Magnetic ground states and the conformal structure of a surface

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- Examples of isospectral, non-isometric planar domains (Gordon, Webb, Wolpert)
- $\ldots$

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However the Laplacian spectrum tells us: the volume, the boundary measure, the dimension, the Euler characteristic for surfaces, etc.

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Remark: if $A = 0$, $d^A = d$ and $\delta^A = \delta$, hence $\Delta_A = \Delta$ is the Laplacian (in $\mathbb{R}^2$, $\delta = -\mathrm{div}$ and hence $\Delta = -\partial_{xx}^2 - \partial_{yy}^2$).

The magnetic Laplacian: spectrum and gauge invariance

Fix g (hence Σ_g). Given a metric h and a closed 1-form A, we have that Δ_A has discrete spectrum $\{\lambda_n(h, A)\}_{n=1}^{\infty}$: $\Delta_A u_n = \lambda_n(h, A) u_n$.

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Therefore we can confine to harmonic 1-forms (for the metric h).

A first inequality

Let $Har(h)$ be the space of h-harmonic 1-forms on Σ_g: Euclidean space of dimension $2g$ with the L^2-norm $\|\cdot\|_h$ on forms.

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$$L^\star = \left\{ A \in Har(h) : \frac{1}{2\pi} \oint_\gamma A \in \mathbb{Z} \,, \forall \gamma \text{ closed curve} \right\}$$

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be the lattice of harmonic 1-forms with integral flux. For $A \in Har(h)$ we define

$$d_h(A, L^\star) := \min\{\|\omega - A\|_h : \omega \in L^\star\}.$$

This is the L^2-distance of A from the lattice $L^\star$.

A first inequality

Theorem 1

$$|h|\lambda_1(h, A) \leq d_h(A, L^\star)^2$$

where $|h|$ denotes the volume of Σ_g with the metric h.

Remark: $|h|\lambda_1(h, A)$ is the normalized eigenvalue, scale invariant.

Some conformal spectral invariants

Let h be a metric on Σ_g and A be a h-harmonic 1-form. We define

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Finally, we define

$$\Lambda_1 = \inf_{[h]} \Lambda_1([h]).$$

This last invariant depends only on the genus g of the surface.

Conformal invariants and the Jacobian torus

All these invariants can be estimated by the geometry of the Jacobian torus of (Σ_g, h):

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- It is a $2g$-dimensional flat torus.
- It is conformally invariant: two conformal metrics h, h' on Σ_g have the same Jacobian torus.

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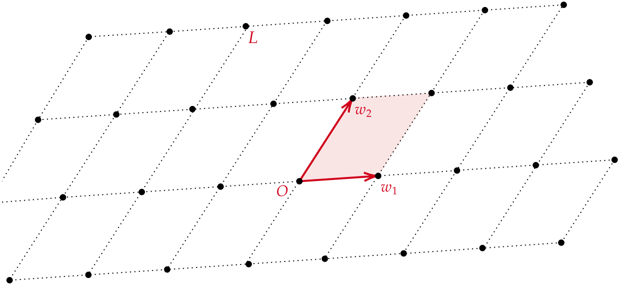
A flat torus is the quotient of $\mathbb{R}^2/L$ by a lattice $L = \ell\, w_1 + m\, w_2$, $\ell, m \in \mathbb{Z}$, with the inherited Euclidean metric.



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Given a metric h on Σ_1 we have

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$$\Lambda_1([h], A) = |\hat{h}|\lambda_1(\hat{h}, A).$$

So we can compute the conformal invariants if we understand the spectrum of flat tori.

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Consider then the flat torus $\mathbb{R}^2/L$ with basis (w_1, w_2).

- The dual lattice $L^\star$ of L is

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- The first eigenvalue is

$$\lambda_1(\hat{h}, A) = \min_{w^\star \in L^\star} |P_A - w^\star|^2.$$

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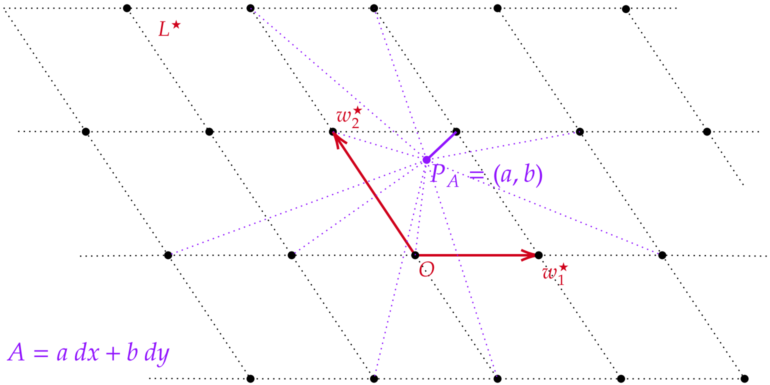

