Resonances in various quantum waveguides systems

Sylwia Kondej
(joint work with K. Ślipko)
University of Zielona Góra, Institute of Physics

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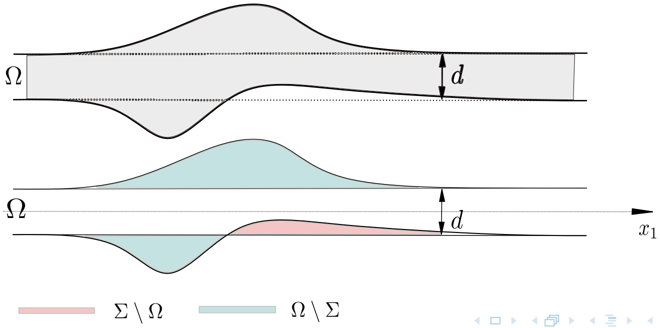
Model: soft waveguide with local deformation

Soft waveguide in $\mathbb{R}^2$

$\Omega \subset \mathbb{R}^2$ asymptotically coincides with straight strip Σ:

$$\Sigma := \Big\{ (x_1, x_2) \ : \ x_1 \in \mathbb{R}, \ \ x_2 \in [-d/2, d/2] \Big\} .$$

Compact deformation



Model

The Hamiltonian

$$H_{\Omega,V} = -\frac{\hbar^2}{2m}\Delta - V \cdot \chi_\Omega \,, \quad V > 0, \,, \qquad \frac{\hbar^2}{2m} = 1$$

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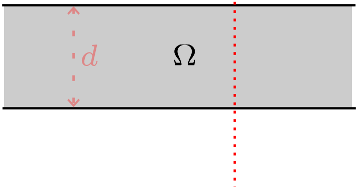
- Essential Spectrum
- Symmetric waveguides and embedded eigenvalues (bound states in continuum BIC).
- I Breaking symmetry and resonances (lifetime, dependence of breaking symmetry parameter)
- II Resonances in waveguides due to the tunneling (lifetime, dependence of waveguide architecture)

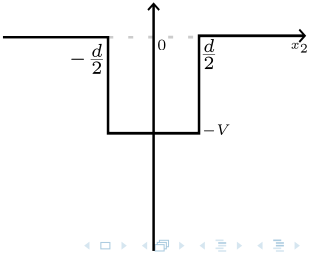
Preliminary facts on the spectrum. Straight waveguide. Essential spectrum.

Straight strip. Translational decomposition

$H_{\Sigma,V}$ can be decomposed:

$$H_{\Sigma,V} = -\frac{d^2}{dx_1^2} \otimes \mathrm{I} + \mathrm{I} \otimes \mathrm{h}_V \,. \tag{1}$$



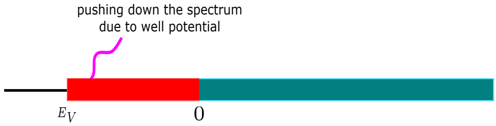


The essential spectrum

The essential spectrum of $H_{\Sigma,V}$

The threshold of the essential threshold of $H_{\Sigma,V}$:

$$\sigma_{\text{ess}}(H_{\Sigma,V}) = [E_V, \infty)\,, E_V \ \ ground.\ \ st\ \ en.\ \ \mathrm{h}_V\,.$$



Essential spectrum of $H_{\Omega,V}$

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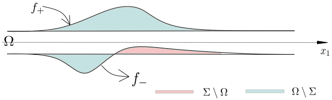
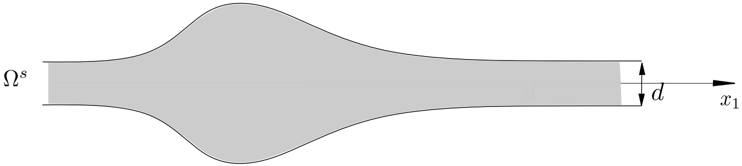


Model. Symmetric waveguide.

Mirror symmetry of Ω.

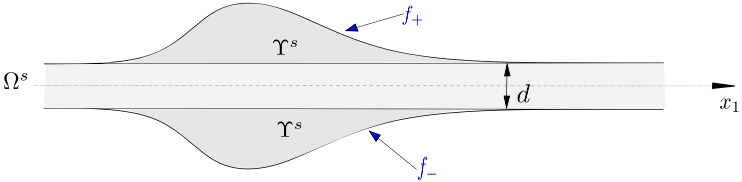


Model. Symmetric waveguide.

Namely, we assume

$$f_+(x_1) = -f_-(x_1)\,, \quad \text{for all} \;\; x_1 \in \mathbb{R},$$

which leads to the mirror symmetry of Ω.



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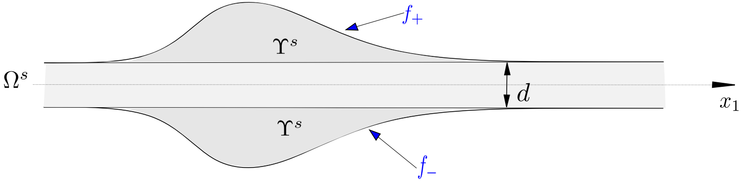
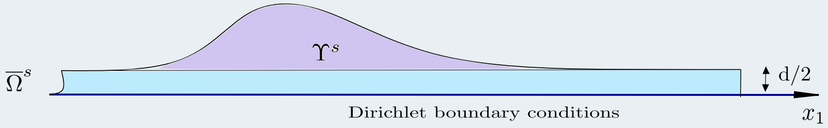


Figure: An example of the symmetric waveguide with the mirror symmetry with respect to the horizontal axis $\mathbb{R} \times \{0\}$. The straight strip Σ is marked in light gray color. $\Omega^s = \Sigma \cup \Upsilon^s$.

Model. Symmetric waveguide: decomposition in half-planes.

