Comparison Principles for Non-local Operators

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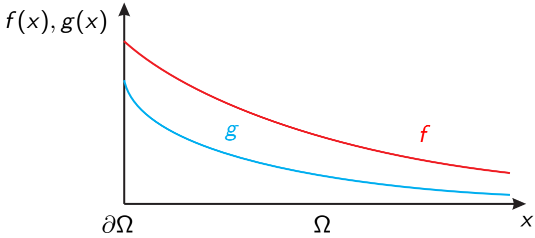
Outline

- Comparison Principles
- Agmon's Subharmonic Comparison Lemma
- Discrete Variant
- Non-Local Variant

Subharmonic Comparison Principles

- developed to study solutions of Schrödinger equation by Agmon
- compare subsolution and supersolution of given equation
- Let $E \in \mathbb{R}$.
  - We call f a supersolution of H at energy E in Ω if
$$\langle \phi, (H-E)f \rangle \geq 0\,, \quad \forall \phi \in C_c^\infty(\Omega), \phi \geq 0\,.$$
  - We call g a subsolution of H at energy E in Ω if
$$\langle \phi, (H-E)g \rangle \leq 0\,, \quad \forall \phi \in C_c^\infty(\Omega), \phi \geq 0\,.$$



Agmon's Subharmonic Comparison Lemma

Setting of the Problem

- Let $H = -\Delta + V$ be a Schrödinger operator defined by the quadratic form

$$\langle \varphi, H\psi \rangle := \langle \nabla\varphi, \nabla\psi \rangle + \langle \varphi, V\psi \rangle$$

  for $\varphi, \psi \in H^1(\mathbb{R}^d)$
- V is an infinitesimally form small perturbation of $\langle \nabla\cdot, \nabla\cdot \rangle$
- $\langle \varphi, V\psi \rangle = \langle \mathrm{sgn}(V)|V|^{1/2}\varphi, |V|^{1/2}\psi \rangle$ where $\mathrm{sgn}(t) = t/|t|$ when $t \neq 0$ and $\mathrm{sgn}(0) := 0$.

Inner Boundary Layer $\widetilde{\partial\Omega}$

- Let $\Omega \subset \mathbb{R}^d$ be an open set with boundary $\partial\Omega$
- $\widetilde{\partial\Omega}$ is a subset of the closure of Ω
- $\widetilde{\partial\Omega}$ contains the boundary of Ω
- $\Omega \setminus \widetilde{\partial\Omega}$ has locally positive distance from the boundary $\partial\Omega$.

Agmon's Subharmonic Comparison Lemma

Agmon's Subharmonic Comparison Lemma

Let Ω be an open subset of $\mathbb{R}^d$, f a positive supersolution and g a subsolution of H at energy E in Ω. Assume that for some $\lambda > 1$

$$\liminf_{L\to\infty} L^{-2} \int_{\Omega\cap\{L\leq|x|\leq\lambda L\}} g_+^2\, dx = 0$$

where $g_+ = \sup(g, 0)$ is the positive part of g, and

$$f \geq g \quad \text{for (almost) all } x \in \widetilde{\partial\Omega}$$

where $\widetilde{\partial\Omega}$ is an inner boundary layer of Ω. Then $f \geq g$ almost everywhere in Ω.

- in original Agmon result Ω is a neighborhood of infinity

Sketch of Proof

Main Ingredients of the Proof

IMS formula

$$\mathrm{Re}\langle\nabla(\xi^2\varphi),\nabla\varphi\rangle = \langle\nabla(\xi\varphi),\nabla(\xi\varphi)\rangle - \langle\varphi,|\nabla\xi|^2\varphi\rangle$$

where $\varphi \in H^1(\mathbb{R}^d)$, $\xi \in W^{1,\infty}(\mathbb{R}^d)$
especially

$$\mathrm{Re}\langle\xi^2\varphi, H\varphi\rangle = \langle\xi\varphi, H\xi\varphi\rangle - \langle\varphi,|\nabla\xi|^2\varphi\rangle$$

Positive Part as Subsolution

Let g be a real subsolution of H at energy E in Ω then $g_+ := \max\{g, 0\}$ is a positive subsolution of H at energy E in Ω

