Spectral properties of electromagnetic waveguides

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Joint work with

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A curve in $\mathbb{R}^3$

Let us consider a curve $\Gamma \subset \mathbb{R}^3$ parametrized by $\gamma : \mathbb{R} \to \mathbb{R}^3$ such that:

- $\gamma : \mathbb{R} \to \mathbb{R}^3$ is of class $\mathscr{C}^2(\mathbb{R})$,
- γ is an arc-length parametrization : $\forall s \in \mathbb{R}, \quad |\gamma'(s)| = 1$,

Assumption 1: The curvature of Γ at a point $\gamma(s)$ is defined as $\kappa(s) = |\gamma''(s)|$ and verifies

$$\boxed{\lim_{|s| \to +\infty} \kappa(s) = 0.}$$

To construct the waveguide we need:

- a frame adapted to the curve Γ,
- a bounded, simply connected, Lipschitz domain $\omega \subset \mathbb{R}^2$.

There is more than one way to frame a curve ! (1/2)

A standard choice is the Frenet frame defined at a point $\gamma(s) \in \Gamma$ as

$$\begin{cases} e_1(s) &= \gamma'(s), \\ e_2(s) &= \frac{1}{\kappa(s)}\gamma''(s), \\ e_3(s) &= e_1(s) \times e_2(s). \end{cases}$$

The frame is adapted because each derivative $\frac{d}{ds} e_j \in \operatorname{span}(e_k, e_p)$ with $p, k \neq j$.

$$\begin{cases} \frac{d}{ds} e_1(s) &= \kappa(s) e_2(s), \\ \frac{d}{ds} e_2(s) &= -\kappa(s) e_1(s) + \tau(s) e_3(s), \\ \frac{d}{ds} e_3(s) &= -\tau(s) e_2(s). \end{cases}$$

Here $\tau(s)$ is the torsion of the curve Γ at the point $\gamma(s)$.

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Here $\tau(s)$ is the torsion of the curve Γ at the point $\gamma(s)$.

$\boxed{\textit{Problem}}$: when $\kappa(s) = 0$ this is ill-defined.

In particular if the curve contains part of a straight line one cannot define this frame.

There is more than one way to frame a curve ! (2/2)

Instead, we fix $e_1(s) = \gamma'(s)$ and construct a relatively adapted parallel frame (e_1, e_2, e_3) which always exists and verify

$$\left\{\begin{array}{rcl} \frac{d}{ds} e_1(s) & = & k_1(s) e_2(s) + k_2(s) e_3(s), \\ \frac{d}{ds} e_2(s) & = & -k_1(s) e_1(s), \\ \frac{d}{ds} e_3(s) & = & -k_2(s) e_1(s). \end{array}\right.$$

with $k_1, k_2 \in \mathscr{C}^0(\mathbb{R})$ such that $k_1(s)^2 + k_2(s)^2(s) = \kappa(s)^2$.

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$\boxed{\text{Relatively parallel}}$:

the derivatives of e_2 and e_3 have a component only on e_1.

The existence of such a frame can be found in

R. L. Bishop
Am. Math. Mon. (1975)

Such a frame is unique up to fixed rotations in the plane $\mathrm{span}(e_2, e_3)$.

Construction of the waveguide (1/2)

Take $\omega \subset \mathbb{R}^2$ a Lipschitz and simply connected domain and consider the map

$$\Phi : \left\{ \begin{array}{ccc} \Omega_0 := \mathbb{R} \times \omega & \rightarrow & \mathbb{R}^3 \\ (s, y_2, y_3) & \mapsto & \gamma(s) + y_2 e_2(s) + y_3 e_3(s). \end{array} \right.$$

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Assumption 2:

$$\boxed{\left(\sup_{y \in \omega} |y|\right) \|\kappa\|_{L^\infty(\mathbb{R})} < 1.}$$

Under this condition Φ is a $\mathscr{C}^1$-diffeomorphism from Ω_0 to its image.

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Problem : as constructed the cross-section ω is not allowed to turn around the curve.

Construction of the waveguide (2/2)

To remedy to this problem, let $\theta : \mathbb{R} \to \mathbb{R}$ be of class $\mathscr{C}^1(\mathbb{R})$ and define for $p \in \{2, 3\}$:

$$e_p^{\theta}(s) := \begin{pmatrix} \cos(\theta(s)) & -\sin(\theta(s)) \\ \sin(\theta(s)) & \cos(\theta(s)) \end{pmatrix} e_p(s).$$

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- θ' is called the twist: it is the rotation speed of (e_2^θ, e_3^θ) in the plane $\mathrm{span}(e_1)^\perp$.

