Magnetic quantum transport

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To move the particle from one place to another, possibly at a large distance, we need to perturb such a 'free' Hamiltonian. In the most classical example the perturbation is a *variation of the field*.

Iwatsuka transport

Suppose that the variation is *translationally invariant* being, e.g., of the form $B(x) = B + b(x)$ with a fixed $B > 0$ where $b(\cdot)$ is compactly supported

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$$H(p) = -\partial_x^2 + (p + Bx + a(x))^2, \quad a(x) := \int_0^x b(t)\,\mathrm{d}t.$$

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There is a conjecture saying that the spectrum of the corresponding magnetic Laplacian is purely absolutely continuous for *any* nontrivial field variation of this type but it remains open

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What we know is that the claim is true in a number of particular cases, e.g.

M. Mantoiu, R. Purice: Some propagation properties of the Iwatsuka model, *Commun. Math. Phys.* **188** (1997), 691–708

P.E., H. Kovařík: Magnetic strip waveguides, *J. Phys. A: Math. Gen.* **33** (2000), 3297–3311

P. Miranda, N. Popoff: Spectrum of the Iwatsuka Hamiltonian at thresholds, *J. Math. Anal. Appl.* **460** (2018), 516–545

N. Raymond, J. Royer: Absence of embedded eigenvalues for translationally invariant magnetic Laplacians, *J. Math. Phys.* **60** (2019), 073506

Transport along an obstacle

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As long as the curve is finite, $|\Gamma| < \infty$, there is *no real transport*; the spectrum contains now discrete eigenvalue clusters accumulating at a known rate to the Landau levels, but $\sigma_{\mathrm{c}}(H) = \emptyset$.

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The situation change when the obstacle is infinitely extended. As in the Iwatsuka case we will suppose that it is *translationally invarient*, that is, Γ is a *straight line*. The invariance motivates us to use the *Landau gauge*, $\vec{A} = (0, Bx, 0)$, obtaining

$$H = -\partial_x^2 + (-i\partial_y + Bx)^2 + \alpha\delta(x).$$

By a partial Fourier transformation, we get then the unitary equivalence to the direct integral $H \approx \int_{\mathbb{R}}^{\oplus} H(p)\,\mathrm{d}p$ with the fibers

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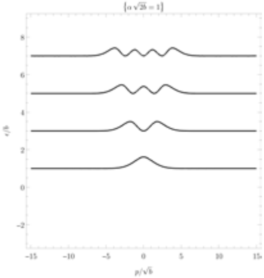

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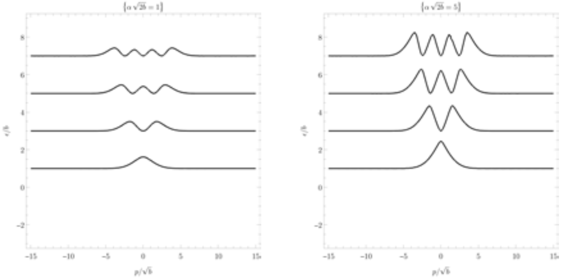

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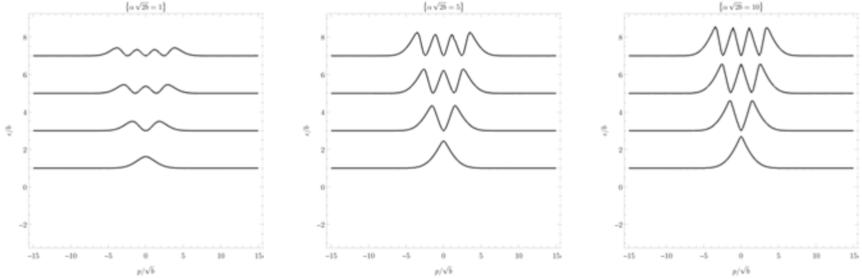

M. Grňo: *Magnetic transport along translationally invariant obstacles*, Bachelor thesis, CTU 2021

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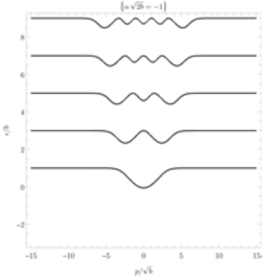
In a similar way, the dispersion curves for $\alpha < 0$ are below the Landau levels; being again more ‘pronounced’ as $|\alpha|$ increases:



