Analytic and algebraic methods in physics

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Approximation of non-local interface ~~and boundary~~ conditions via homogenization

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University of Hradec Králové
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P. Exner, A.K., arXiv:2505.19016

Introduction

δ' interaction in $1d$

$$u'(0+) = u'(0-), \quad u'(0) = K \cdot (u(0+) - u(0-)), \quad K \in \mathbb{R}.$$

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Surface δ' interaction

The surface δ' interaction on the hypersurface $\Gamma \subset \mathbb{R}^n$:

$$\left(\frac{\partial u}{\partial \nu}\right)^{\pm} = \pm K \cdot (u^{\pm} - u^{\mp}).$$

Here $u^{\pm}$ denote the traces of the wave function on either side of Γ, $\left(\frac{\partial u}{\partial \nu}\right)^{\pm}$ are the corresponding traces of the normal derivative taken with respect to the unit normal vector field directed from the 'minus' side of Γ towards the 'plus' side, $K : \Gamma \to \mathbb{R}$.

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[J. Behrndt, M. Langer, V. Lotoreichik, AHP 14 (2013)]
[J. Behrndt, P. Exner, V. Lotoreichik, Rev. Math. Phys. 26 (2014)

Non-local δ'-type surface interaction

$$\left(\frac{\partial u}{\partial \nu}\right)^{\pm}(x) = \pm \int_{\Gamma} K(x,y)(u^{\pm}(x) - u^{\mp}(y))\,\mathrm{d}s_y$$

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Remark

Along with the Laplacian with these non-local interface conditions, we may also consider (not in this talk though!) the Laplacian subject to non-local Robin-type boundary conditions of the form

$$\frac{\partial u}{\partial \nu}(x) = \pm \int_\Gamma K(x,y)(u(x) - u(y))\,\mathrm{d}s_y$$

Here Γ is a subset of $\partial\Omega$, ν is the outward unit normal to $\partial\Omega$.

The analysis of Laplacians with abstract non-local boundary conditions (covering those mentioned above) has been carried out in

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[W. Feller, Ann. Math. 55 (1952)]

[W. Feller, Trans. Amer. Math. Soc. 77 (1954)]

[A.D. Ventsel, Theory Probab. Appl. 4 (1959)]

and some key follow-ups

[K.-i. Sato, T. Ueno, J. Math. Kyoto Univ. 4 (1965)]

[K. Taira, Mem. Am. Math. Soc. 99 (1992)]

[E.I. Galakhov, A.L. Skubachevskii, J. Differ. Eqs. 176 (2001)]

[W. Arendt, S. Kunkel, M. Kunze, J. Math. Soc. Japan 70 (2018)]

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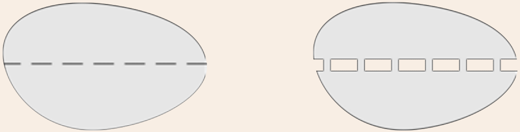
Approximation of $1d$ δ'-interactions

One can approximate δ'-interactions by Schrödinger operators with a **triple** of properly scaled δ-like regular potentials.

[T. Cheon, T. Shigehara, Phys. Lett. A 43 (1998)]

[P. Exner, H. Neidhardt, V.A. Zagrebnov, CMP 224 (2001)]

(Geometric) approximation of surface δ'-interactions



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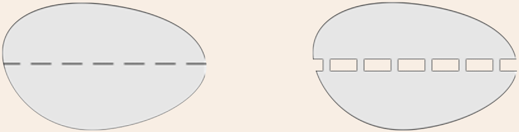

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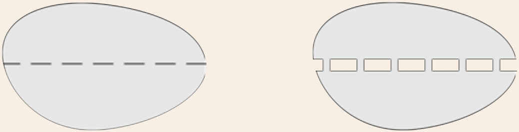

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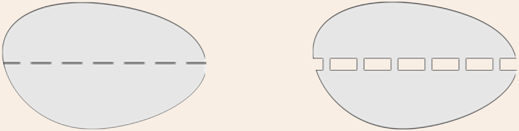

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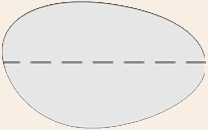

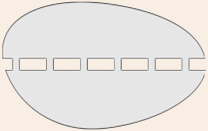

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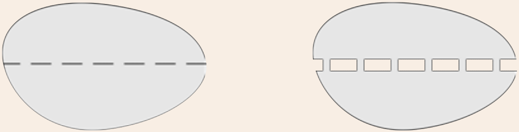

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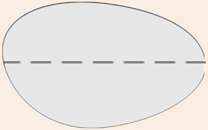

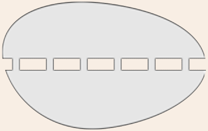

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(also, Attauch, Damlamian, Murat, Picard, Cioranescu, K. ...)

