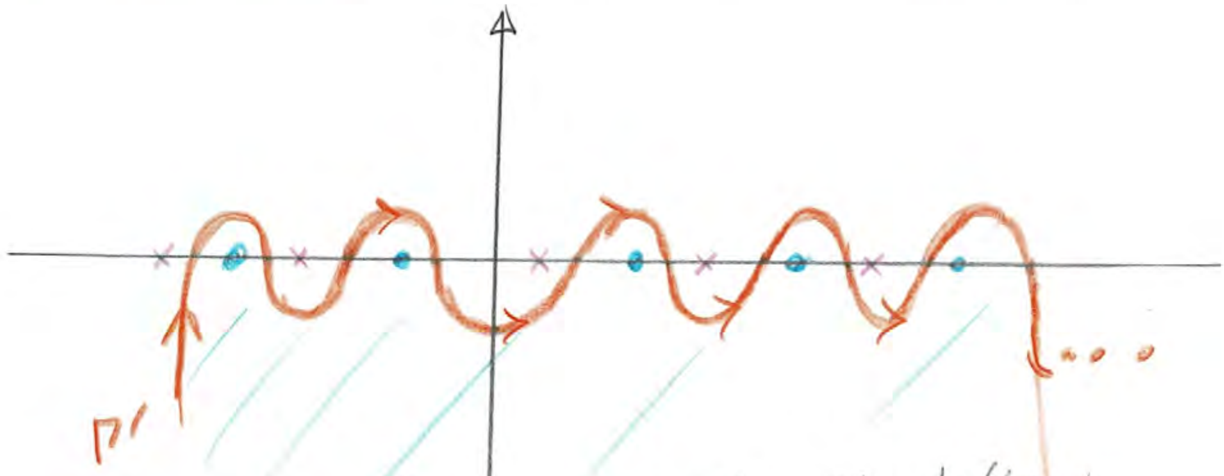
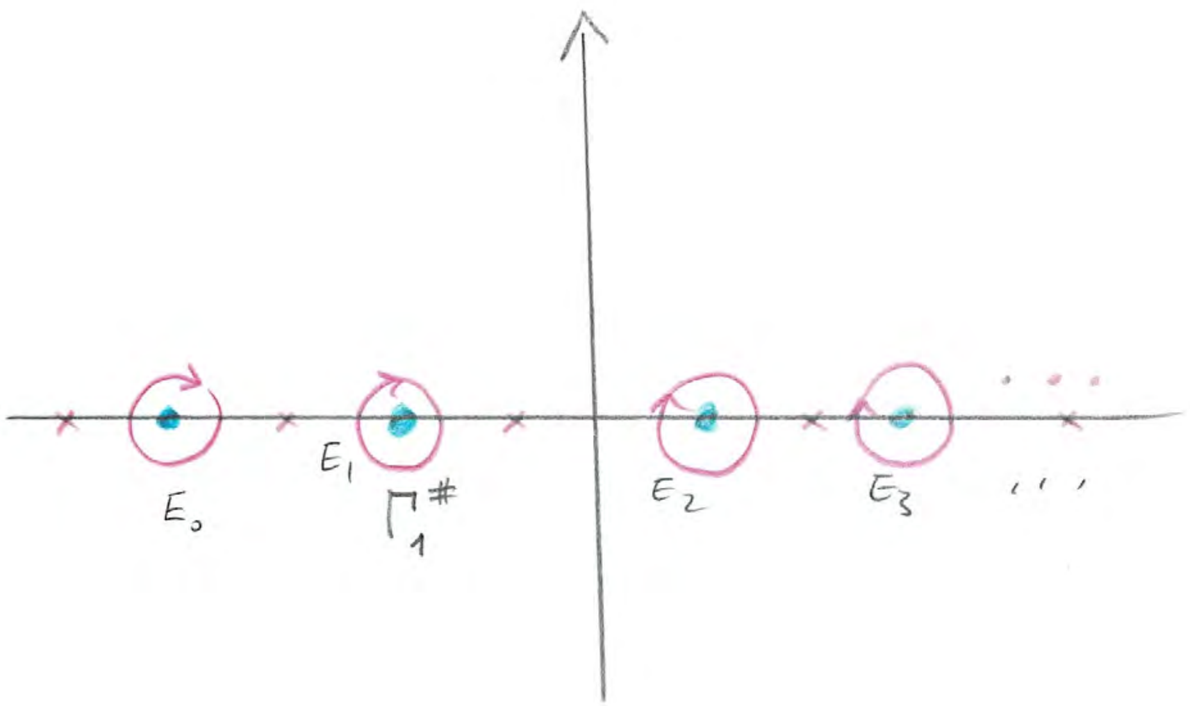


opposite orientation

$$\oint_{\Gamma'} G_0 \, \Phi^{-1} G_0 \, dE$$

• Note that we have *no poles of the Green's function on the left part of the line $E_0^* + i\mathbb{R}$ nor any zeros of* $\Phi(E)$, the product of two Green's functions decay rapidly as $|E| \to \infty$ along the negative real direction as well as along the imaginary directions, hence we have no contributions from the contours at infinity for these deformations. This observation allows us to change the contour as delineated below.