Sketch of Proof

- f supersolution, g subsolution of $H - E$
- define $u := (g - f)_+$ which is subsolution of $H - E$
- take $\xi \in C_0^\infty(\mathbb{R}^d)$
- u is a subsolution
  $\langle \xi u, (H-E)\xi u\rangle - \langle u, |\nabla \xi|^2 u\rangle = \mathrm{Re}\langle \xi^2 u, (H-E)u\rangle = \langle \xi^2 u, (H-E)u\rangle \leq 0$
  i.e. $\langle \xi u, (H-E)\xi u\rangle \leq \langle u, |\nabla \xi|^2 u\rangle \quad$ for all $\xi \in C_0^\infty(\Omega)$
- f is a supersolution, $\rho = \frac{\xi u}{f}$
  $0 \leq \langle \varphi, (H-E)f\rangle = \langle \rho^2 f, (H-E)f\rangle = \langle \xi u, (H-E)\xi u\rangle - \langle f, \left|\nabla\left(\frac{\xi u}{f}\right)\right|^2 f\rangle$
  i.e. $\langle f, \left|\nabla\left(\frac{\xi u}{f}\right)\right|^2 f\rangle \leq \langle \xi u, (H-E)\xi u\rangle \quad$ for all $\xi \in C_0^\infty(\Omega)$
- the previous gives us
  $\langle f, \left|\nabla\left(\frac{u}{f}\right)\right|^2 f\rangle \leq \langle u, |\nabla \xi|^2 u\rangle$
- considering a sequence $\xi_L \to 1$ we get

$$u = cf \quad \text{in} \quad x \in \Omega$$

which completes the proof because $u = 0$ in $\widetilde{\partial\Omega}$ and $f > 0$ everywhere

Discrete Setting

Discrete Laplacian

$$(-\Delta\psi)_n := \sum_{\substack{m\in\mathbb{Z}^d \\ |m-n|=1}} (\psi_n - \psi_m) = 2d\psi_n - \sum_{\substack{m\in\mathbb{Z}^d \\ |m-n|=1}} \psi_m, \quad n \in \mathbb{Z}^d,$$

Discrete Schrödinger Operator

$$H_V := -\Delta + V$$

where $(V\psi)_n := V_n\psi_n, n \in \mathbb{Z}^d$ is real multiplication operator on maximal domain $\mathrm{Dom}\, V := \{\psi \in \ell^2(\mathbb{Z}^d) \mid V\psi \in \ell^2(\mathbb{Z}^d)\}$

Subsolutions and Supersolutions

Let $\Omega \subset \mathbb{Z}^d$ and $\lambda \in \mathbb{R}$. Functions $u, w : \mathbb{Z}^d \to \mathbb{R}$ are called *subsolution* and *supersolution* of the equation $(H_V - \lambda)\psi = 0$ in Ω, if

$$[(H_V - \lambda)u]_n \leq 0 \quad \text{and} \quad [(H_V - \lambda)w]_n \geq 0,$$

for all $n \in \Omega$, respectively.

Discrete Agmon's Comparison Principle

Discrete Comparison Principle (Jex, Štampach 2025)

Let $N \in \mathbb{N}$, $\lambda \in \mathbb{R}$, and w be a strictly positive supersolution of $(H_V - \lambda)\psi = 0$ in $\mathbb{Z}^d_{\geq N}$. Suppose further that u is a subsolution of $(H_V - \lambda)\psi = 0$ in $\mathbb{Z}^d_{\geq N}$ satisfying

$$\liminf_{M\to\infty} \frac{1}{M^2} \sum_{M\leq|n|\leq\alpha M} \sum_{j=1}^{d} |u_n u_{n-\delta_j}| = 0$$

for some $\alpha > 1$. Then, for all $n \in \mathbb{Z}^d_{\geq N}$, one has

$$u_n \leq C w_n,$$

where C is any positive constant such that

$$C \geq \max\left\{ \frac{u_n}{w_n} \;\middle|\; N-1 \leq |n| < N+1 \right\}.$$

Sketch of Proof

Main Ingredients of the Proof

Summation by Parts Identity

Let $\phi, \psi \in C_c(\mathbb{Z}^d)$. If $\phi_n = 0$ for all $n \in \mathbb{Z}^d$ such that $N - 1 \le |n| < N + 1$, then the identity

$$\sum_{|n| \ge N} \phi_n \left(D_j^* \psi\right)_n = \sum_{|n| \ge N} (D_j \phi)_n \psi_n$$

holds for all $j \in \{1, \dots, d\}$.

Positive Part as Subsolution

If u is a subsolution of $(H_V - \lambda)\psi = 0$ in $\Omega \subset \mathbb{Z}^d$, then $u_+ := \max(0, u)$ is also a subsolution of $(H_V - \lambda)\psi = 0$ in Ω.

Setting of Problem

Hamiltonian

$$H = K + V$$

for suitable multiplication potential V

Non-local part of operator
defined by quadratic form

$$\langle \varphi, K\psi \rangle := \int K(x,y)\overline{(\varphi(x) - \varphi(y))}(\psi(x) - \psi(y))\mathrm{d}x\mathrm{d}y$$

where $K(x,y) \geq 0$ for all x, y.