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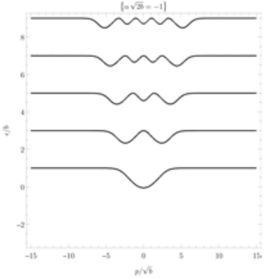

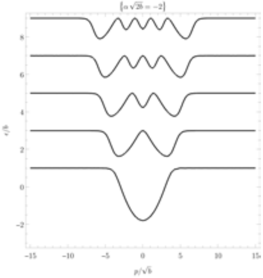

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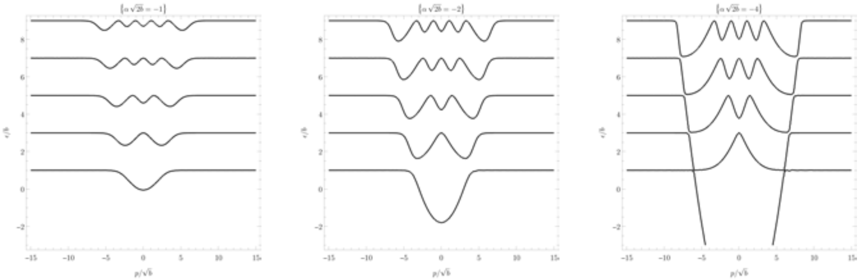

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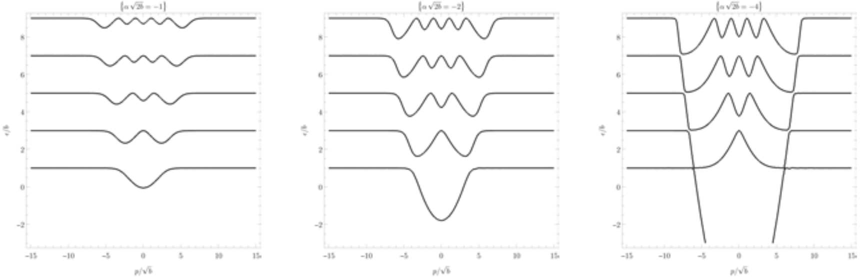

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Obviously, as $|\alpha|$ increases, the dispersion curves in most part of the spectrum resemble those of magnetic transport in *Dirichlet halfplane*, however, the pictures suggests that the gaps below/above the Landau levels *remain open*.

Transport along a regular potential obstacle

Replacing the singular interaction by a nontrivial *regular* potential v, the problem reduces to investigation of the fiber operators

$$h_v(p) = -\partial_x^2 + (p + Bx)^2 + v(x), \quad p \in \mathbb{R},$$

on $L^2(\mathbb{R})$. If we assume that

- $v_+ \in L^2_{\mathrm{loc}}(\mathbb{R})$ and $v_- \in (L^2 + L^\infty)(\mathbb{R})$, where $v_\pm(x) := \max(\pm v(x), 0)$, and $\liminf_{|x|\to\infty}(v(x) + \beta B^2 x^2) > -\infty$ for some $\beta < 1$

the operator $h_v(p)$ is for any $p \in \mathbb{R}$ essentially self-adjoint on $C_0^\infty(\mathbb{R})$ and its spectrum is *purely discrete* and simple consisting of eigenvalues $\epsilon_n(p)$, $n = 0, 1, 2, \dots$. If needed we write $\epsilon_n(p, v)$; the corresponding normalized real-valued eigenfunctions will be $\phi_n(\cdot; p)$ or $\phi_n(\cdot; p, v)$.

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To prove the *pure absolute continuity*, one has to check that

- the family $\{h_v(p) : \ p \in \mathbb{R}\}$ is analytic with respect to p,
- no eigenvalue branch $\epsilon_n(\cdot)$ is constant.

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To prove the claim one should note that $\epsilon_n(p)$ coincide with eigenvalues of the 'shifted' operator, a perturbed harmonic oscillator

$$\tilde{h}_v(p) = -\partial_x^2 + B^2x^2 + v\left(x - \tfrac{p}{B}\right)$$

and check that $\lim_{p\to\pm\infty} \epsilon_n(p) = v_\pm + B(2n+1)$ holds for any $n \in \mathbb{N}_0$.

