Making sense of Green's function integrals

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$$\int_{\mathbb{R}} e^{iax^2} f(x) dx.$$

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The Schrödinger equation

For some potential V and initial conditions F, we consider

$$\begin{aligned} i\frac{\partial}{\partial t}\Psi(t,x) &= \Big(-\frac{\partial^2}{\partial x^2}+V(t,x)\Big)\Psi(t,x), && t>0,\ x\in\mathbb{R}, \\ \Psi(0,x) &= F(x), && x\in\mathbb{R}. \end{aligned}$$

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We want to solve it using the Green's function integral

$$\Psi(t,x) = \int_{\mathbb{R}} G(t,x,y)F(y)dy.$$

What kind of integrals are we faced with?

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$$G(t,x,y) = -\frac{c}{2}\Lambda\left(\frac{|x|+|y|}{2\sqrt{it}} + c\sqrt{it}\right) e^{\frac{i(|x|+|y|)^2}{4t}}$$ δ-potential $V \sim \delta(x)$

Oscillatory integrals

In all the cases we are faced with oscillatory integrals of the form

$$\int_{\mathbb{R}} e^{iy^2} f(y)dy, \tag{1}$$

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The aim of this talk is now:

- Two methods (with variations) to give meaning to (1).
- Different assumptions on the function f.
- Find absolute convergent representations of (1).

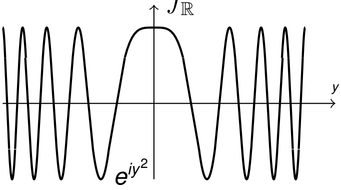
Method I:

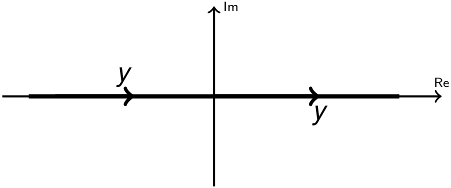
Fresnel integral

Fresnel integral

The idea is to rotate the integration path $\mathbb{R}$ into the complex plane

$$\int_{\mathbb{R}} e^{iy^2} f(y)dy$$





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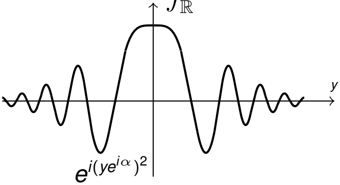

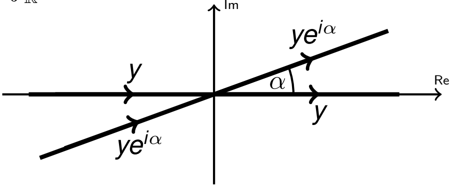

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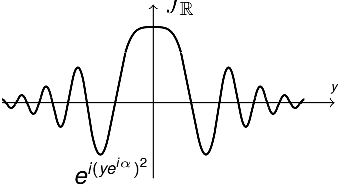

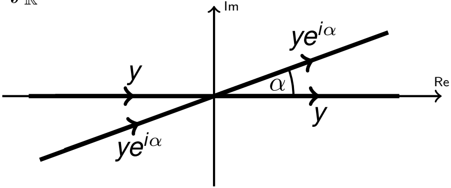

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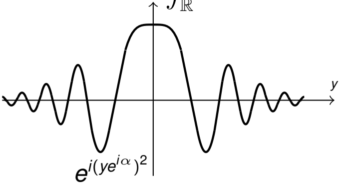

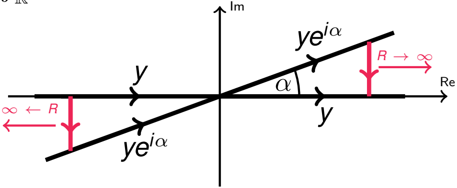

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Strictly speaking this is the Cauchy theorem along a triangle path

$$\int_{-R}^{R} e^{iy^2} f(y)dy = e^{i\alpha} \int_{-R}^{R} e^{i(ye^{i\alpha})^2} f(ye^{i\alpha})dy$$
$$+ \int_{-R}^{-R-iR\tan\alpha} e^{iz^2} f(z)dz + \int_{R}^{R+iR\tan\alpha} e^{iz^2} f(z)dz$$

Assumptions on f

We need the vertical red paths to vanish when $R \to \infty$!

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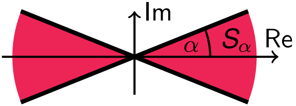
- The function f extends holomorphically to the double sector S_α.