Main results

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To avoid these troubling features, we consider the Neumann Laplacian not on a domain in $\mathbb{R}^n$, but on a **Riemannian manifold**.

- $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain containing the origin.

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In $L^2(\Omega)$ we consider the sesquilinear form $\mathfrak{h}$ given by

$$\mathrm{dom}(\mathfrak{h}) = H^1(\Omega \setminus \Gamma),$$

$$\mathfrak{h}[u,v] = \int_{\Omega\setminus\Gamma} \nabla u \cdot \overline{\nabla v}\, \mathrm{d}x + \int_\Gamma \int_\Gamma K(x,y)(u^+(x) - u^-(y))\overline{(v^+(x) - v^-(y))}\, \mathrm{d}s_x\, \mathrm{d}s_y$$

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We denote by $\mathscr{H}$ the self-adjoint positive operator associated with $\mathfrak{h}$.

Note that $u = (\mathscr{H} + \mathrm{Id})^{-1} f$ with $f \in L^2(\Omega)$ is the weak solution to the following boundary value problem:

$$
\begin{cases}
-\Delta u + u = f & \text{in } \Omega^\pm, \\
\dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \\
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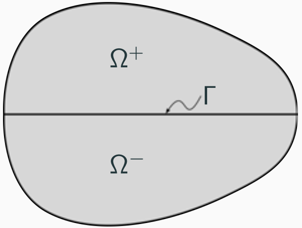
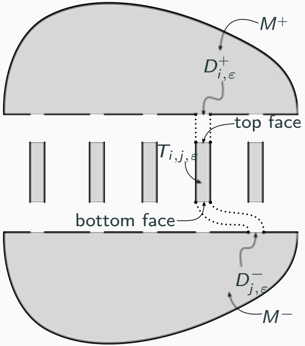
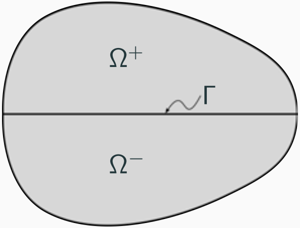
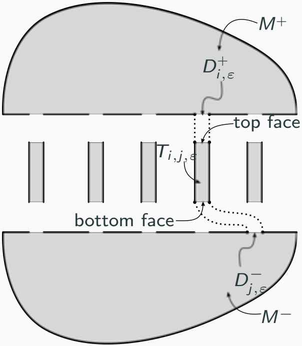
Our goal is to approximate the resolvent $(\mathscr{H} + \mathrm{Id})^{-1}$ of the operator $\mathscr{H}$ by the resolvent $(\mathscr{H}_\varepsilon + \mathrm{Id})^{-1}$ of the Neumann Laplacian $\mathscr{H}_\varepsilon$ on an appropriately constructed Riemannian manifold M_ε.


Ω^+
Γ
Ω^-


M^+
$D^+_{i,\varepsilon}$
top face
$T_{i,j,\varepsilon}$
bottom face
$D^-_{j,\varepsilon}$
M^-





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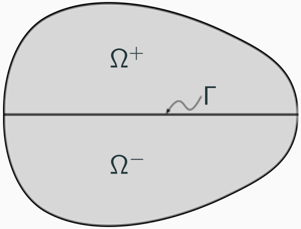

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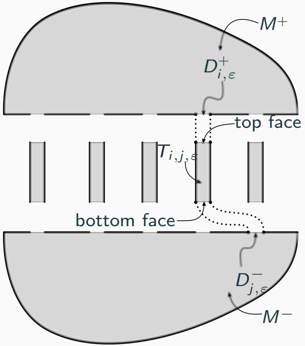

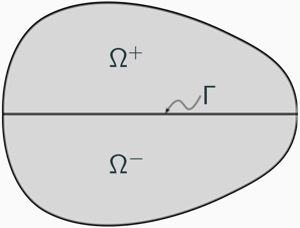

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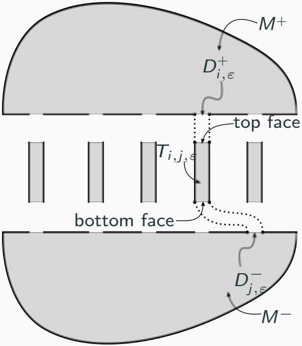