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Φ is a $\mathscr{C}^1(\mathbb{R})$ diffeomorphism and we set

$$\Omega := \Phi(\mathbb{R} \times \omega).$$

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$$\Omega := \Phi(\mathbb{R} \times \omega).$$

Assumption 3:

$$\boxed{\lim_{|s| \to +\infty} \theta'(s) = \beta, \quad \beta \in \mathbb{R} \quad \text{(constant twist at infinity).}}$$

Some pictures !

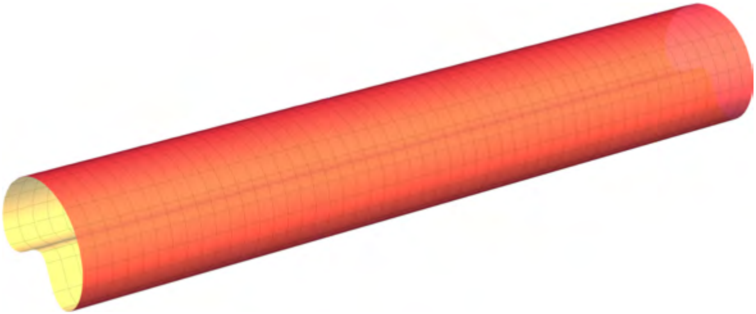

Figure: A straight waveguide $\Omega_0 := \mathbb{R} \times \omega$.

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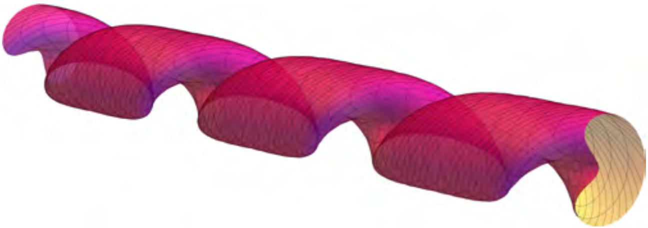

Figure: A constantly twisted waveguide.

Some pictures !

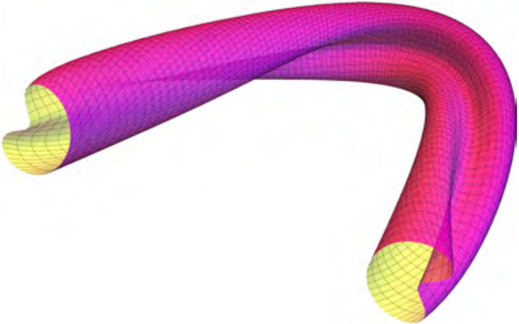

Figure: A curved and twisted waveguide.

Physical context and Mathematical challenges

Goal: Define the Maxwell operator in a waveguide Ω built with a homogeneous and isotropic material and embedded in a perfect conductor.

- $\varepsilon_0 > 0$: permittivity of the material,
- $\mu_0 > 0$: permeability of the material,
- E : electric field,
- H : magnetic field
- $\mathsf{E} \times \mathsf{n} = 0$ and $\mathsf{H} \cdot \mathsf{n} = 0$ on $\partial\Omega$ (perfectly conducting boundary condition).

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The waveguide $\Omega \subset \mathbb{R}^3$ is an unbounded Lipschitz domain:

Problem: how to define (and in which sense) the boundary condition to obtain a self-adjoint Maxwell operator.

Minimal and maximal curl operators

1. Consider the curl operator acting in $L^2(\Omega, \mathbb{C}^3)$ defined as

$$S_0 := \mathrm{curl}, \quad \mathrm{Dom}(S_0) = \mathscr{C}_0^\infty(\Omega, \mathbb{C}^3).$$

2. As S_0 is symmetric (thus closable), one can define the minimal curl operator $S = (S_0^*)^* = \overline{S_0}$ acting as

$$S = \mathrm{curl}, \quad \mathrm{Dom}(S) := H_0(\mathrm{curl}, \Omega).$$

   By definition $H_0(\mathrm{curl}, \Omega)$ is the closure of $\mathscr{C}_0^\infty(\Omega, \mathbb{C}^3)$ with respect to the graph norm for the curl operator.