Upper half-plane component with Dirichlet boundary conditions



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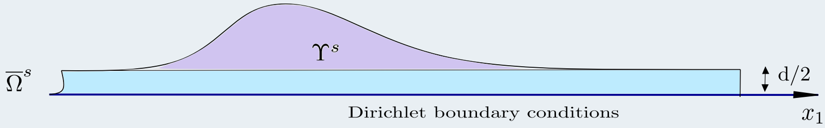
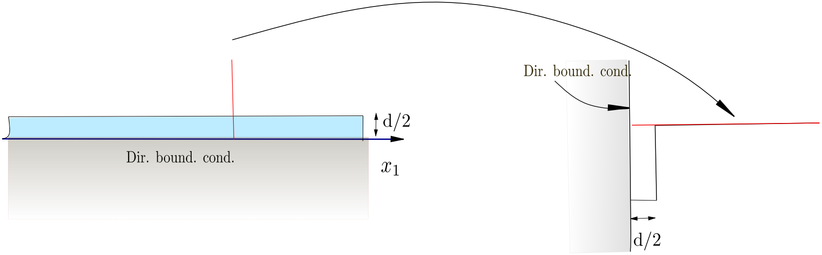




$$H_{\Sigma}^{D} = -\frac{d^2}{dx_1^2} \otimes \mathrm{I} + \mathrm{I} \otimes \mathrm{h}_V^D : W^{2,2}(\mathbb{R}) \otimes W_0^{2,2}(\mathbb{R}_+)$$

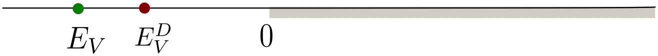
Spectral facts in half-plane system

Minimum of h_V^D

Denote

$$E_V^D := \min \sigma(\mathrm{h}_V^D)\,.$$

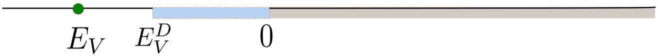
We have $E_V^D = 0$ if $V \leq (\pi/d)^2$ and $E_V^D < 0$ otherwise.



The essential and discrete spectrum of $H^D_{\Sigma,V}$

The Hamiltonian $H^D_{\Sigma,V}$ has the essential spectrum:

$$\sigma_{\mathrm{ess}}(H^D_{\Sigma,V}) = [E_V^D, \infty)\,.$$

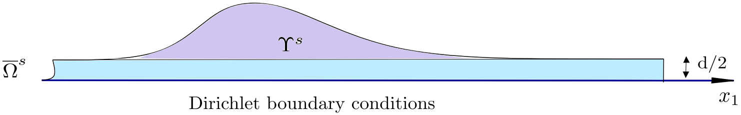


Spectral facts in half-plane system

Essential spectrum of $H^{D}_{\overline{\Omega}^s,V}$

The stability of the essential spectrum under the compact deformation

$$\sigma_{\mathrm{ess}}(H^{D}_{\overline{\Omega}^s,V}) = \sigma_{\mathrm{ess}}(H^{D}_{\overline{\Sigma},V}) = [E^{D}_{V},\infty)\,.$$



Discrete spectrum

$H^{D}_{\overline{\Omega}^s,V}$ admits discrete eigenvalues below E^{D}_{V}, i.e.

$$\sigma_{\mathrm{d}}(H^{D}_{\overline{\Omega}^s,V}) \neq \emptyset\,,$$

Embedded eigenvalues of $H_{\Omega^s,V}$.

Antisymmetric eigenfunctions

Let $E \in \sigma_{\mathrm{d}}(H^D_{\Omega^s,V})$ and ϕ^D denote the corresponding eigenfunction. Let

$$\phi(x_1,x_2) = \phi^D(x_1,x_2) \quad \text{for } x_2 > 0$$
$$\phi(x_1,x_2) = -\phi^D(x_1,x_2) \text{ for } x_2 < 0\,.$$

Embedded eigenvalues of $H_{\Omega^s,V}$.

Antisymmetric eigenfunctions

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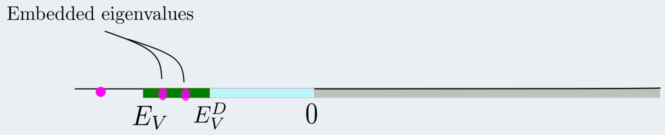
Embedded eigenvalues

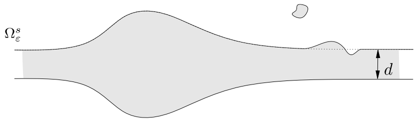
If $E \in [E_V, E_V^D)$ then it gives a rise to the embedded eigenvalues of $H_{\Omega^s,V}$.



Normalized eigenfunctions of $H_{\Omega^s,V}$ $\{E_k^s\}_{k\in\mathcal{N}}$, $\{\phi_k^s\}_{k\in\mathcal{N}}$.

Small deformation of Ω^s

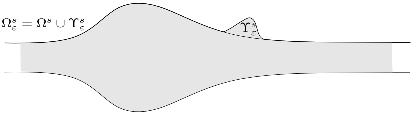
I.

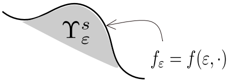


II. Assume $\Omega^s \subset \Omega^s_\varepsilon$ (more general case will be discussed), $\Omega^s_\varepsilon \subset \Omega^s_{\varepsilon_0}$ and the sets

$$\Upsilon^s_\varepsilon := \Omega^s_\varepsilon \setminus \Omega^s$$

are compactly supported.





Moreover,

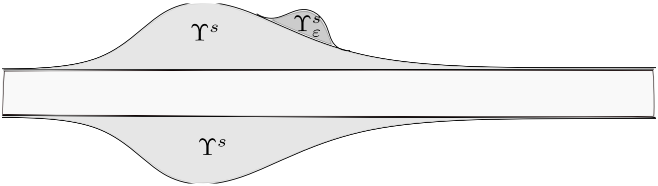
$$|\Upsilon^s_\varepsilon| \to 0 \quad \text{for} \quad \varepsilon \to 0\,. \tag{3}$$

Small deformation of Ω^s

The Hamiltonian with a symmetric waveguide Ω^s –unperturbed system.

We can express $H_{\Omega^s_\varepsilon, V}$ in the following way

$$H_{\Omega^s_\varepsilon, V} = H_{\Omega^s, V} - V \cdot \chi_{\Upsilon^s_\varepsilon}\,, \quad \text{where} \quad H_{\Omega^s, V} = H_{\Sigma, V} - V \cdot \chi_{\Upsilon^s}\,.$$



Resovent

Hamiltonian

$$H_{\Omega,V} = H_{\Sigma,V} - V \cdot \chi_\Upsilon\,, \qquad \Upsilon = \Omega \setminus \Sigma\,.$$

Resovent and Birman–Schwinger operator

Let z be such that

$$K_\Upsilon(z) := \mathrm{I} - V\chi_\Upsilon R_\Sigma(z)\chi_\Upsilon \in \mathcal{B}(L^2(\Upsilon))$$

is invertible. Then

$$R_\Omega(z) = R_\Sigma(z) + V R_\Sigma(z)\chi_\Upsilon(\mathrm{I} - V\chi_\Upsilon R_\Sigma(z)\chi_\Upsilon)^{-1}\chi_\Upsilon R_\Sigma(z)\,.$$