A perturbative result

If we strengthen assumption (i) and suppose that

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First of all, we need to check that $\lim_{p\to\pm\infty} \epsilon_n(p) = B(2n+1)$. To this aim, we use that quadratic form associated with $\tilde{h}_v(p)$,

$$\tilde{q}_v(p)[\psi] = \int_{\mathbb{R}} |\psi'(x)|^2 \mathrm{d}x + \int_{\mathbb{R}} |Bx\psi(x)|^2 \mathrm{d}x + \int_{\mathbb{R}} (\psi, v(x - \tfrac{p}{B})\psi)$$

and check that on their common core, $C_0^\infty(R)$, they converge to $\tilde{q}_0(p)$; this claim is by Kato equivalent to the strong resolvent convergence of the operators, $\tilde{h}_v(p) \to h_0$ as $|p| \to \infty$.

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Next we fix a j and replace v by λv; denoting then the normalized real-valued oscillator eigenfunctions by φ_j, we get

$$\epsilon_j(p;\lambda v) = B(2j+1) + \lambda \int_{\mathbb{R}} v\big(x - \tfrac{p}{B}\big)\, \varphi_j(x)^2 \,\mathrm{d}x + \mathcal{O}(\lambda^2)$$

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The first-order term is a *convolution type expression*; by assumption (ii) and injectivity of the Fourier transformation we infer that it is nonzero, hence $\epsilon_j(\cdot;\lambda v)$ is nonconstant for all small enough λ, and the claim extends to any finite collection of the dispersion curves.

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Corollary

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Before proceeding further, let us mention situations when the potential v does not satisfy the above assumption being *below unbounded*.

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3. If $\alpha = 0$ and $B^2 - \beta^2 < 0$ the last claim remains true but there is a significant difference: the fiber operator is in this case still e.s.a., but the spectrum of the fiber operators is purely a.c., without eigenfunctions, in other words, there are *no states which could be transported along the barrier*.

Attractive potentials

Returning to suitably localized potential, consider *sign preserving* ones. The sign matter here; let us begin with

- iii $v \in L^1(\mathbb{R})$ is *purely attractive*, that is, $v = -v_- \neq 0$.

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To prove the claim, let L_n be the subspace spanned by the first $n+1$ oscillator eigenfunctions φ_j; for any $\psi \in L_n$ we obviously have $\tilde{q}_0(p)[\psi] \leq B(2n+1)\|\psi\|^2$, and consequently,

$$\tilde{q}_v(p)[\psi] \leq B(2n+1)\|\psi\|^2 + \int_{\mathbb{R}} v\left(x - \tfrac{p}{B}\right)|\psi(x)|^2 \mathrm{d}x$$

Each function ψ is bounded and nonzero a.e., hence *the last term is negative*, and the same is true for its maximum over unit sphere in L_n; the claim then follows from the min-max principle.

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Estimating eigenvalues *from below* by min-max is more complicated; to deal with the case $v \geq 0$, we need an additional tool.

Feynman-Hellmann trick

We have to impose stronger regularity requirements on the potential:

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Let $\tilde\phi_n$ be the normalized eigenfunction, $\tilde{h}_v(p)\tilde\phi_n = \epsilon_n(p)\tilde\phi_n$. We can choose it real-valued, in which case we have

$$\epsilon_n'(p) = 2\int_{\mathbb{R}} \Big(\frac{\partial\tilde\phi_n}{\partial p}\Big)(x,p)\tilde{h}_v(p)\,\tilde\phi_n(x,p) + \int_{\mathbb{R}} \tilde\phi_n(x,p)\frac{\mathrm{d}}{\mathrm{d}p}\tilde{h}_v(p)\,\tilde\phi_n(x,p)\,\mathrm{d}x$$

The first term is $\epsilon_n(p)\,\frac{\mathrm{d}}{\mathrm{d}p}(\tilde\phi_n,\tilde\phi_n)$, and since $\tilde\phi_n$ is normalized, it vanishes. In the rest, the only p-dependent term is the last one, thus we obtain, so

$$\epsilon_n'(p) = -\frac{1}{B}\int_{\mathbb{R}} v'(x-\tfrac{p}{B})\,\tilde\phi_n(x,p)^2\,\mathrm{d}x$$

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Unfortunately, we cannot use the convolution trick here to argue that $\epsilon_n' = 0$ implies $v' = 0$, because the functions $\phi_n(\cdot,p)$ now depend on p. The formula is nevertheless useful.

Odd potentials

For instance, we can provide another illustration that an asymmetry supports absolute continuity. Given v satisfying assumption (iv), we define the family of scaled potentials, $v_\eta :\ v_\eta(x) = \eta v(\eta x)$.