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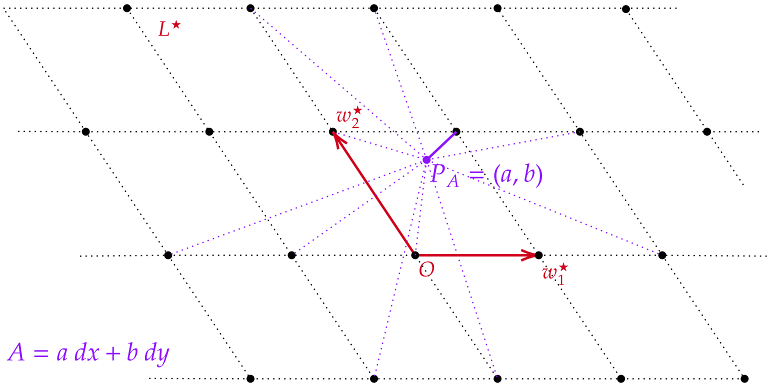

Remark: The dual lattice $L^\star$ is identified with the lattice of harmonic 1-forms with integral fluxes: $\mathbb{R}^2/L^\star = Jac(\hat{h})$ with a homotetic metric of factor $|\hat{h}|$.

Computations in genus $g = 1$

We go back to our spectral invariants:

$$\Lambda_1([h], A) = \sup_{h' \in [h]} |h'| \lambda_1(h', A)$$

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- Conclusion: the first invariant “hears” the distance of a harmonic 1-form from the lattice of 1-forms with integral flux in a given conformal class $[h]$.

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- Conclusion: the second invariant “hears” the diameter of the Jacobian torus of $[h]$.

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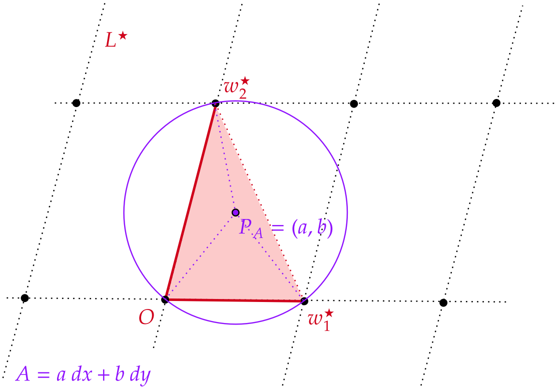

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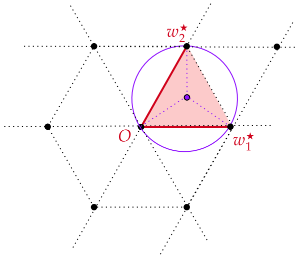

- Λ_1 identifies the conformal class of the equilateral flat torus.

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- Do the three invariants "hear" the same information on the Jacobian torus as in $g = 1$?

Can we hear the conformal class of a surface?

We prove the following:

Theorem 4

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where A is the unique harmonic form for h co-homologous to ω.

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Our main result is the following

Theorem 5

Assume that the metrics h_1, h_2 on Σ_g are such that

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We deduce the following corollary

Corollary

Assume that the metrics in Theorem 5 are hyperbolic. Then they are isometric. For $g = 1$, the statement holds for flat metrics.

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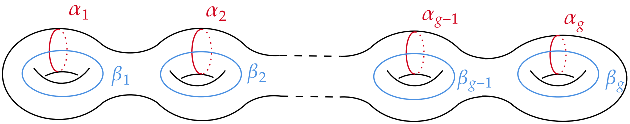

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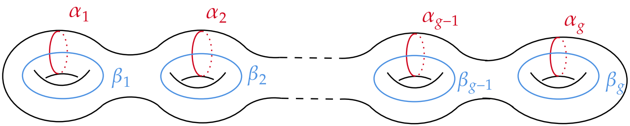

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Can we hear the conformal class of a surface?

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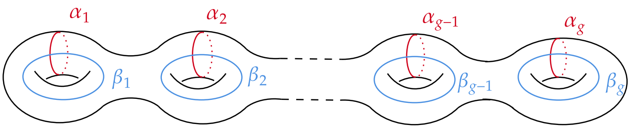

Let $\omega_{i,j}^n = \frac{\omega_i + \omega_j}{n}$. If two metrics h_1, h_2 on Σ_g satisfy

$$\boxed{\lambda_1\left(h_1, \omega_{i,j}^n\right) = \lambda_1\left(h_2, \omega_{i,j}^n\right), \quad \forall n \geq 1, 1 \leq i \leq j \leq 2g,}$$

then they are conformal and have the same volume.

A few words on the proof

- Starting point: given a closed 1-form ω, $\lim_{t\to 0} \frac{\lambda_1(h,t\omega)}{t^2} = \frac{\|A\|_h^2}{|h|}$, where A is h-harmonic co-homologous to ω:

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- If two metrics are "ground state isospectral", they have the same Gram matrix and the same volume by the three points above.
- We prove that if h_1, h_2 have the same Gram matrix, they are conformal. Here we use a deep result in Riemann surfaces theory, the Torelli's Theorem.

Thank you for your attention

B. Colbois, L. Provenzano and A. Savo,
Magnetic ground states and the conformal class of a surface,
preprint (2025).