Resovent $R_\Sigma(z)$

We have

$$R_\Sigma(z) = R^d_\Sigma(z) + R^c_\Sigma(z)$$

$$R^d_\Sigma(z)f(x) = \sum_{n\in\mathcal{N}_V} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}p_1 \, \frac{\widehat{f_1}(p_1)\mathrm{e}^{ip_1x_1}}{p_1^2 + E_{n,V} - z} P_{V,n} f_2(x_2) \tag{4}$$

$$= \sum_{n\in\mathcal{N}_V} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}p_1 \, \frac{\widehat{f_1}(p_1)\mathrm{e}^{ip_1x_1}}{p_1^2 + E_{n,V} - z} \phi_{V,n}(x_2) \Big(\phi_{V,n}, f_2\Big)_{L^2(\mathbb{R})}, \tag{5}$$

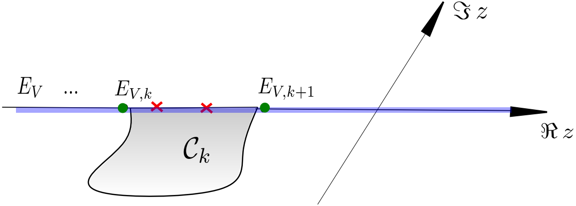
where $f(x_1,x_2) = f_1(x_1)f_2(x_2)$ and $\widehat{f_1}$ is the Fourier transform of f_1 and

$$R^c_\Sigma(z)f = \int_{\mathbb{R}^2_+} \frac{1}{\beta+\beta'-z} \mathrm{d}\mathcal{E}(\beta)f_1 \otimes \mathrm{d}\mathcal{E}_V(\beta')f_2 \,. \tag{6}$$

Resolvent; analytic continuation

Lemma

There exists an open set $\mathcal{C}_k \subset \mathbb{C}_-$ such that the operator valued function $z \mapsto K_\Upsilon(z)$ admits an analytic continuation to $\overline{\mathcal{C}}_k := ((E_{V,k}, E_{V,k+1}) \cup \mathcal{C}_k) \setminus \mathcal{D}_k$, where $\mathcal{D}_k$ is a set of finite number of points. Moreover, $K_\Upsilon(z)$ is analytically invertible for $z \in \overline{\mathcal{C}}_k$, i.e. $K_\Upsilon(z)^{-1}$ exists and $z \mapsto K_\Upsilon(z)^{-1}$ is analytic.



Lemma

For any $k \in \mathcal{N}_V$ and interval $[\alpha, \beta] \subset [E_{V,k}, E_{V,k+1}]$, which is disjoint from $\mathcal{D}_k$, we have $\operatorname{Ran} \mathcal{E}_{\Omega,V}((\alpha,\beta)) \subset \mathcal{H}_{ac}$.

Resovent; analytic continuation of the resolvent.

$R_\Sigma(z)$:

$$R_\Sigma(z)|_{f,g} := fR_\Sigma(z)g \,:\, L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)\,,$$

where $f, g \in C_0^\infty(\mathbb{R}^2)$.
The same holds for $fR_\Sigma(z)\chi_\Upsilon \,:\, L^2(\Upsilon) \to L^2(\mathbb{R}^2)$ and $\chi_\Upsilon R_\Sigma(z)g \,:\, L^2(\mathbb{R}^2) \to L^2(\Upsilon)$.

$R_\Omega(z)$:

For $z \in \overline{\mathcal{C}}_k$ and any $f, g \in C_0^\infty(\mathbb{R}^2)$ the operator

$$R_\Omega(z)|_{f,g} = R_\Sigma(z)|_{f,g} + [\,VR_\Sigma(z)\chi_\Upsilon K_\Upsilon(z)^{-1}\chi_\Upsilon R_\Sigma(z)\,]|_{f,g}\,, \tag{7}$$

can be continued to the lower complex half-plane: $A|_{f,g} = fAg$.

Poles of resolvent. SK'24

Theorem

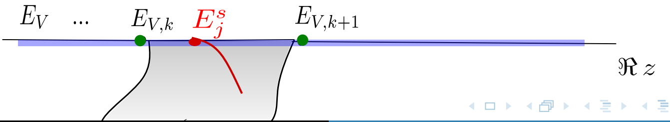
Assume that $E_j^s \in (E_{V,k}, E_{V,k+1})$ *is a simple embedded eigenvalue of* $H_{\Omega^s,V}$. *Then the resolvent of* $H_{\Omega^s_\varepsilon,V}$ *admits the analytic continuation via* $(E_{V,k}, E_{V,k+1})$ *to the lower complex half-plane. Moreover, the analytic continuation of the resolvent has a pole at*

$$z = E_j^s + \Lambda(\varepsilon) + i\Gamma(\varepsilon)\,, \quad \text{with} \quad \Gamma(\varepsilon) < 0\,, \tag{8}$$

where the real and imaginary components: $\Lambda(\varepsilon)$ *and* $\Gamma(z)$ *take the forms*

$$\Lambda(\varepsilon) = -V\int_{\Upsilon^s_\varepsilon} |\phi_j^s(x)|^2 \mathrm{d}x + \mathcal{O}(|\Upsilon^s_\varepsilon|^{3/2})\,, \quad \Gamma(\varepsilon) = \mathcal{O}(|\Upsilon^s_\varepsilon|^2)\,, \tag{9}$$

for $\varepsilon \to 0$. *We have* $\int_{\Upsilon^s_\varepsilon} |\phi_j^s(x)|^2 \mathrm{d}x = \mathcal{O}(|\Upsilon^s_\varepsilon|)$.



Ingredients of proof

We have

$$K_{\Upsilon^s_\varepsilon}(z) = \mathrm{I} - V\chi_{\Upsilon^s_\varepsilon} R_{\Omega^s}(z)\chi_{\Upsilon^s_\varepsilon} \tag{10}$$

This allows to decompose for $z \in \mathcal{S}_j$

$$K_{\Upsilon^s_\varepsilon}(z) = \mathrm{I} + S_\varepsilon(z) + \frac{V}{z - E^s_j}\chi_{\Upsilon^s_\varepsilon} P^s_j \chi_{\Upsilon^s_\varepsilon}\,, \tag{11}$$

where

$$S_\varepsilon(z) := -V\left[\chi_{\Upsilon^s_\varepsilon} R_\Sigma(z)\chi_{\Upsilon^s_\varepsilon} + \chi_{\Upsilon^s_\varepsilon} W(z)\chi_{\Upsilon^s_\varepsilon}\right]. \tag{12}$$

Lemma

Assume that $\varepsilon \to 0$. *For* $z \in \mathcal{S}_j$*:*

$$\|S_\varepsilon(z)\|^2_{L^2(\Upsilon^s_\varepsilon)\to L^2(\Upsilon^s_\varepsilon)} = \mathcal{O}(|\Upsilon^s_\varepsilon|)\,. \tag{13}$$