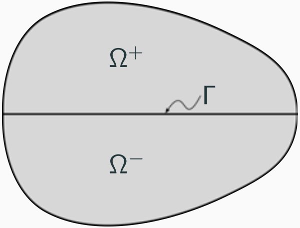

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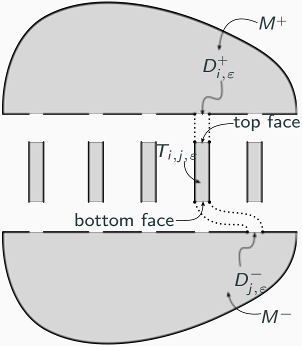
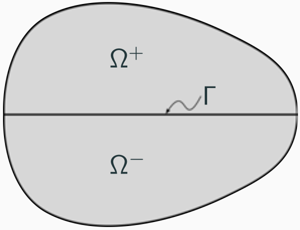
- There is a bijective map $\ell_\varepsilon : \mathbb{I}_\varepsilon \to \mathbb{I}_\varepsilon$ such that

$$\mathbf{D}^+_{i,\varepsilon} \cong \mathbf{D}^-_{j,\varepsilon} \text{ provided } j = \ell_\varepsilon(i).$$





- $M^\pm := \Omega^\pm \pm e_n$, $\Gamma^\pm := \Gamma \pm e_n$, $e_n := (0, \dots, 1)$.
- $\{D^\pm_{i,\varepsilon} \subset \Gamma^\pm,\ i \in \mathbb{I}_\varepsilon\}$ be two families of pairwise disjoint relatively open sets; $\mathbb{I}_\varepsilon$ is a finite set.
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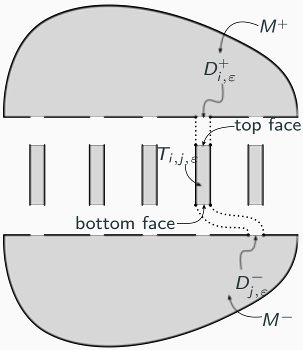

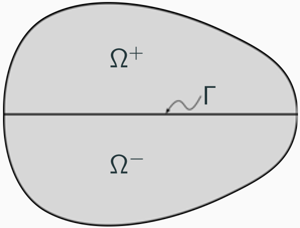

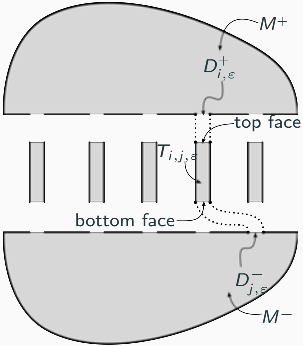

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- For $(i,j) \in \mathbb{L}_\varepsilon$, we glue $T_{i,j,\varepsilon}$ to M^+ (resp., M^-) by identifying the top face of $T_{i,j,\varepsilon}$ and $D^+_{i,\varepsilon}$ (resp., the bottom face of $T_{i,j,\varepsilon}$ and $D^-_{j,\varepsilon}$).

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$$\mathrm{dom}(\mathfrak{h}_\varepsilon) = H^1(M_\varepsilon), \quad \mathfrak{h}_\varepsilon[u,v] = (\nabla u, \nabla v)_{L^2(M_\varepsilon)}$$

$$= \int_{M^+} \sum_{k=1}^n \frac{\partial u}{\partial x^k} \frac{\partial \overline{v}}{\partial x^k} \,\mathrm{d}x + \int_{M^-} \sum_{k=1}^n \frac{\partial u}{\partial x^k} \frac{\partial \overline{v}}{\partial x^k} \,\mathrm{d}x + \sum_{(i,j)\in\mathbb{L}_\varepsilon} \int_{T_{i,j,\varepsilon}} \sum_{k=1}^n \frac{\partial u}{\partial x^k} \frac{\partial \overline{v}}{\partial x^k} \,\mathrm{d}x$$

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$\sup_{i\in\mathbb{I}_\varepsilon} \rho_{i,\varepsilon}^{\pm} \ln d_{i,\varepsilon}^{\pm} \to 0$ as $\varepsilon \to 0$ as $n = 2$.