Proposition

Assume (iii) and (iv), and let v be an odd function with $v'(0) \neq 0$, then to every $n \in \mathbb{N}_0$ there is a $\eta_n > 0$ such that the functions $\epsilon_j(\cdot; v_\eta)$ are non-constant for all $j = 0, 1, \dots, n$.

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Since v_η is odd and $\tilde\phi_j(\cdot,0;v_\eta)^2$ is even in this case, we have

$$\epsilon_j'(0;v_\eta)=-\frac{2\eta^2}{B}\int_0^\infty v'(\eta x)\,\tilde\phi_j(x,0;v_\eta)^2\,\mathrm{d}x.$$

By dominated convergence, the integral tends to $v'(0)$ as $\eta\to 0$, hence there is an $\eta(j)$ such that the derivative is nonzero for $\eta\in(0,\eta(j))$; it is then enough to set $\eta_n:=\min_{0\le j\le n}\eta(j)$.

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Remark: Without any symmetry, the opposite limit, $\eta \to \infty$, produces for $\int_{\mathbb{R}} v(x)\, \mathrm{d}x \neq 0$ a nontrivial δ-potential for which the spectrum is again *ac*.

Using eigenfunction asymptotics

One can combine the Feynman-Hellmann formula with with properties of the eigenfunctions. Let us mention a simple example:

Lemma

No eigenvalue branch $\epsilon_n(\cdot)$ of operator $\tilde{h}_w(p)$ with the potential $w(x) := \max\{a|1-x|, 0\}$, $a > 0$, is constant.

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$$\tilde{\phi}_n(x,p)^2 = c_n\, \mathrm{e}^{-Bx^2}\, |x|^{\epsilon_n(p)/B - 1/2} \left(1 + \mathcal{O}(x^{-1})\right)$$

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with a $c_n > 0$, the error term is uniform in p and $\epsilon_n(p) \to B(2n+1)$ as $p \to \infty$. Function w' is *odd*, however, $(-a, a) \mapsto \tilde{\phi}_n(x + \frac{p}{B}, p)^2$ is *not even* which means that the right-hand side of the FH formula *is nonzero for all p large enough*.

Repulsive potentials

As we will see, we need a slightly stronger regularity of the potential:

- $v \in L^1(\mathbb{R})$ is *purely repulsive*, $v = v_+$, and there is an open interval $J \subset \mathbb{R}$ such that $v(x) \geq a > 0$ for all $x \in J$.

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Proposition

Under the assumption (v), no eigenvalue branch $\epsilon_n(\cdot, v)$ is constant.

Let w be the potential of the lemma, properly shifted. Then $v(x) \geq w(x)$ holds a.e. in $\mathbb{R}$ and the inequalities between the respective quadratic forms in combination with min-max principle give

$$\epsilon_n(p, v) \geq \epsilon_n(p, w) \geq B(2n+1) \quad \text{for all } p \in \mathbb{R} \text{ and } n \in \mathbb{N}_0$$

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The second inequality is strict at least for some p, which yields the claim.

Example

Set v equal to $\sqrt{2}$ at points of the 'fat Cantor' set and zero elsewhere in $\mathbb{R}$. The measure of the set is $\frac{1}{2}$ so such a function is a normalized element of $L^2(\mathbb{R})$, however, there is no interval on which it would be nonzero.

One more condition for sign changing potentials

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Assume (v) and let v be compactly supported with the first derivative not vanishing at a support endpoint, then no ev branch $\epsilon_n(\cdot)$ is constant.

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We again combine FH relations with the asymptotic behavior of $\tilde{\phi}_n$. Without loss of generality we may suppose that $\mathrm{supp}\, v = [0, a]$ and the condition is satisfied at $y = 0$. Was $\epsilon_n'(p) = 0$ *constant for all* p, then up to the overall factor $\frac{1}{B}\big(\frac{p}{B}\big)^{\mu-1}\mathrm{e}^{-p^2/B}$ we must have

$$p\int_0^a v'(y)\Big(1+\frac{By}{p}\Big)^{\mu}\,\mathrm{e}^{-2py-By^2}\,\big(1+\mathcal{O}\big((y+\tfrac{p}{B})^{-1}\big)\,\mathrm{d}y = 0$$

for all large p, where we denoted $\mu = \epsilon_n(p)/B - 1/2$. Passing to the integration variable $z = py$ we see that the assumption requires