Moreover: $\Re\left(\chi_{\Upsilon^s_\varepsilon}\phi^s_j, S_\varepsilon(E^s_j)\chi_{\Upsilon^s_\varepsilon}\phi^s_j\right)_{L^2(\Upsilon^s_\varepsilon)} = \mathcal{O}(|\Upsilon^s_\varepsilon|^{3/2})$

$$\Im\left(\chi_{\Upsilon^s_\varepsilon}\phi^s_j, S_\varepsilon(E^s_j)\chi_{\Upsilon^s_\varepsilon}\phi^s_j\right)_{L^2(\Upsilon^s_\varepsilon)} = \mathcal{O}(|\Upsilon^s_\varepsilon|^2)\,. \tag{14}$$

Ingredients of proof

For $z \in \mathcal{S}_j$ we conclude that $\ker K_{\Upsilon_\varepsilon^s}(z) \neq \emptyset$ if and only if

$$\ker \left[\mathrm{I} + \frac{V}{z - E_j^s} \big(\mathrm{I} + S_\varepsilon(z) \big)^{-1} \chi_{\Upsilon_\varepsilon^s} P_j^s \chi_{\Upsilon_\varepsilon^s} \right] \neq \emptyset \,. \tag{15}$$

Lemma

Assume that $E_j^s \in (E_{V,k}, E_{V,k+1})$ *is a simple embedded eigenvalue of* $H_{\Omega^s,V}$. *For* ε *sufficiently small, there exists* $z \in \mathcal{S}_j$ *such that* $\ker K_{\Upsilon_\varepsilon^s}(z) \neq \emptyset$ *if and only if* z *is a solution of*

$$z - E_j^s + V\eta(z, \varepsilon) = 0 \,, \tag{16}$$

where

$$\eta(z, \varepsilon) := \Big(\chi_{\Upsilon_\varepsilon^s} \phi_j^s, \big(\mathrm{I} + S_\varepsilon(z)\big)^{-1} \chi_{\Upsilon_\varepsilon^s} \phi_j^s \Big)_{L^2(\Upsilon_\varepsilon^s)} \,.$$

Ingredients of proof

$$z - E_j^s + V\left(\chi_{\Upsilon_\varepsilon^s}\phi_j^s, (\mathrm{I} + S_\varepsilon(z))^{-1}\chi_{\Upsilon_\varepsilon^s}\phi_j^s\right)_{L^2(\Upsilon_\varepsilon^s)} = 0\,.$$

The operator $\mathrm{I} + S_\varepsilon(z)$ is invertible and the inverse

$$(\mathrm{I} + S_\varepsilon(z))^{-1} = \mathrm{I} - S_\varepsilon(z) + S_\varepsilon(z)^2 + \dots \tag{17}$$

$$z - E_j^s + V\left(\chi_{\Upsilon_\varepsilon^s}\phi_j^s, \chi_{\Upsilon_\varepsilon^s}\phi_j^s\right)_{L^2(\Upsilon_\varepsilon^s)} \tag{18}$$

$$- V\left(\chi_{\Upsilon_\varepsilon^s}\phi_j^s, S_\varepsilon(z)\chi_{\Upsilon_\varepsilon^s}\phi_j^s\right)_{L^2(\Upsilon_\varepsilon^s)} + \dots = 0\,. \tag{19}$$

$$\Re\left(\chi_{\Upsilon_\varepsilon^s}\phi_j^s \;\; S_\varepsilon(E_j^s)\chi_{\Upsilon_\varepsilon^s}\phi_j^s\right)_{L^2(\Upsilon_\varepsilon^s)} = \mathcal{O}(|\Upsilon_\varepsilon^s|^{3/2})$$

$$\Im\left(\chi_{\Upsilon_\varepsilon^s}\phi_j^s, S_\varepsilon(E_j^s)\chi_{\Upsilon_\varepsilon^s}\phi_j^s\right)_{L^2(\Upsilon_\varepsilon^s)} = \mathcal{O}(|\Upsilon_\varepsilon^s|^2$$

The Fermi golden rule

- $\Gamma(z)$ preserves the Fermi golden rule

$$\Gamma(\varepsilon) = -\pi V^2 \frac{\mathrm{d}(\chi_{\Upsilon^s_\varepsilon}\phi^s_j, \mathcal{E}_{\Omega^s}(\lambda)\chi_{\Upsilon^s_\varepsilon}\phi^s_j)_{L^2(\Upsilon^s_\varepsilon)}}{\mathrm{d}\lambda}\Big|_{\lambda=E^s_j} + \mathcal{O}(|\Upsilon^s_\varepsilon|^{5/2})\,. \tag{20}$$

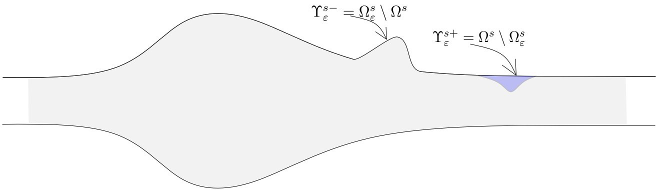
- The resonance width

$$\Gamma_w = -2\,\Im z$$

- The resonance lifetime

$$\tau = \frac{1}{\Gamma_w}$$

Generalization



The Hamiltonian takes the form

$$H_{\Omega_\varepsilon^s} = H_{\Omega^s} - V\chi_{\Omega_\varepsilon^s \setminus \Omega^s} + V\chi_{\Omega^s \setminus \Omega_\varepsilon^s} .$$

$$z = E_j^s + \Lambda(\varepsilon) + i\Gamma(\varepsilon) ,$$

where $\Gamma(\varepsilon) = \mathcal{O}(|\Upsilon_\varepsilon^s|^2)$ and $\Lambda(\varepsilon)$ and

$$\Lambda(\varepsilon) = \left(-V \int_{\Upsilon_\varepsilon^{s-}} |\phi_j^s(x)|^2 \mathrm{d}x + V \int_{\Upsilon_\varepsilon^{s+}} |\phi_j^s(x)|^2 \mathrm{d}x \right) + \mathcal{O}(|\Upsilon_\varepsilon^s|^{3/2}) .$$

$\Lambda(\varepsilon)$ feels ‘the sign of the deformation’ and it reflexes the behaviour of the first perturbation term.