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(C8) For each $v \in C(\overline{\Gamma} \times \overline{\Gamma})$ one has

$$\sum_{(i,j)\in\mathbb{L}_\varepsilon} |\mathbf{D}^+_{i,\varepsilon}|h^{-1}_{i,j,\varepsilon} v(x^+_{i,\varepsilon}, x^-_{j,\varepsilon}) \to \int_{\Gamma\times\Gamma} K(x,y)v(x,y)\,\mathrm{d}s_x\,\mathrm{d}s_y \text{ as } \varepsilon \to 0,$$

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Since $\mathscr{H}_\varepsilon$ and $\mathscr{H}$ act in different Hilbert spaces $L^2(M_\varepsilon)$ and $L^2(\Omega)$, we need a suitable operator $\mathscr{J}_\varepsilon : L^2(M_\varepsilon) \to L^2(\Omega)$.

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Theorem 1

Let $\{f_\varepsilon \in L^2(M_\varepsilon)\}_\varepsilon$ be a family of functions satisfying

$$\lim_{\varepsilon\to 0}\sum_{(i,j)\in\mathbb{L}_\varepsilon}\|f\|^2_{L^2(T_{i,j,\varepsilon})} = 0,\quad \mathscr{J}_\varepsilon f_\varepsilon \rightharpoonup f \text{ in } L^2(\Omega) \text{ as } \varepsilon\to 0 \qquad (\star)$$

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with some $f\in L^2(\Omega)$. Then, the following holds:

$$\mathscr{J}_\varepsilon(\mathscr{H}_\varepsilon+\mathrm{Id})^{-1}f_\varepsilon \rightharpoonup (\mathscr{H}+\mathrm{Id})^{-1}f \text{ in } H^1(\Omega\setminus\Gamma) \text{ as } \varepsilon\to 0.$$

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$$\|\mathscr{J}_\varepsilon u_\varepsilon - u\|_{L^2(\Omega)} \to 0.$$

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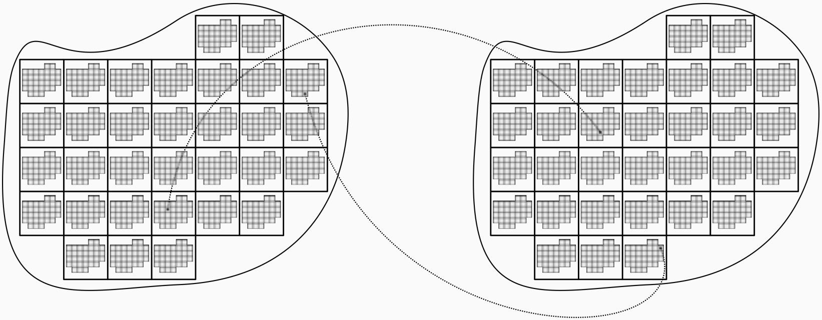
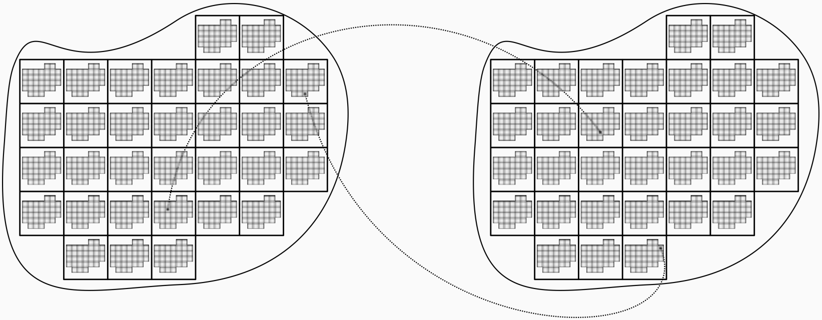
Then the following holds:

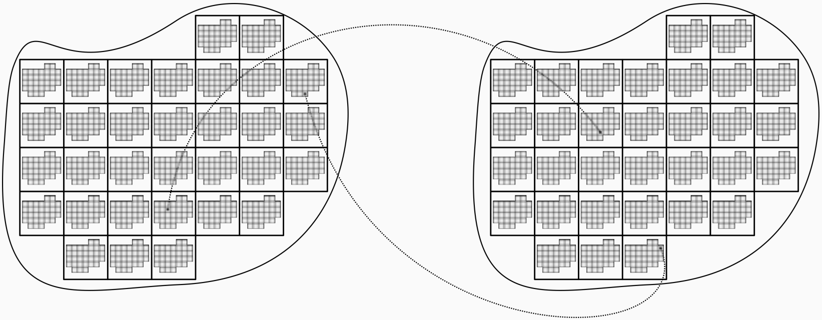
$$\forall T > 0: \lim_{\varepsilon \to 0} \max_{t \in [0,T]} \left\| \mathscr{J}_\varepsilon(\exp(-\mathscr{H}_\varepsilon t) f_\varepsilon) - \exp(-\mathscr{H} t) f \right\|_{L^2(\Omega)} = 0.$$