$$\int_0^{pa} v'\big(\tfrac{z}{p}\big)\Big(1+\tfrac{Bz}{p^2}\Big)^{\mu}\,\mathrm{e}^{-2z-Bz^2/p^2}\,\big(1+\mathcal{O}\big((\tfrac{z}{p}+\tfrac{p}{B})^{-1}\big)\,\mathrm{d}z = 0$$

for all large p which is a *contradiction*, however, because the left-hand side of the last relation converges to $\int_0^\infty v'(0)\,\mathrm{e}^{-2z}\,\mathrm{d}z = \frac{1}{2}v'(0) \neq 0$ as $p\to\infty$.

Examples

Let us add a few examples starting from barrier/well potentials

$$v(x) = \frac{\lambda}{\pi} \frac{a}{x^2 + a^2}$$

for which the Schrödinger operators converge in the norm-resolvent sense as $a \to 0$ to the one with δ-interaction of strength λ at $x = 0$.

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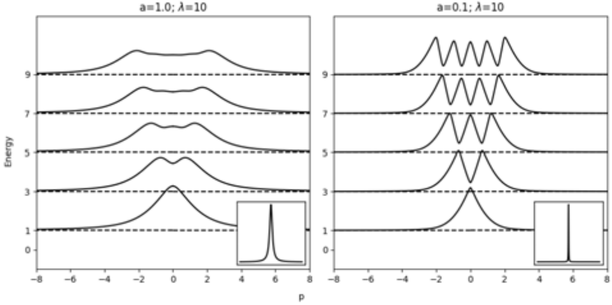

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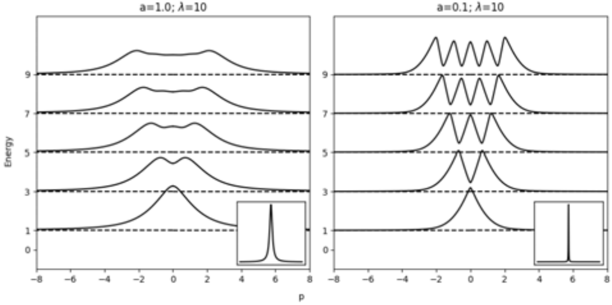

Note that a spectral gap closed for regular potential may open in the limit

Examples

Here is the same for attractive potentials. Recall that the spectrum is simple so the the dispersion curves cannot cross. As $|\lambda|a^{-1}$ becomes large, however, distances between the curves become small and there are points when they have narrowly avoided crossings.

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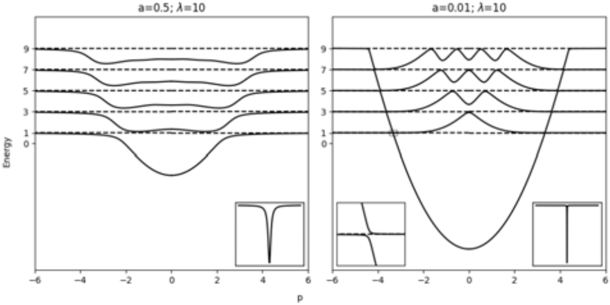

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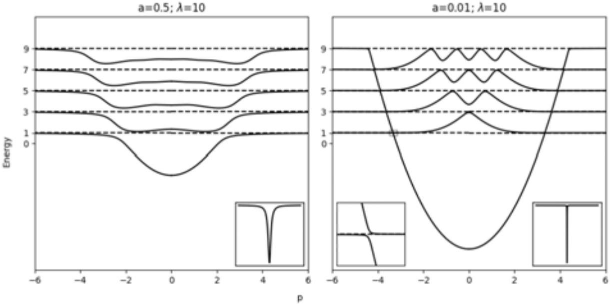

Again, for small values of a the curves resemble the δ-interaction pattern, however, the spectral gaps which are open for the latter may close for the approximating potentials.