The flat layer

$$\Sigma := \left\{(x_1, x_2, x_3) \,:\, (x_1, x_2) \in \mathbb{R} \times \mathbb{R}\,,\ \ x_3 \in [-d/2, d/2]\right\}.$$

Let Ω stand for a compact deformation of Σ, i.e. the boundaries $\partial\Omega_\pm$ are defined as the graphs of smooth functions $f_\pm : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ that are non-constant, however, for sufficiently large values of $\rho = \sqrt{x_1^2 + x_2^2}$, we have $f_\pm(x_1, x_2) = \pm d/2$. The layer is given by

$$\Omega := \{(x_1, x_2, y(x_1, x_2)) \,:\, (x_1, x_2) \in \mathbb{R} \times \mathbb{R}\,, \tag{21}$$

$$f_-(x_1, x_2) \le y(x_1, x_2) \le f_+(x_1, x_2)\}\,. \tag{22}$$

Resonances induced by tunelling

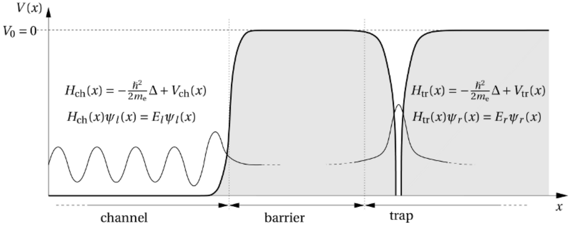


Figure: https://www.iue.tuwien.ac.at/phd/goes/dissse33.html

Model: strip with distant well

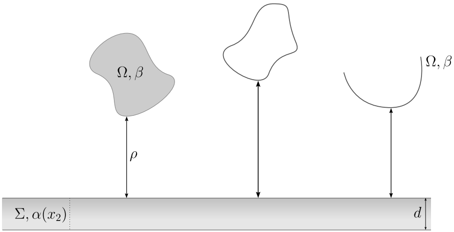
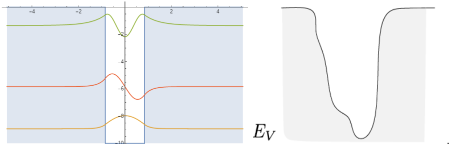


Figure: Strip and distant well, wire.

Distance $\rho := \min_{x\in\Sigma, y\in\Omega} |x-y|$.



Model: strip with distant well

The Hamiltonian

$$H_{\alpha,\beta} = -\Delta + V_\alpha + V_\beta \,. \tag{23}$$

Straight strip

$$V_\alpha = -\alpha \chi_\Sigma(x) \,, \quad \text{where} \quad \alpha > 0 \,. \tag{24}$$

'Perturbation' localized on a distant well

$$V_\beta = -\beta \chi_\Omega(x) \,, \quad \text{where} \quad \beta > 0 \,. \tag{25}$$

Generalization: Kato class trap

$$\mathcal{E}_{\alpha,\beta\mu}[f] = \int_{\mathbb{R}^2} |\nabla f(x)|^2 \, \mathrm{d}x - \int_{\Sigma} \alpha(x_2)|f(x)|^2 \, \mathrm{d}x - \beta \int_{\mathbb{R}^2} |I_\mu f(x)|^2 \, \mathrm{d}\mu(x) \tag{26}$$

where $f \in W^{1,2}(\mathbb{R}^2)$.

- $I_\mu \, : \, W^{1,2}(\mathbb{R}^2) \to L^2(\mathbb{R}^2 \, , \mu)$ continuous embedding.

- μ Kato class measure, for example $\mu(\mathcal{B}) = \mathrm{lin}(\mathcal{B} \cap \Omega)$, where $\mathrm{lin}(\cdot)$ stands for the linear measure and $\mathcal{B} \in \mathbb{R}^2$ - Borel set. Then the perturbation corresponds to $-\beta\delta(x - \Omega)$

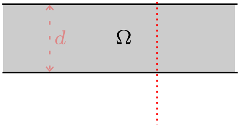
- P. Exner, SK, JoP A 2004,

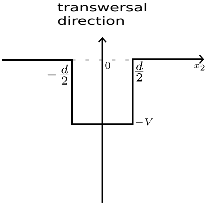
$H_{\alpha,0}$ strip only

Straight strip. Translational decomposition

$H_{\alpha,0}$ can be decomposed:

$$H_{\alpha,0} = -\frac{d^2}{dx_1^2} \otimes \mathrm{I} + \mathrm{I} \otimes \mathrm{h}_\alpha \,. \tag{27}$$





- $\{E_{\alpha;1} \leq \ldots \leq E_{\alpha,N_\alpha}\}$ discrete eigenvalues of h_α;
- The spectrum of $H_{\alpha,0}$ covers

$$\sigma(H_{\alpha,0}) = \sigma_{\mathrm{ess}}(H_{\alpha,0}) = [E_\alpha \,, \infty) \,, \quad E_\alpha = E_{1;\alpha} \,. \tag{28}$$

Well without strip

Hamiltonian $H_{0,\beta\mu}$ has the eigenvalues $\{\mathcal{E}_{\beta;1} \leq \ldots \leq \mathcal{E}_{\beta;N_\beta}\}$, with a corresponding eigenfunction $\{\omega_{\beta;n}\}_{n\in\mathcal{N}_\beta}$.

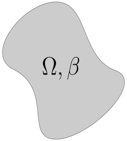
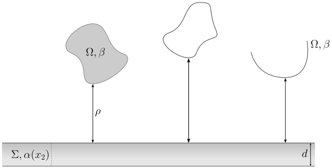


Figure: Distant well.

Spectrum

Assume that $H_{0,\beta}$ has an eigenvalue $\mathcal{E}_\beta \in (E_{\alpha;j}\,, E_{\alpha;j+1})$. Then the resonances resides

$$z = \mathcal{E}_\beta + \Lambda(\rho) + i\Psi(\rho) = \mathcal{E}_\beta + \mathcal{O}\left(\frac{-\sqrt{2|\mathcal{E}_\beta|}\rho}{\rho}\right). \qquad (29)$$

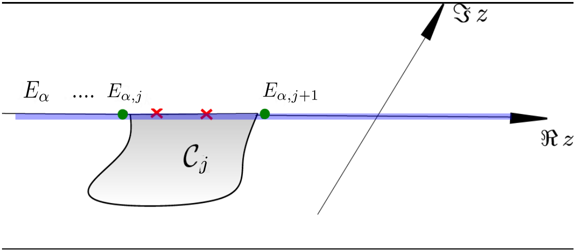


Ingredients of the strategy: Resolvent of $H_{\alpha,\beta}$

$$R_{\alpha,\beta}(z) = R_{\alpha,0}(z)+\beta R_{\alpha,0}(z)\chi_\Omega(\mathrm{I} - \beta\chi_\Omega R_{\alpha,0}(z)\chi_\Omega)^{-1}\chi_\Omega R_{\alpha,0}(z)\,, \tag{30}$$

$$K_\Omega(z) = \mathrm{I} - \beta\chi_\Omega R_{\alpha,0}(z)\chi_\Omega$$

• Analytic extension of $\chi_\Omega R_{\alpha,0}(z)\chi_\Omega$, $z \mapsto K_\Omega(z)$, $z \mapsto K_\Omega(z)^{-1}$ (Fredholm Theorem).