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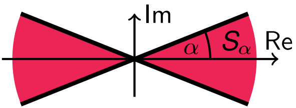

- The function f is exponentially bounded by

$$|f(z)| \leq Ae^{B|\operatorname{Im}(z)|}, \qquad z \in S_\alpha.$$

Then we obtain the convergence

$$\Big|\int_{R}^{R+iR\tan\alpha} e^{iz^2} f(z)dz\Big| \leq A\int_{0}^{R\tan\alpha} e^{-2Ry} e^{By} dy \leq \frac{A}{2R-B} \xrightarrow{R\to\infty} 0.$$

Definition of the Oscillatory integral

Definition 1a) of Oscillatory integral

If f is holomorphic on S_α and bounded by $|f(z)| \leq Ae^{B|\operatorname{Im}(z)|}$, then

$$\int_{\mathbb{R}} e^{iy^2} f(y)dy := \lim_{R\to\infty} \int_{-R}^{R} e^{iy^2} f(y) = \lim_{R\to\infty} e^{i\alpha} \int_{-R}^{R} e^{i(ye^{i\alpha})^2} f(ye^{i\alpha})dy$$

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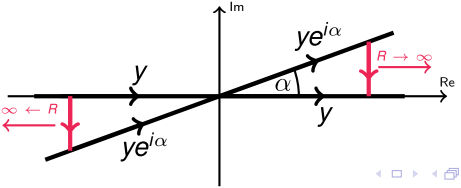

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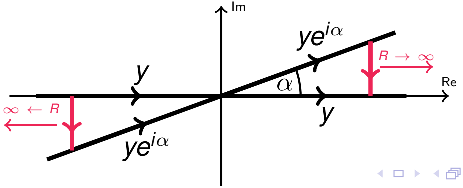

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If we now assume

$$|f(z)| \leq Ae^{B|z|}, \qquad z \in S_\alpha,$$

there vanishes the terms

$$\left| \int_{R}^{R+iR\tan\alpha} e^{-\varepsilon z^2} e^{iz^2} f(z)dz \right| \leq Ae^{-\varepsilon R^2} \int_0^R e^{\varepsilon y^2 - 2Ry} e^{B|R+iy|} dy \xrightarrow{R\to\infty} 0.$$

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P. S., *Time evolution of superoscillations for the Schrödinger equation on* $\mathbb{R} \setminus \{0\}$, Quantum Stud. Math. Found. **9** (2022).

Method II:

Integration by parts

The basic formula

The main idea is to use the integration by parts formula

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$$\int_{\mathbb{R}} e^{iy^2} f(y) dy = \frac{1}{2i} \int_{\mathbb{R}} \frac{d}{dy}\left(e^{iy^2}\right) \frac{f(y)}{y} dy$$

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The following diagram visualizes this method.

$$\begin{tikzcd} f \arrow[r] \arrow[d] & \frac{f'}{y} \\ \frac{f}{y^2} & \end{tikzcd}$$

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The following diagram visualizes this method.

$$\begin{array}{ccc} f & \longrightarrow & \frac{f'}{y} \\ \downarrow & & \\ \frac{f}{y^2} & & \end{array}$$

Instead of f, we have to integrate the functions $\frac{f'}{y}$ and $\frac{f}{y^2}$, with the additional decaying factors $\frac{1}{y}$ and $\frac{1}{y^2}$.

Inductively applying this formula

$$
\begin{array}{ccc}
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\downarrow & & \\
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\end{array}
$$

Inductively applying this formula

$$\begin{array}{ccccc} f & \longrightarrow & \frac{f'}{y} & \longrightarrow & \frac{f''}{y} \\ \downarrow & & \downarrow & & \\ \frac{f}{y^2} & & \frac{f'}{y^3} & & \end{array}$$

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$$\begin{array}{ccccccc}
f & \longrightarrow & \dfrac{f'}{y} & \longrightarrow \cdots \longrightarrow & \dfrac{f^{(n-1)}}{y^{n-1}} & \longrightarrow & \color{red}{\dfrac{f^{(n)}}{y^{n}}} \\
\downarrow & & \downarrow & & \downarrow & & \\
\color{red}{\dfrac{f}{y^2}} & & \color{red}{\dfrac{f'}{y^3}} & \cdots & \color{red}{\dfrac{f^{(n-1)}}{y^{n+1}}} & &
\end{array}$$

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$$
\begin{tikzcd}
f \arrow[r] \arrow[d] & \frac{f'}{y} \arrow[r] \arrow[d] & \cdots \arrow[r] & \frac{f^{(n-1)}}{y^{n-1}} \arrow[r] \arrow[d] & \frac{f^{(n)}}{y^{n}} \\
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\frac{f}{y^4} & & & &
\end{tikzcd}
$$

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\downarrow & & \downarrow & & \downarrow & & \\
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The integrals on the right hand side of

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Then the estimates of the lower derivatives follow automatically

$$f^{(k)}(y) = \mathcal{O}(|y|^{n-1-\delta+(n-k)}), \quad \text{as } |y| \to \infty, \ k \in \{0, \dots, n-1\}.$$

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Definition 2) Oscillatory integral

If $f \in C^n(\mathbb{R})$ and $f^{(n)}(y) = \mathcal{O}(|y|^{n-1-\delta})$ as $|y| \to \infty$, then

$$\begin{aligned} \int_{\mathbb{R}} e^{iy^2} f(y) dy &:= \lim_{\varepsilon \to 0+} \int_{\mathbb{R}} e^{-\varepsilon y^2} e^{iy^2} f(y) dy \\ &= \sum_{k=0}^{m} C_k \int_{\mathbb{R}} e^{iy^2} \frac{f^{(n)}(y)}{y^{n+2k}} dy + \sum_{k=0}^{n-1} D_k \int_{\mathbb{R}} e^{iy^2} \frac{f^{(k)}(y)}{y^{k+2+2m}} dy \end{aligned}$$