3. Remark that the maximal curl operator is S^* and acts as

$$S^* = \mathrm{curl}, \quad \mathrm{Dom}(S^*) = H(\mathrm{curl}, \Omega) := \left\{ \mathrm{F} \in L^2(\Omega, \mathbb{C}^3) : \mathrm{curl}\, \mathrm{F} \in L^2(\Omega, \mathbb{C}^3) \right\}.$$

4. The operator with domain $H_0(\mathrm{curl}, \Omega) \times H(\mathrm{curl}, \Omega) \subset L^2(\Omega, \mathbb{C}^3 \times \mathbb{C}^3)$ acting as

$$\begin{pmatrix} 0 & \mathrm{curl} \\ \mathrm{curl} & 0 \end{pmatrix} = \begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix},$$

   is self-adjoint by construction.

The boundary condition (electric field)

- If $F_1, F_2 \in \mathscr{C}^\infty(\overline{\Omega}, \mathbb{R}^3)$:

$$\int_\Omega ((\operatorname{curl} F_1)\cdot F_2)\,dx = \int_\Omega (F_1\cdot(\operatorname{curl} F_2))\,dx + \int_{\partial\Omega} (F_1 \times n)\cdot F_2\,d\sigma$$

- $F_1 \in H_0(\operatorname{curl}, \Omega)$ if and only if for all $F_2 \in H(\operatorname{curl}, \Omega)$ there holds

$$\int_\Omega ((\operatorname{curl} F_1)\cdot F_2)\,dx = \int_\Omega (F_1\cdot(\operatorname{curl} F_2))\,dx.$$

Hence, by $F_1 \times n = 0$ on $\partial\Omega$, we mean that $F_1 \in H_0(\operatorname{curl}, \Omega)$.

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Physical motivations

When there is an (exterior) current density $\mathscr{J}$ the time-dependent Maxwell's equations in the waveguide Ω read

$$\begin{cases} \varepsilon_0 \partial_t \mathrm{E}(x,t) - \operatorname{curl} \mathrm{H}(x,t) = -\mathscr{J}(x,t) & \text{for} \quad (x,t) \in \Omega \times \mathbb{R}_+^*, \\ \mu_0 \partial_t \mathrm{H}(x,t) + \operatorname{curl} \mathrm{E}(x,t) = 0 & \text{for} \quad (x,t) \in \Omega \times \mathbb{R}_+^*, \\ \operatorname{div}(\varepsilon_0 \mathrm{E})(x,t) = \operatorname{div}(\mu_0 \mathrm{H})(x,t) = 0 & \text{for} \quad (x,t) \in \Omega \times \mathbb{R}_+^*, \\ \mathrm{E}(x,t) \times \mathrm{n}(x) = 0 & \text{for} \quad (x,t) \in \partial\Omega \times \mathbb{R}_+^*, \\ \mathrm{H}(x,t) \cdot \mathrm{n}(x) = 0 & \text{for} \quad (x,t) \in \partial\Omega \times \mathbb{R}_+^*, \\ (E(\cdot,0), H(\cdot,0)) = (E_0, H_0) & \text{with} \quad (E_0, H_0) \in L^2(\Omega, \mathbb{R}^3 \times \mathbb{R}^3). \end{cases}$$

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\mathrm{E}(x,t)\times\mathrm{n}(x) = 0 & \text{for} & (x,t)\in\partial\Omega\times\mathbb{R}_+^* \\
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\end{array}\right.$$

Physical motivations

It can be rewritten as a Schrödinger type equation

$$\begin{cases} \partial_t \psi + i\mathscr{A}\psi = f & \text{for all} \quad t > 0, \\ \psi(0) = \psi_0 & \text{initial data in} \quad L^2(\Omega, \mathbb{R}^3 \times \mathbb{R}^3). \end{cases}$$

Here $\psi = (\mathrm{E}, \mathrm{H})^\top \in H_0(\mathrm{curl}, \Omega) \times H(\mathrm{curl}, \Omega)$,

- $\mathscr{A}$ is the self-adjoint Maxwell operator

$$\mathscr{A} := i \begin{pmatrix} 0 & \varepsilon_0^{-1}\,\mathrm{curl} \\ -\mu_0^{-1}\,\mathrm{curl} & 0 \end{pmatrix}, \quad \mathrm{Dom}(\mathscr{A}) = H_0(\mathrm{curl}, \Omega) \times H(\mathrm{curl}, \Omega).$$