Zeros of Birman–Schwinger operator

z satisfies $\ker[\mathrm{I} - \beta\chi_\Omega R_{\alpha,0}(z)\chi_\Omega] \neq \emptyset$ if and only if

$$\ker[\mathrm{I} - \beta\chi_\Omega R(z)\chi_\Omega - G_{\alpha,\beta}(z)] \neq \emptyset\,, \tag{31}$$

where

$$G_{\alpha,\beta}(z) = \beta\,\alpha\,\chi_\Omega R(z)\chi_\Sigma(\mathrm{I}-\alpha\chi_\Sigma R(z)\chi_\Sigma)^{-1}\chi_\Sigma R(z)\chi_\Omega \;:\; L^2(\Omega)\to L^2(\Omega)\,. \tag{32}$$

Lemma

$$\|G_{\alpha,\beta}(z)\|_{\mathrm{HS}} \leq C_1\,\frac{e^{-\sqrt{2}\mathcal{T}_z\rho}}{\rho}\,, \tag{33}$$

where C_1 is a positive constant which depends on z; $\mathcal{T}_z := \Im\sqrt{z} > 0$

Lemma: spectral equation from operator to scalar eq.

$z \neq \mathcal{E}_{\beta;n}$

$$\ker[\mathrm{I} - \beta \chi_\Omega R(z) \chi_\Omega - G_{\alpha,\beta}(z)] \neq \emptyset \tag{34}$$

Spectral equation

(34) is equivalent to

$$z - \mathcal{E}_{\beta;n} + \frac{1}{\beta}\Big(w_{\beta;n} \,, G_{\alpha,\beta}(z) \big[\mathrm{I} - A_n(z) G_{\alpha,\beta}(z) \big]^{-1} w_{\beta;n} \Big)_{L^2(\Omega)} = 0 \,. \tag{35}$$

$w_{\beta;n} := \beta \chi_\Omega \omega_{\beta;n}$

$A_n(z)$ is analytic in $\mathcal{S}_n$.

Pole of the resolent. SK, Ślipko'24

Theorem

Assume that $H_{0,\beta}$ has a simple, discrete eigenvalue $\mathcal{E}_{\beta;n} \in (E_{\alpha;j}\,, E_{\alpha;j+1})$. Then the second sheet analytic continuation of $R_{\alpha,\beta}(z)$ and has a unique pole $z_n \in \mathcal{S}_n$, i.e. $\ker[\mathrm{I} - \beta\chi_\Omega R_{\alpha,0}(z_n)\chi_\Omega] \neq \emptyset$, with the asymptotics

$$z_n(\rho) = \mathcal{E}_{\beta;n} + \mathcal{O}\left(\frac{-\sqrt{2|\mathcal{E}_{\beta;n}|}\rho}{\rho}\right). \tag{36}$$

Step 1. Auxiliary statement: the Neumann expansion. We adopt the notation $b := \frac{1}{\rho}$, which means that $b \to 0$ for $\rho \to \infty$.

$$\eta(z,b) := \Big(w_{\beta;n}\,, G_{\alpha,\beta}(z) [\,\mathrm{I} - A_n(z) G_{\alpha,\beta}(z)\,]^{-1} w_{\beta;n} \Big)_{L^2(\Omega)} \,. \quad (37)$$

• Spectral eq.

$$z - \mathcal{E}_{\beta;n} + \frac{1}{\beta} \eta(z,b) = 0\,.$$

We have $\lim_{b\to 0} \eta(z,b) = 0$ for any $z \in \mathcal{S}_n$ and η assuming that $\eta(z,0) = 0$ - continuous extension.

• The Neumann series

$$[\mathrm{I} - A_n(z) G_{\alpha,\beta}(z)]^{-1} = \mathrm{I} + A_n(z) G_{\alpha,\beta}(z) + [\,A_n(z) G_{\alpha,\beta}(z)\,]^2 + \ldots \quad (38)$$

which implies

$$\eta(z,b) = \Big[(w_{\beta;n}\,, G_{\alpha,\beta}(z) w_{\beta;n})_{L^2(\Omega)} + (w_{\beta;n}\,, G_{\alpha,\beta}(z) A_n(z) G_{\alpha,\beta}(z) w_{\beta;n})_{L^2(\Omega)} + (w_{\beta;n}\,, G_{\alpha,\beta}(z) [A_n(z) G_{\alpha,\beta}(z)]^2 \, w_{\beta;n}\,)_{L^2(\Omega)} + \ldots \Big] = 0\,.$$

Ingredients of proof

Step 2. Unique solution of the spectral equation.

• $\partial_z \eta(z;b)$ analytic w.r.t. z.

• This gives $\partial_z \eta(z;b)|_{b=0} = 0$, where we understand the symbol $|_{b=0}$ as the limiting value for $b \to 0$.
For $z = \mathcal{E}_{\beta;n}$ and $b = 0$ the spectral eq.

$$z - \mathcal{E}_{\beta;n} + \beta^{-1}\eta(z,b) = 0$$

is satisfied.

• Applying the Implicit Function Theorem we conclude that for b small enough the spectral eq. equation has a unique solution $z_n(b)$ such that $\eta(z_n(b), b) = 0$.

Ingredients of proof

Step 3. Asymptotics of the solution. We have

$$z_n(\rho) = \mathcal{E}_{\beta;n} - \frac{1}{\beta}\Big[(w_{\beta;n}\,, G_{\alpha,\beta}(z) w_{\beta;n})_{L^2(\Omega)} + (w_{\beta;n}\,, G_{\alpha,\beta}(z) A_n(z) G_{\alpha,\beta}$$

$$+ (w_{\beta;n}\,, G_{\alpha,\beta}(z) [A_n(z) G_{\alpha,\beta}(z)]^2\, w_{\beta;n}\,)_{L^2(\Omega)} + \ldots \Big]\,. \tag{40}$$

As follows from the above discussion, the dominated term of $z_n(\rho)$ comes from the term

$$-\beta^{-1} (w_{\beta;n}\,, G_{\alpha,\beta}(z) w_{\beta;n})_{L^2(\Omega)}$$

for $z \in \mathcal{S}_n$.