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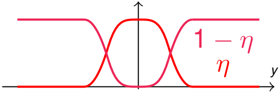

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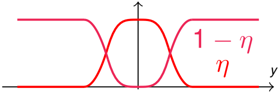

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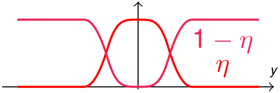

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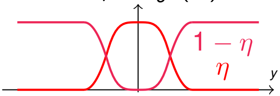

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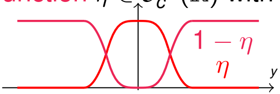

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J. Behrndt, P. S.: *On a class of oscillatory integrals and their application to the time dependent Schrödinger equation*. J. Math. Anal. and Appl. **543** (2025).

Summary

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- At $y = 0$: Splitting up the integral or multiplying cutoff

Application:

Superoscillations

The standard superoscillatory function

For $a > 1$ and $n \in \mathbb{N}$ consider

$$F_n(x) = \left(\cos\left(\frac{x}{n}\right) + ia\sin\left(\frac{x}{n}\right)\right)^n$$

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Moreover, for $\frac{|x|}{n} \ll 1$ these functions approximately behave as

$$F_n(x) \approx \left(1 + ia\frac{x}{n}\right)^n \approx e^{iax}.$$

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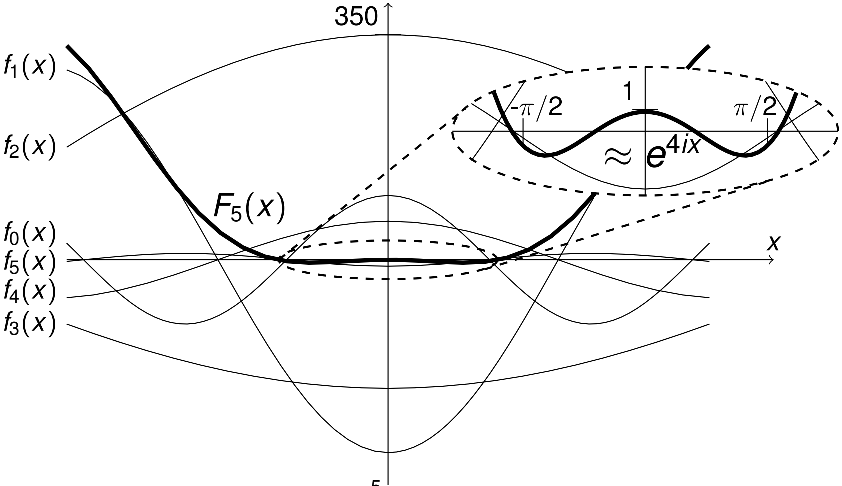
In particular one can show that there converges

$$\lim_{n\to\infty} F_n(x) = e^{iax}.$$



$$F_5(x) = \sum_{j=0}^{5} \underbrace{C_j(5) e^{ik_j(5)x}}_{f_j(x)}.$$

Time evolution of superoscillations

Consider superoscillating initial conditions F_n of the time dependent Schrödinger equation

$$\begin{aligned} i\frac{\partial}{\partial t}\Psi(t,x) &= \Big(-\frac{\partial^2}{\partial x^2}+V(t,x)\Big)\Psi(t,x), && t>0,\ x\in\mathbb{R}, \\ \Psi(0,x) &= F_n(x), && x\in\mathbb{R}, \end{aligned}$$

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\lim_{n\to\infty}\Psi(t,x;F_n) &= \lim_{n\to\infty}\int_{\mathbb{R}} G(t,x,y)F_n(y)dy\\
&= \int_{\mathbb{R}} G(t,x,y)\lim_{n\to\infty}F_n(y)dy\\
&= \int_{\mathbb{R}} G(t,x,y)e^{iay}dy = \Psi(t,x;e^{ia\cdot}).
\end{aligned}
\tag{2}
$$

Time evolution of superoscillations

Consider superoscillating initial conditions F_n of the time dependent Schrödinger equation

$$\begin{aligned} i\frac{\partial}{\partial t}\Psi(t,x) &= \Big(-\frac{\partial^2}{\partial x^2} + V(t,x)\Big)\Psi(t,x), && t>0,\ x\in\mathbb{R},\\ \Psi(0,x) &= F_n(x), && x\in\mathbb{R}, \end{aligned}$$

Is $\Psi_n(t,x;F_n)$ still superoscillating for $t>0$?

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- The previous theory allows us to interchange the limit in (2).
- Oscillatory behaviour of $\Psi(t,x;e^{ia\cdot})$ needs to be investigated.

Open questions

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Thank you for your attention!