Example

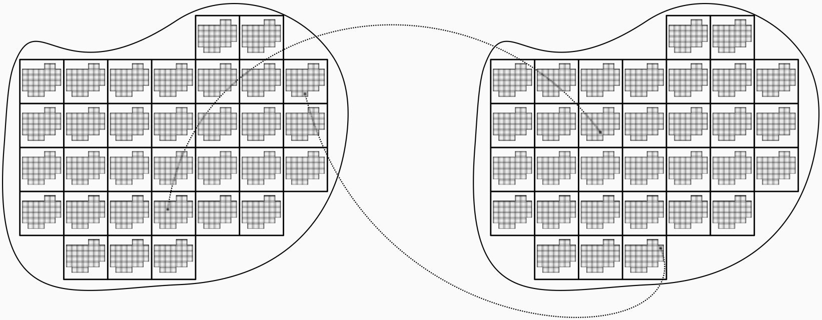




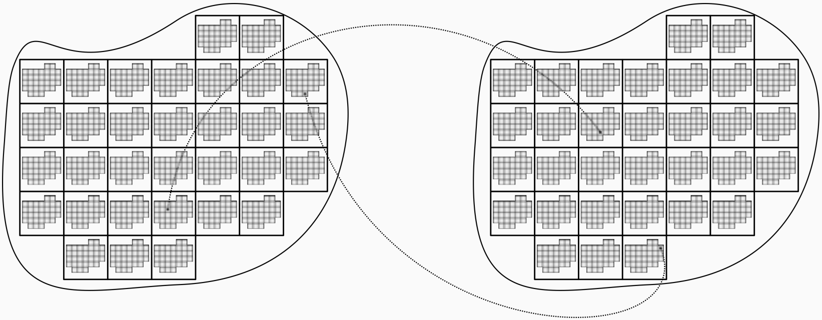
- The sets Γ^+ (left) and Γ^- (right).



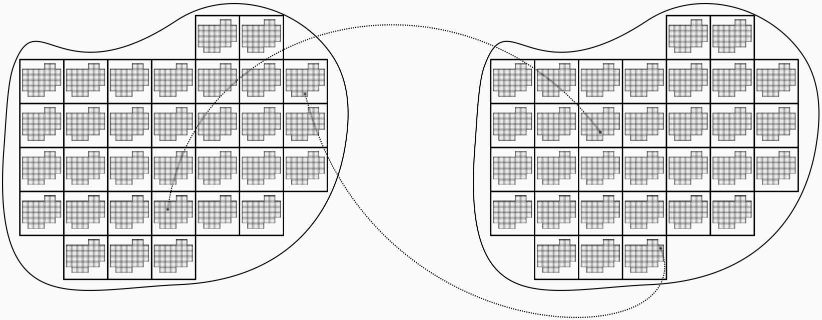
- The sets Γ^+ (left) and Γ^- (right).
- The big cubes have a side length ε.



- The sets Γ^+ (left) and Γ^- (right).
- The big cubes have a side length ε.
- The small cubes have a side length $C\varepsilon^2$.



- The sets Γ^+ (left) and Γ^- (right).
- The big cubes have a side length ε.
- The small cubes have a side length $C\varepsilon^2$.
- The sets $D_{i,\varepsilon}^{\pm}$ are placed in the centers of the small cubes.



- The sets Γ^+ (left) and Γ^- (right).
- The big cubes have a side length ε.
- The small cubes have a side length $C\varepsilon^2$.
- The sets $D_{i,\varepsilon}^{\pm}$ are placed in the centers of the small cubes.
- The dotted lines represent pairs of small cubes such that the sets $D_{i,\varepsilon}^{+}$ and $D_{j,\varepsilon}^{-}$ located within these cubes, are connected by $T_{i,j,\varepsilon}$.

Thank you for the attention !