Examples

Consider next a potential well with a flat bottom, with smoothed edges for numerical reasons, give by the formula

$$v(x; a, b, \lambda) = \begin{cases} \lambda & \text{if } |x| < a \\ \lambda \cos\left(\frac{\pi}{2} \frac{|x|-a}{(b-a)}\right) & \text{if } a \leq |x| \leq b \\ 0 & \text{otherwise} \end{cases}$$

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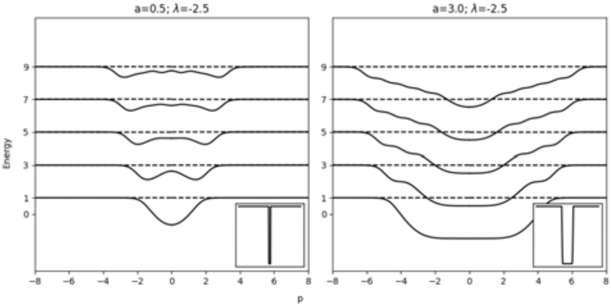

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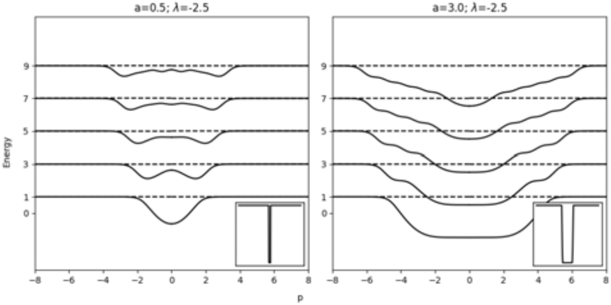

As expected, the central part is close to the shifted Landau level.

Examples

Finally, consider a potential without the mirror symmetry, for instance

$$v(x;\lambda,a)=\begin{cases}\lambda\,\sin\left(\frac{x}{a}\right) & \text{if } |x|<a\pi\\ 0 & \text{otherwise}\end{cases}$$

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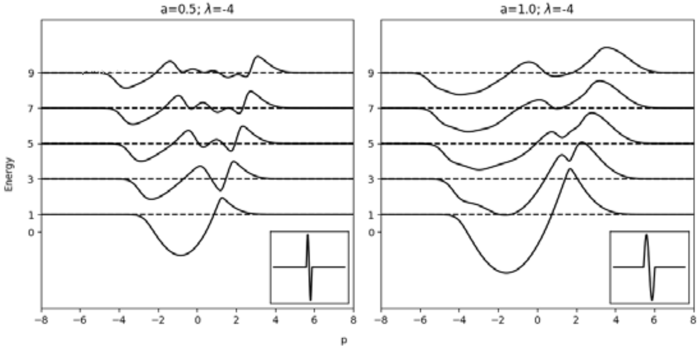

To conclude the potential obstacle case

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In the rest of the talk we are going to deal with situations where no classical analogue exists.

Transport with no classical analogue

There are situations when the classical transport through motion on segments of cyclotron orbits bouncing from the obstacle does not occur, but quantum transport still exists. As an the first example, let us replace the δ interaction line by a *periodic array* of *2D point interactions*.

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Without loss of generality we may suppose that they are situated along the x-axis; using Landau gauge we can write the Hamiltonian formally as

$$H_{\alpha,\ell} = (-i\partial_x - By)^2 - \partial_y^2 + \sum_{j\in\mathbb{Z}} \tilde{\alpha}\, \delta(x - x_0 - j\ell), \quad \ell > 0.$$

The interaction term with 'coupling constant' $\tilde{\alpha}$ naturally has a symbolic meaning only; as well known, the perturbation is non-additive and the proper way to introduce it relies on self-adjoint extensions.

S. Albeverio, F. Gesztesy, R. Høegh-Krohn, H. Holden: *Solvable Models in Quantum Mechanics*, 2nd edition with an appendix by P. Exner, AMS Chelsea Publishing, Providence, R.I., 2005.

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The Landau levels in the spectrum survive because there are generalized eigenfunctions having zeros at the points $x_0 + j\ell,\ j \in \mathbb{Z}$.

Transport with no classical analogue

However, the perturbation gives rise to an additional spectral component:

Theorem

The spectrum of $H_{\alpha,\ell}$ consists for any $\alpha \in \mathbb{R}$ of the Landau levels $B(2n+1)$, $n \in \mathbb{N}_0$, and an infinite family of absolutely continuous spectral bands situated between any two adjacent Landau levels and below B.

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Proof idea: Using Floquet decomposition, $H \approx \int_{\mathbb{R}}^{\oplus} H(p)\,\mathrm{d}p$, the problem reduces to analysis of the fiber operators $H(p)$ describing a single point interaction in an infinite strip of width ℓ with quasi-periodic boundary conditions,

$$\partial_x^i \psi(\ell-, y) = \mathrm{e}^{i\theta\ell} \partial_x^i \psi(0+, y), \quad i = 0, 1.$$

The resolvent of $H(p)$ can be expressed explicitly by Krein's formula in terms of degenerate hypergeometric functions; this allows one to prove the existence of absolutely continuous bands.