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- It acts in $L^2_{\varepsilon_0,\mu_0}(\Omega, \mathbb{R}^3 \times \mathbb{R}^3)$ the space of square integrable vector fields. The scalar product is defined as

$$\left\langle \begin{pmatrix} \mathrm{E}_1 \\ \mathrm{H}_1 \end{pmatrix}, \begin{pmatrix} \mathrm{E}_2 \\ \mathrm{H}_2 \end{pmatrix} \right\rangle_{L^2_{\varepsilon_0,\mu_0}(\Omega,\mathbb{R}^3\times\mathbb{R}^3)} = \int_\Omega (\varepsilon_0 \mathrm{E}_1 \cdot \mathrm{E}_2 + \mu_0 \mathrm{H}_1 \cdot \mathrm{H}_2)\, dx.$$

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Motivation:

$$\boxed{\text{The spectrum of } \mathscr{A} \text{ gives the dynamics of the Maxwell system.}}$$

Divergence free fields and the magnetic boundary condition

To look for divergence-free fields we introduce the solenoidal subspace

$$\mathscr{J}(\Omega) := \left\{ (\mathrm{E}, \mathrm{H}) \in L^2(\Omega, \mathbb{R}^3 \times \mathbb{R}^3) : \mathrm{div}(\varepsilon_0 \mathrm{E}) = \mathrm{div}(\mu_0 \mathrm{H}) = 0, (\mathrm{H} \cdot \mathrm{n})|_{\partial\Omega} = 0 \right\}.$$

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Here $(\mathrm{H}\cdot\mathrm{n})|_{\partial\Omega} = 0$ means that for all $\varphi \in H^1(\Omega)$ there holds

$$0 = \int_\Omega \mathrm{div}(\mathrm{H})\varphi dx = -\int_\Omega \mathrm{H}\cdot(\nabla\varphi) dx.$$

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Lemma

There holds $\ker(\mathscr{A})^\perp = \mathscr{J}(\Omega)$. In particular, $\mathscr{J}(\Omega)$ is a closed subspace left stable by $\mathscr{A}$.

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Essential spectrum

Recall that we have assumed

$$\boxed{\lim_{|s|\to+\infty} \kappa(s) = 0, \qquad \lim_{|s|\to+\infty} \theta'(s) = \beta, \qquad \beta \in \mathbb{R}.}$$

Theorem (structure of the essential spectrum)

There exists $a_\beta > 0$ such that

$$Sp_{\text{ess}}(\mathscr{A}) = (-\infty, -a_\beta] \cup \{0\} \cup [a_\beta, +\infty).$$

If $\beta = 0$, one has $a_\beta = \sqrt{\frac{\lambda_2^N(\omega)}{\varepsilon_0 \mu_0}}$.

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$\boxed{\text{Problem}}$: Focus on $\mathscr{A}_\perp$ and we need to get rid of 0 !

Essential spectrum (2/2)

Assume further

$$\theta' = 0 \quad \text{(no twist)}, \quad \left(\sup_{y\in\omega}|y|\right)\|\kappa\|_{L^\infty(\mathbb{R})} < \frac{1}{2}.$$

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Proposition

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The key-point is a Poincaré-type inequality of the form:

$$\|\mathrm{E}\|_{L^2_{\varepsilon_0}(\Omega)} + \|\mathrm{H}\|_{L^2_{\mu_0}(\Omega)} \leq C\big(\|\mu_0^{-1}\operatorname{curl}\mathrm{E}\|_{L^2_{\mu_0}(\Omega)} + \|\varepsilon_0^{-1}\operatorname{curl}\mathrm{H}\|_{L^2_{\varepsilon_0}(\Omega)}\big).$$

for some constant $C > 0$ uniform in $(\mathrm{E},\mathrm{H})^\top \in \mathscr{J}(\Omega) \cap \mathrm{Dom}(\mathscr{A})$.