$\sharp$ Using the interlacing theorem, we can, so to speak, flip the contour while preserving the result of integration and then deform the contour to the one $\Gamma^\sharp$ that consists of isolated closed contours $\Gamma_n^\sharp$ around each isolated eigenvalue E_n of the initial Hamiltonian H_0 with opposite orientation, as shown.



E_0
E_1
$\Gamma_1^{\#}$
E_2
E_3

♣ Applying the residue theorem, we obtain

$$\sum_{n=0}^{\infty} \overline{\psi_n(x)}\psi_n(y) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \frac{\phi_n(x)\overline{\phi_n(y)}}{-|\phi_n(a)|^2}[-2\pi i|\phi_n(a)|^2],$$

where the minus sign is due to the opposite orientation of the contour Γ'. Finally (which should be done in a more rigorous way by taking a limit of truncated expressions), we prove

$$\sum_{n=0}^{\infty} \overline{\psi_n(x)}\psi_n(y) = \sum_{n=0}^{\infty} \overline{\phi_n(x)}\phi_n(y) = \delta(x-y).$$

♣ Some Ideas:

• We can write the resulting renormalized Hamiltonian in this basis:

$$H = \sum_{k\notin\mathcal{N}}^{\infty} E_k^* (H_0 - E_k^*)^{-1}|a\rangle \left(\frac{d\Phi(E)}{dE}\bigg|_{E_k^*} \right)^{-1} \langle a|(H_0 - E_k^*)^{-1} + \sum_{k\in\mathcal{N}} E_k|\phi_k\rangle\langle\phi_k|.$$

It is clear that the resulting (renormalized) operator cannot be expressed as a differential operator, but only as an integral operator. Or equivalently,

$$\sum_{k\notin\mathcal{N}}^{\infty} E_k^* \left(\frac{d\Phi(E)}{dE}\bigg|_{E_k^*} \right)^{-1} G_0(x,a|E_k^*)G_0(a,y|E_k^*) + \sum_{k\in\mathcal{N}} E_k\overline{\phi_k(x)}\phi_k(y).$$

Incidentally, the above integral kernel can be utilized to show that the operator H, defined through this kernel, is *essentially self-adjoint*.

♠ Suppose we have a symmetric operator A (physicists call Hermitian) which has a complete set of eigenvectors, then the closure of operator A [that is if we define A on a slightly larger set, by adding all vectors for which A acts continuously to its domain] becomes a self-adjoint operator.

• Remark: The set of functions $G_0(x, a|E_k^*) - G_0(x, a|E_l^*)$ are in the domain of the initial Hamiltonian H_0.

♠ Sudden Approximation: let us suppose that initially the system is prepared in the eigenstate $G_0(x, a|E_k^*(a))$, $E_k^*(a)$ referring to the energy for this case. A sudden perturbation means that the system has no time to readjust itself, so the wave function remains as it is, but should be decomposed in terms of the new eigenbasis $G_0(x, b|E_m^*(b))$'s to calculate the probability of finding the system in the new energy eigenstate $E_m^*(b)$.

This means that the conditional probability of finding the system in $E^*_m(b)$, given that it was in $E^*_k(a)$ initially, is

$$
\begin{aligned}
p(m,b|k,a) &= \left[\frac{d\Phi(E|a)}{dE}\bigg|_{E^*_k} \frac{d\Phi(E|b)}{dE}\bigg|_{E^*_m} \right]^{-1} \times \\
&\left| \int_{\mathcal{M}} d\mu(x) \overline{G_0(x,b|E^*_m(b))} G_0(x,a|E^*_k(a)) \right|^2 \\
&= \left[\frac{d\Phi(E|a)}{dE}\bigg|_{E^*_k} \frac{d\Phi(E|b)}{dE}\bigg|_{E^*_m} \right]^{-1} \times \\
&\left| \frac{G_0(a,b|E^*_m(b)) - G_0(a,b|E^*_k(a))}{E^*_m(b) - E^*_k(a)} \right|^2,
\end{aligned}
$$

where the energy eigenstates $E^*_m(b)$ are found from the solutions of

$$
\Phi(E|b) = \frac{1}{\alpha_R} - \sum_k \frac{|\phi_k(b)|^2 (E + \mu^2)}{(E_k + \mu^2)(E_k - E)} = 0,
$$

whereas $E^*_k(a)$ refers to the zeros of $\Phi(E|a)$.

• It is possible to imagine a sudden change of a and μ_a to b and μ_b, without any difficulty.

• The explicit result presented seems to be suitable for developing a perturbation theory for the time dependent problems (ongoing work with FE and KG).

♠ There are other potential applications of these ideas.

THANKS!