Subsolutions and Supersolutions

Let $\alpha \in \mathbb{R}$. We call u a α-subsolution of H in Ω if

$$\langle \phi, (H - \alpha)u \rangle \leq 0\,, \quad \forall \phi \in C_c^\infty(\Omega), \phi \geq 0\,.$$

We call v a α-supersolution of H in Ω if

$$\langle \phi, (H - \alpha)v \rangle \geq 0\,, \quad \forall \phi \in C_c^\infty(\Omega), \phi \geq 0\,.$$

Non-Local Comparison Principle

Hypothesis 1

Let $u \in \mathcal{D}(K)$. We say that u is admissible if there exists a sequence $\chi_R \in C_c^\infty(\Omega)$, $\chi_R \xrightarrow{R\to\infty} 1$ pointwise such that

$$\liminf_{R\to\infty} \int_{\Omega^2} K(x,y) \left(\chi\left(\frac{x}{R}\right) - \chi\left(\frac{y}{R}\right)\right)^2 u_+(x) u_+(y) = 0$$

holds.

Comparison Principle

Let u be a α-subsolution of $K + V$ in Ω and v be positive α-supersolution of $K + V$ in Ω. Furthermore assume that u satisfies Hypothesis 1 and there exists $C, \delta \in \mathbb{R}^+$

$$u(x) \le Cv(x)\,, \quad \forall x \in \{y \in \Omega | d(y, \partial\Omega) < \delta\}$$

then

$$u(x) \le Cv(x)\,, \quad \forall x \in \Omega$$

“IMS” formula

“IMS” formula

Let K be as above, then

$$\mathrm{Re}\langle \xi^2\varphi, K\varphi\rangle = \langle \xi\varphi, K\xi\varphi\rangle - \mathcal{L}_\xi(\varphi,\varphi)$$

where $\mathcal{L}_\xi(\varphi,\psi) = \int K(x,y)\overline{(\xi(x)-\xi(y))}(\xi(x)-\xi(y))\overline{\varphi}(x)\psi(y)\mathrm{d}x\mathrm{d}y$.

Sketch of Proof

- write left hand side as

$$\mathrm{Re}\langle \xi^2\varphi, K\varphi\rangle = \int K(x,y)\mathrm{Re}\left[\overline{(\xi(x)^2\varphi(x)-\xi(y)^2\varphi(y))}(\varphi(x)-\varphi(y))\right]\mathrm{d}x\mathrm{d}y$$

- use the identity for complex numbers

$$\mathrm{Re}\overline{(a^2c-b^2d)}(c-d) = |ac-bd|^2 - (a-b)^2\mathrm{Re}(c\overline{d})$$

  where $c,d\in\mathbb{C}$ and $a,b\in\mathbb{R}$
- $\mathcal{L}_\xi(\varphi,\varphi)$ being symmetric completes the proof

Subsolution - Positive Part

Positive part

Let K be as above and let ψ be a real valued α-subsolution of $H = K + V$. Then ψ_+ is α-subsolution of H.

Sketch of Proof

- introduce notation

$$\psi_\varepsilon = \sqrt{|\psi|^2 + \epsilon} = \sqrt{\psi^2 + \epsilon}$$

- one can see that $2\psi_+ = \psi + |\psi|$ and $2\tilde{\psi}_+ = \psi + \psi_\varepsilon \xrightarrow{\varepsilon \to 0} 2\psi_+$
- calculate

$$2\langle \varphi, K\tilde{\psi}_\varepsilon \rangle = \langle \varphi, K\psi \rangle + \langle \varphi, K\psi_\varepsilon \rangle = \left\langle \varphi \left(1 + \frac{\psi}{\psi_\varepsilon} \right), K\psi \right\rangle + \mathcal{E}_\mathrm{r}$$

where $\mathcal{E}_\mathrm{r} = \langle \varphi, K\psi_\varepsilon \rangle - \left\langle \varphi \frac{\psi}{\psi_\epsilon}, K\psi \right\rangle$

- show that $\mathcal{E}_\mathrm{r} \leq 0$

Sketch of Proof

- analogous to previous theorems
- use IMS
- use the fact that positive part of subsolution is again subsolution
- define $w(x) = (u(x) - Cv(x))_+$
- take

$$0 \geq \langle \xi^2 w, (K + V - \mu) w \rangle = \langle \xi w, (K + V - \mu) \xi w \rangle - \mathcal{L}_\xi(w, w)$$

and for $\rho = \frac{\xi w}{v}$

$$0 \leq \langle \rho^2 v, (K + V - \mu) v \rangle = \langle \xi w, (K + V - \mu) \xi w \rangle - \mathcal{L}_\rho(v, v)$$

References

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- nonlocal variant
  Hundertmark, Jex: on arXiv soon

Thank you for your attention