Fermi's golden rule

The first order $\Im z(\rho)$:
• $\varphi_k(x,p_1) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{ip_1x_1}\phi_{\alpha;k}(x_2)$ generalized eigenfunctions of $H_{\alpha,0}$,
• $\omega_{\beta;n}$ bound state of $H_{0,\beta}$. For the special case of Hamiltonian the dominated term turns to

$$\Gamma_n(\rho) = -\beta^2 \sum_{k=1}^{j} \frac{\pi}{2\sqrt{\mathcal{E}_{\beta;n} - E_{\alpha;k}}} \Big[\Big| \int_\Omega \omega_{\beta;n}(x)^* \varphi_k(x,p_1) \mathrm{d}x \Big|^2_{p_1=(\mathcal{E}_{\beta;n}-E_{\alpha;k})^{1/2}} + \Big| \int_\Omega \omega_{\beta;n}(x)^* \varphi_k(x,p_1) \mathrm{d}x \Big|^2_{p_1=-(\mathcal{E}_{\beta;n}-E_{\alpha;k})^{1/2}} \Big],$$

$\mathcal{E}_{\beta;n} \in (E_{\alpha;j}, E_{\alpha;j+1})$. Realization of

$$\pi \sum_{k=1}^{j} \frac{1}{2\sqrt{\mathcal{E}_\beta - E_{\alpha;k}}} \Big| \Big(V_\beta \omega_\beta \,, \, \varphi_k(\cdot, p_1) \Big) \Big|^2_{p_1=\pm(\mathcal{E}_\beta - E_{\alpha;k})^{1/2}}, \quad (41)$$

Fermi's golden rule: a trap defined by the Kato measure

- The dominated term of the imaginary component of pole takes the form

$$\Gamma_n(\rho) = -\beta^2 \sum_{k=1}^{j} \frac{\pi}{2\sqrt{\mathcal{E}_{\beta;n} - E_{\alpha;k}}} \Big[\Big| \int_\Omega I_\mu(\omega_{\beta;n}(x)^* \varphi_k(x,p_1)) \mathrm{d}\mu(x) \Big|^2_{p_1 = (\mathcal{E}_{\beta;n} - E_{\alpha;k})^{1/2}} + \Big| \int_\Omega I_\mu(\omega_{\beta;n}(x)^* \varphi_k(x,p_1)) \mathrm{d}\mu(x) \Big|^2_{p_1 = -(\mathcal{E}_{\beta;n} - E_{\alpha;k})^{1/2}} \Big], \tag{42}$$

where $\varphi_k(x,p_1)$ are the generalized eigenfunctions of $H_{\alpha,0}$ and let us recall that $\omega_{\beta;n}$ stand for the eigenfunctions of $H_{0,\beta\mu}$; the above integrals depend on the argument p_1 and the indexes mean that they are determined at the specific points $p_1 = \pm(\mathcal{E}_{\beta;n} - E_{\alpha;k})^{1/2}$.

Fermi's golden rule

• The imaginary component of pole plays the main rule in the time evolution of metastable state. It describes the rate of transition from an energy eigenstate of a quantum system to an energy continuum as an effect of a weak perturbation.
• From the physical perspective, distantly located trap allows for controlling scattering processes and quantum transport in the waveguide, influencing tunneling and dissipative processes.
• The distance between the trap and the waveguide can be adjusted to impact the system dynamics (crucial for optimizing such systems).

Waveguide with semitransparent cavity

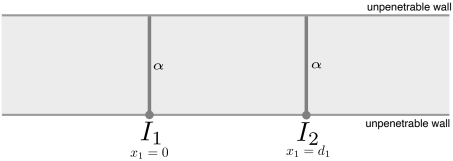


Figure: The structure of the waveguide with semi-transparent barriers.

The Hamiltonian of this system can be formally written as

$$\hat{H}_\alpha = -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} + \alpha\delta(x - I_1) + \alpha\delta(x - I_2)\,,$$

where $\alpha > 0$ denotes the the strength of the delta potential, and $x = (x_1, x_2)$.

Boundary conditions

$$\lim_{x_1 \to 0^-} f(x_1, x_2) = \lim_{x_1 \to 0^+} f(x_1, x_2) = f(0, x_2)$$

$$\lim_{x_1 \to d_1^-} f(x_1, x_2) = \lim_{x_1 \to d_1^+} f(x_1, x_2) = f(d_1, x_2)$$

$$\lim_{x_1 \to 0^-} \frac{\partial f(x_1, x_2)}{\partial x_1} - \lim_{x_1 \to 0^+} \frac{\partial f(x_1, x_2)}{\partial x_1} = \alpha\, f(0, x_2)\,,$$

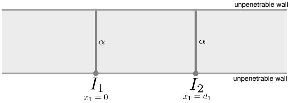
and

$$\lim_{x_1 \to d_1^-} \frac{\partial f(x_1, x_2)}{\partial x_1} - \lim_{x_1 \to d_1^+} \frac{\partial f(x_1, x_2)}{\partial x_1} = \alpha\, f(d_1, x_2)\,.$$

Stating the problem: resonances for $\alpha \to \infty$.

If '$\alpha = \infty$' then unpenetrable cavity produces eigenvalues:

$$\epsilon_{m,n} := \left(\frac{\pi m}{d_1}\right)^2 + \left(\frac{\pi n}{d_2}\right)^2$$



Essential spectrum:

$$\left[\left(\frac{\pi}{d_2}\right)^2, \infty\right)$$

Resonances:

Existence of resonances for $\alpha \to \infty$.

Spectral equation

$$K_{\alpha,n}(\epsilon_{m,n}-\delta) = \det \begin{bmatrix} 1+\alpha \frac{i}{2\sqrt{(\pi m/d_1)^2-\delta}} & \alpha \frac{i \exp\left(i\sqrt{(\pi m/d_1)^2-\delta}\, d_1\right)}{2\sqrt{(\pi m/d_1)^2-\delta}} \\ \alpha \frac{i \exp\left(i\sqrt{(\pi m/d_1)^2-\delta}\, d_1\right)}{2\sqrt{(\pi m/d_1)^2-\delta}} & 1+\alpha \frac{i}{2\sqrt{(\pi m/d_1)^2-\delta}} \end{bmatrix}$$