Essential spectrum (2/2)

Assume further

$$\theta' = 0 \quad \text{(no twist)}, \qquad \left(\sup_{y\in\omega}|y|\right)\|\kappa\|_{L^\infty(\mathbb{R})} < \frac{1}{2}.$$

Proposition

There holds

$$\sigma_{\mathrm{ess}}(\mathscr{A}_\perp) = \left(-\infty, -\sqrt{\frac{\lambda_2^N(\omega)}{\varepsilon_0\mu_0}}\right] \cup \left[-\sqrt{\frac{\lambda_2^N(\omega)}{\varepsilon_0\mu_0}}, +\infty\right).$$

Should be true with non-zero twist and a condition involving $\|\theta' - \beta\|_{L^\infty(\mathbb{R})}$ but ask for tedious computations...

Discrete spectrum

- ψ is a normalized eigenmode associated with $\lambda_2^N(\omega)$, the first non-trivial Neumann eigenvalue in ω.

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Theorem (existence of discrete spectrum)

Assume $\theta' = 0$ (no twist) and that the Poincaré inequality holds. Let $\kappa \in L^1(\mathbb{R})$ be such that

$$X \cdot \left(\int_{\mathbb{R}} k^\theta(s) ds \right) > \frac{2\lambda_2^N(\omega)}{\varepsilon_0} \frac{b^2 \|\kappa\|_{L^\infty(\mathbb{R})}}{1 - b\|\kappa\|_{L^\infty(\mathbb{R})}} \|\kappa\|_{L^1(\mathbb{R})}$$

then

$$\sigma_{\mathrm{dis}}(\mathscr{A}) \neq \emptyset.$$

Here $b := \sup_{y \in \Omega}(|y|)$.

Some remarks on the geometric condition

Geometric condition:

$$\boxed{X \cdot \left(\int_{\mathbb{R}} k^{\theta}(s) ds \right) > \frac{2\lambda_2^N(\omega)}{\varepsilon_0} \frac{b^2 \|\kappa\|_{L^\infty(\mathbb{R})}}{1 - b\|\kappa\|_{L^\infty(\mathbb{R})}} \|\kappa\|_{L^1(\mathbb{R})}}$$

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- In this case, as θ is a constant, there is an optimal choice of $\theta_\star$ maximize the quantity

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$$X \cdot \left(\int_{\mathbb{R}} k^{\theta_\star}(s)ds \right) > 0 \quad \text{for } \theta = \theta_\star.$$

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- It means that there is an optimal way to attache the cross-section ω to the curve Γ.

Slightly curved waveguides

Geometric condition:

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$$\gamma_\delta : \mathbb{R} \to \mathbb{R}^3, \quad \gamma_\delta(s) = \frac{1}{\delta} \gamma(\delta s).$$

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Geometric condition:

$$\boxed{X \cdot \left(\int_{\mathbb{R}} k_\delta^{\theta\star}(s) ds \right) > \frac{2\lambda_2^N(\omega)}{\varepsilon_0} \frac{b^2 \|\kappa_\delta\|_{L^\infty(\mathbb{R})}}{1 - b\|\kappa_\delta\|_{L^\infty(\mathbb{R})}} \|\kappa_\delta\|_{L^1(\mathbb{R})}}$$

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When $\delta \to 0$, the condition is satisfied.

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Given a curve for which

- $\int_{\mathbb{R}} \begin{pmatrix} k_1(s) \\ k_2(s) \end{pmatrix} ds \neq 0$,
- a cross-section ω for which $X \neq 0$,

one can always build a waveguide which traps electromagnetic modes.

(Unitary) Piola transform

$\boxed{\text{Key-point}}$: Rewrite the problem in a straight waveguide $\Omega_0 = \mathbb{R} \times \omega$.

$$\mathbb{U} : L^2_{\varepsilon_0,\mu_0}(\Omega) \to L^2_{\varepsilon,\mu}(\Omega_0), \quad \begin{pmatrix} \mathsf{E} \\ \mathsf{H} \end{pmatrix} \mapsto \begin{pmatrix} J_\Phi^\top (\mathsf{E} \circ \Phi) \\ J_\Phi^\top (\mathsf{H} \circ \Phi) \end{pmatrix}$$

$$\text{where} \quad \varepsilon = \varepsilon_0 \mathbb{G}^{-1}, \quad \mu = \mu_0 \mathbb{G}^{-1} \text{ with } \mathbb{G} = \frac{J_\Phi J_\Phi^\top}{\det J_\Phi}.$$

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Unitary equivalent operator $\widehat{\mathscr{A}} := \mathbb{U} \mathscr{A} \mathbb{U}^*$ acting in $L^2_{\varepsilon,\mu}(\Omega_0) \to L^2_{\varepsilon,\mu}(\Omega_0)$.