Determinant:

$$K_{\frac{1}{\rho},n}(\epsilon_{m,n}-\delta) = \rho^2+\rho \frac{i}{\sqrt{(\frac{\pi m}{d_1})^2-\delta}} - \frac{1}{4(\sqrt{(\frac{\pi m}{d_1})^2-\delta})^2} + \frac{\exp\left(2id_1\sqrt{(\frac{\pi m}{d_1})^2-\delta}\right)}{4(\sqrt{(\frac{\pi m}{d_1})^2-\delta})^2} ,$$

where we again adopt rescaling

$$\rho = \frac{1}{\alpha} .$$

Zeros of the determinant

$$f_n(\rho,\delta) := K_{\frac{1}{\rho},n}(\epsilon_{m,n}-\delta)$$

Note that

$$f_n(\rho,0)=0\,,$$

We apply the implicit function theorem.

$$\begin{aligned}\frac{\mathrm{d}^2\delta}{\mathrm{d}\rho^2}\bigg|_{\rho=0} &= -\frac{(\partial_\rho^2 f)_*(\partial_\delta f)_*^2 - 2(\partial_\rho\partial_\delta f)_*(\partial_\rho f)_*(\partial_\delta f)_* + (\partial_\delta^2 f)_*(\partial_\rho f)_*^2}{(\partial_\delta f)_*^3}\\ &= \frac{8\pi^2 m^2\,(i\pi m-3)}{d_1^4}\,. \end{aligned} \qquad (43)$$

The system governed by $\hat{H}_\alpha$ given by exhibits resonances, which manifest as the poles of the resolvent. $\epsilon_{m,n}$ is an eigenvalue of the system which formally corresponds to $\alpha = \infty$. Then the resonance is located at

$$z_{m,n}(\rho) = \left(\frac{\pi m}{d_1}\right)^2 + \left(\frac{\pi n}{d_2}\right)^2 + \mu_m(\rho) + i\nu_m(\rho)\,, \quad \text{where} \quad \rho = \frac{1}{\alpha} \tag{44}$$

and μ_n and ν_n determine the real and imaginary components of the with the following asymptotics

$$\mu_m(\rho) = -\frac{4\pi^2 m^2}{d_1^3}\rho + \mathcal{O}(\rho^2)\,, \quad \text{and} \quad \nu_m(\rho) = -\frac{4\pi^3 m^3}{d_1^4}\rho^2 + \mathcal{O}(\rho^3)\,. \tag{45}$$

Conclusions:

1. The energy of the unpenetrable cavity (explicitly when $\alpha = \infty$ is defined by
$$\left(\frac{\pi m}{d_1}\right)^2 + \left(\frac{\pi n}{d_2}\right)^2 .$$
The result shows when $\alpha \neq \infty$ the resonances emerge. The factors related to α are both real $\mu_m(\rho)$ and imaginary $\nu_m(\rho)$.
2. We see that the factor $\mu_m(\rho)$, representing the real part of the resolvent's pole is linearly dependent on ρ, similarly as in one dimensional system. The negativity of the term $\frac{4\pi^2 m^2}{d_1^3}$ gives us the decreasing function of ρ, however, we get the increasing function of α.
3. The imaginary part represents the resonance width, which is connected to its lifetime by relation
$$\tau = \frac{1}{2\Im z_{m,n}(\rho)} \approx \frac{1}{2\nu_m(\rho)} = \mathcal{O}(\alpha^2) .$$
4. Both $\mu_m(\rho)$ and $\nu_m(\rho)$ do not depend on transversal quantum number. They depend only on longitudinal quantum number which is related to the width of d_1. Therefore, for the increasing longitudinal m the resonance width is increasing.

Dirichlet cavity with small aperture

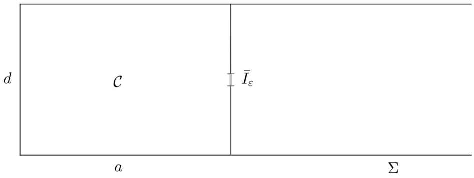


Figure: Geometry of the waveguide Σ with a cavity $\mathcal{C}$ containing a gap $\bar{I}\varepsilon$.

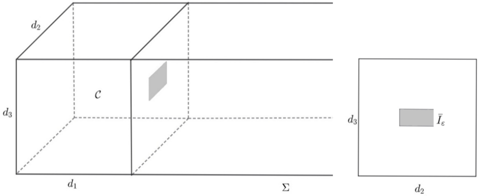


Figure: Geometry of the waveguide Σ with a cavity $\mathcal{C}$ containing a gap $\bar{I}\varepsilon$.

Dirichlet cavity with small aperture

$$\sigma_{\mathrm{ess}}(-\Delta_{I_\varepsilon}^{D}) = \left[\left(\frac{\pi}{d_2}\right)^2, \infty\right). \tag{46}$$

For a closed cavity, i.e., when $\varepsilon = 0$, the trapped modes give rise to discrete energy levels

$$\epsilon_{l,k} = \left(\frac{\pi l}{d_1}\right)^2 + \left(\frac{\pi k}{d_2}\right)^2,$$

which, in fact, represent eigenvalues embedded in the essential spectrum.

Conjecture. SK, N. Kurtskhalia'25

Theorem

Assume that $\epsilon_{l,k}$ is a simple eigenvalue of $-\Delta^D(\mathcal{C})$. Then there exists a uniquely determined function $\varepsilon \mapsto z(\varepsilon)$ such that $\ker K^\varepsilon\big(z(\varepsilon)\big) \neq \emptyset$, with the following asymptotic expansion

$$z(\varepsilon) = \epsilon_{l,k} + \mu(\varepsilon) + i\,\nu(\varepsilon), \tag{47}$$

where the real-valued functions $\mu(\varepsilon)$ and $\nu(\varepsilon)$ satisfy

$$\mu(\varepsilon) = \mathcal{O}(\varepsilon^2) \quad \text{and} \quad \nu(\varepsilon) = \mathcal{O}(\varepsilon^2)\,.$$

Now, suppose that $\epsilon_{l,k}$ has multiplicity N. In this case, there exist N functions $\varepsilon \mapsto z_j(\varepsilon)$, for $j = 1, \ldots, \mathrm{N}$, such that $\ker K^\varepsilon\big(z_j(\varepsilon)\big) \neq \emptyset$, with each function admitting the asymptotic expansion

$$z_j(\varepsilon) = \epsilon_{l,k} + \mu_j(\varepsilon) + i\,\nu_j(\varepsilon),$$

where the corresponding functions $\mu_j(\varepsilon)$ and $\nu_j(\varepsilon)$ behave as $\mathcal{O}(\varepsilon^2)$.

Open question:

1. Higher dimensional systems with attractive delta interaction on loop and $\alpha \to \infty$. Resonances with resonant width $\mathcal{O}(\alpha^{-2})$?
2. Properties of the survival probability.

Thank you