$$\mathscr{A} = \begin{pmatrix} 0 & i\varepsilon_0^{-1} \operatorname{curl} \\ -i\mu_0^{-1} \operatorname{curl} & 0 \end{pmatrix} \qquad \widehat{\mathscr{A}} = \begin{pmatrix} 0 & i\varepsilon^{-1} \operatorname{curl} \\ -i\mu^{-1} \operatorname{curl} & 0 \end{pmatrix}$$

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The Piola transform preserves the Maxwell structure:

- $\operatorname{Dom}(\widehat{A}) = H_0(\operatorname{curl}, \Omega_0) \times H(\operatorname{curl}, \Omega_0)$,
- $\mathbb{U} \mathscr{J}(\Omega) = \mathscr{J}_{\varepsilon,\mu}(\Omega_0)$ where

$$\mathscr{J}_{\varepsilon,\mu}(\Omega_0) := \{(\mathsf{E}, \mathsf{H})^\top \in L^2_{\varepsilon,\mu}(\Omega_0, \mathbb{R}^3 \times \mathbb{R}^3) : \operatorname{div}(\varepsilon \mathsf{E}) = \operatorname{div}(\mu \mathsf{H}) = 0, \\ (\mu \mathsf{H}) \cdot \mathsf{n} = 0\}.$$

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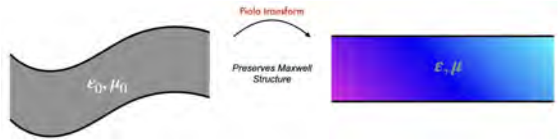
Look for electromagnetic waves with
non-constant permittivity and permeability.



Essential spectrum - constant twist β

Assume $\kappa = 0, \quad \theta' = \beta.$

Observe that

$$\mathbb{G}_\beta(s, y_2, y_3) = \mathbb{G}_\beta(y_2, y_3) := \begin{pmatrix} 1 + \beta^2 |y|^2 & -y_3\beta & y_2\beta \\ -y_3\beta & 1 & 0 \\ y_2\beta & 0 & 1 \end{pmatrix}$$

depends only on $(y_2, y_3) \in \omega$.

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N. Filonov,
St. Petersbrug Math. J., (2019)

Comparison of $\widehat{\mathscr{A}}$ and $\widehat{\mathscr{A}_\beta}$

We would like to compare the resolvents of $\widehat{\mathscr{A}}$ and $\widehat{\mathscr{A}_\beta}$.

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Problem: They do not act in the same Hilbert space !

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Another unitary transform

$$\mathbb{V} : L^2_{\varepsilon,\mu}(\Omega_0) \to L^2_{\varepsilon_0,\mu_0}(\Omega_0), \quad \begin{pmatrix} \mathsf{E} \\ \mathsf{H} \end{pmatrix} \mapsto \begin{pmatrix} \mathbb{G}^{-1/2}\mathsf{E} \\ \mathbb{G}^{-1/2}\mathsf{H} \end{pmatrix}$$

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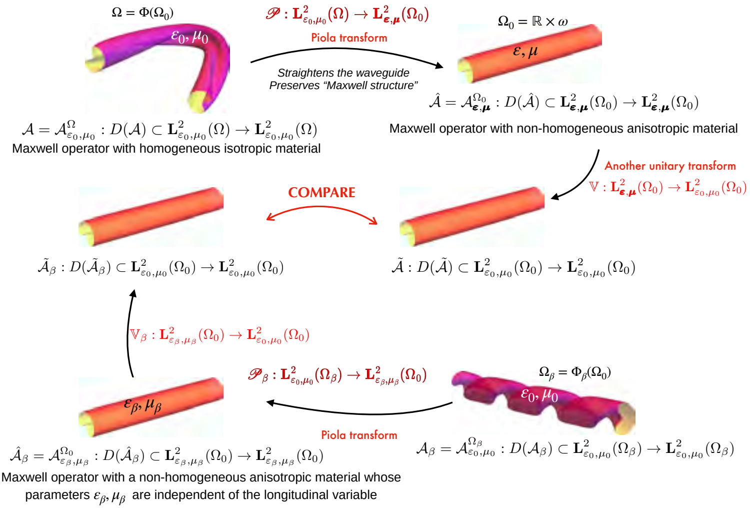
Define the unitary equivalent operators acting in $L^2_{\varepsilon_0,\mu_0}(\Omega_0)$:

- $\widetilde{\mathscr{A}} := \mathbb{V}\widehat{\mathscr{A}}\,\mathbb{V}^*$,
- $\widetilde{\mathscr{A}_\beta} := \mathbb{V}_\beta\widehat{\mathscr{A}_\beta}\,\mathbb{V}_\beta^*$

Comparison of operators: a picture !



Resolvent identity

The rest of the proof is by investigating, for $\lambda \in \rho(\widetilde{\mathscr{A}}) \cap \rho(\widetilde{\mathscr{A}_\beta})$, the resolvents

$$\widetilde{R}(\lambda) := (\widetilde{\mathscr{A}} - \lambda)^{-1}, \quad \widetilde{R}_\beta(\lambda) := (\widetilde{\mathscr{A}_\beta} - \lambda)^{-1}$$

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- $\mathscr{Q}(\lambda) = \mathscr{Q}(\lambda_0)(Id + \mathscr{K}(\lambda))$ with $\mathscr{K}(\lambda)$ is a compact operator.
- $\mathscr{Q}_\beta(\lambda) = \mathscr{Q}_\beta(\lambda_0)(Id + \mathscr{K}_\beta(\lambda))$ with $\mathscr{K}_\beta(\lambda)$ is a compact operator.
- It yields a Birman-Schwinger principle.
- We conclude studying the analytic family of compact operators $(\lambda \mapsto \mathscr{K}_\beta(\lambda))$ and the finitely meromorphic family of compact operators $(\lambda \mapsto \mathscr{K}(\lambda))$.

Poincaré inequality

$$\boxed{\text{Assume } \theta' = 0, \quad b\|\kappa\|_{L^\infty(\mathbb{R})} < \frac{1}{2}.}$$

There exists $C > 0$ such that for all $(\mathrm{E}, \mathrm{H})^\top \in \mathscr{J}_{\varepsilon,\mu}(\Omega_0) \cap (H_0(\mathrm{curl}, \Omega_0) \times H(\mathrm{curl}, \Omega_0))$:

$$\|\mathrm{E}\|_{L^2_\varepsilon(\Omega_0)} + \|\mathrm{H}\|_{L^2_\mu(\Omega_0)} \leq C\left(\|\mu^{-1}\,\mathrm{curl}\,\mathrm{E}\|_{L^2_\mu(\Omega_0)} + \|\varepsilon^{-1}\,\mathrm{curl}\,\mathrm{H}\|_{L^2_\varepsilon(\Omega_0)}\right)$$

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$$\|\mathrm{E}\|_{L^2_\varepsilon(\Omega_0)} + \|\mathrm{H}\|_{L^2_\mu(\Omega_0)} \leq C\left(\|\mu^{-1}\,\mathrm{curl}\,\mathrm{E}\|_{L^2_\mu(\Omega_0)} + \|\varepsilon^{-1}\,\mathrm{curl}\,\mathrm{H}\|_{L^2_\varepsilon(\Omega_0)}\right)$$

- It is a consequence of the Birman-Schwinger principle.
- A consequence is that $0 \notin Sp(\widehat{\mathscr{A}_\perp})$.

Strategy of the proof (discrete spectrum)

- Work with the quadratic form associated with $(\widehat{\mathscr{A}_\perp})^2$

$$q[\mathsf{E},\mathsf{H}] := \int_{\Omega_0} \left(\mu^{-1}\operatorname{curl}\mathsf{E}\cdot\operatorname{curl}\mathsf{E} + \varepsilon^{-1}\operatorname{curl}\mathsf{H}\cdot\mathsf{H}\right)dx,$$
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- Look for a test function of the form $(\mathrm{E},0)^\top \neq (0,0)^\top$ verifying:

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- Conclude with the min-max principle.

Choice of the trial function

Set F of the form

$$\mathrm{F}(s,y) = \varphi(s)\begin{pmatrix} 0 \\ \partial_3\psi(y) \\ -\partial_2\psi(y) \end{pmatrix}.$$

$\boxed{\text{Problem}}$: $(\mathrm{F},0)^\top \notin \mathrm{Dom}(q)$ because $\mathrm{div}(\varepsilon \mathrm{F}) \neq 0$

We correct it as

$$\mathrm{E} = \mathrm{F} + \nabla u$$

with $u \in H_0^1(\Omega)$ weak-solution of

$$\begin{cases} \mathrm{div}(\varepsilon \nabla u) = -\mathrm{div}(\varepsilon \mathrm{F}) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

With this choice we control $\|\nabla u\|_{L^2_\varepsilon(\Omega_0)}$ using $\mathrm{div}(\mathrm{F}) = 0$ and compute

$$q[\mathrm{E},0] - \frac{\lambda_2^N(\omega)}{\varepsilon_0\mu_0}\|\mathrm{E}\|^2_{L^2(\Omega_0)} < -X \cdot \left(\int_{\mathbb{R}} k^\theta(s)ds\right) + \frac{2\lambda_2^N(\omega)}{\varepsilon_0}\,\frac{b^2\|\kappa\|_{L^\infty(\mathbb{R})}}{1 - b\|\kappa\|_{L^\infty(\mathbb{R})}}\|\kappa\|_{L^1(\mathbb{R})}.$$

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- For ω being an isosceles right triangle $X \neq 0$! The modes can be computed by symmetry using the one of the square.
- Plenty many numerical examples of $X \neq 0$...

Domains for which $X \neq 0$

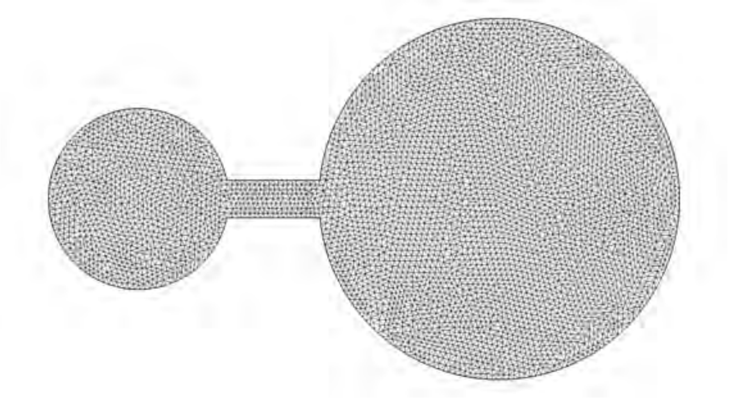

Figure: A Dumbbell domain. $X_{\text{dumbbell}} = \begin{pmatrix} -0.1255 \\ 0 \end{pmatrix}$

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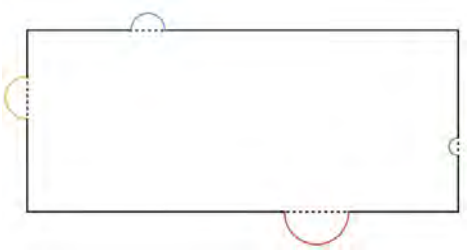

Figure: Rectangle of base 2π and height π with bumps. The bumps are half-disks of varying radius and positions.

$$X_{\text{blue}} = \begin{pmatrix} 0.00315511 \\ 0.00126526 \end{pmatrix}, \quad X_{\text{yellow}} = \begin{pmatrix} -0.000798985 \\ 0.00301619 \end{pmatrix}$$

$$X_{\text{red}} = \begin{pmatrix} -0.0152116 \\ 0.00285829 \end{pmatrix}, \quad X_{\text{green}} = \begin{pmatrix} 0.0000322362 \\ 0.000374781 \end{pmatrix}$$

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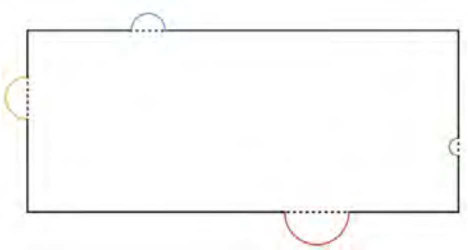

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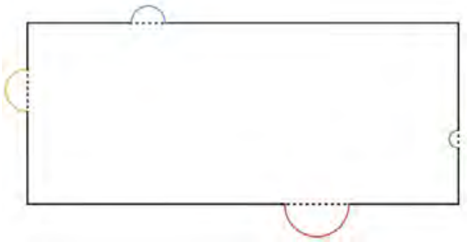

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- $\boxed{\text{Intuition}}$: Condition $X = 0$ is not stable under perturbations of ω.
- Theoretical proof of this when ω is a rectangle *via* shape derivatives.

Conclusion and perspectives

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Thank you for your attention !