P.E., A. Joye, H. Kovařík: Edge currents in the absence of edges, *Phys. Lett.* **A264** (1999), 124–130.

Making the perturbation random

Real-life material samples typically contains *impurities*; this motivates the investigation of systems in which the perturbation is *random*. For magnetic transport along a hard obstacle this was studied in numerous papers, as an example one can mention

J. Fröhlich, G.M. Graf, J. Walcher: On the extended nature of edge states of quantum Hall Hamiltonians, *Ann. Henri Poincaré* **1** (2000), 405–442.

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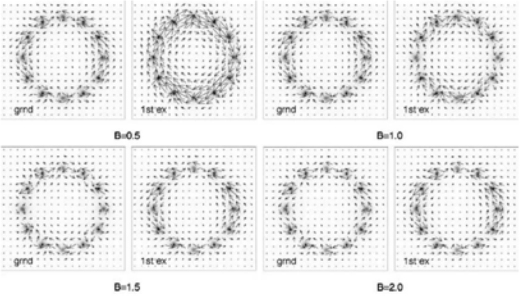
For point obstacles an inspiration can derived from inspection a finite number of perturbations arranged *along a loop*:



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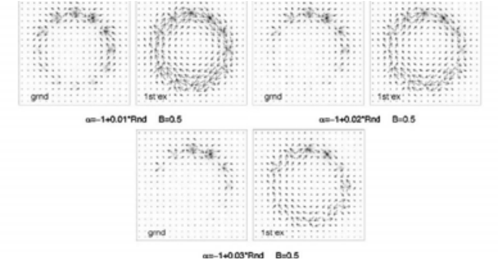

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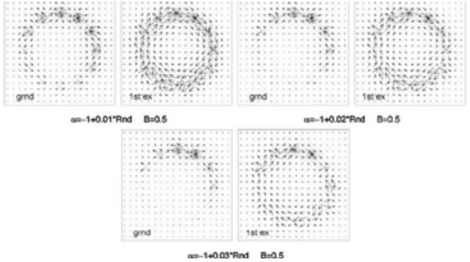

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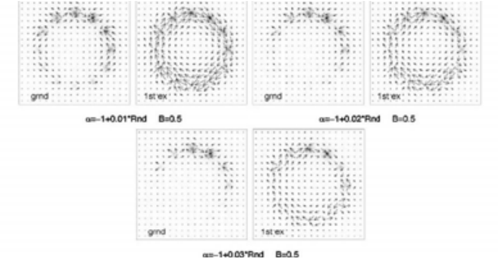

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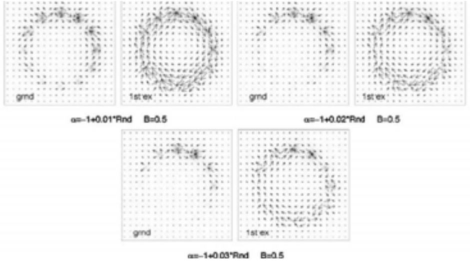

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- The similar problem in which the randomness comes from 'wiggling' the *point interaction positions*; in this case one expects also the Landau levels to get smeared into a pp spectrum

Magnetic layers

Magnetic transport can come also from *geometric perturbations*. This can happen if the system is essentially two-dimensional but a restricted motion in the direction of the field is allowed. In the trivial case, the particle is supposed to be confined to a *flat layer* of width $d > 0$.

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In presence of a nonzero magnetic field, the motion is again *localized* since the variables in the problem separate: the spectrum of the corresponding Hamiltonian consists of the infinitely degenerate eigenvalues

$$\lambda_{n,m} = B(2n+1) + \left(\frac{\pi m}{d}\right)^2, \quad n \in \mathbb{N}_0,\ m \in \mathbb{N}.$$

Suppose now that the layer is *locally* or *globally bent* is a *translationally invariant way*, then the spectrum may become a.c.

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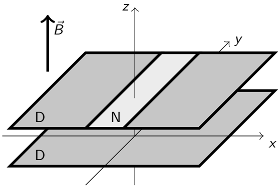
As in the Iwatsuka model we do not know whether it happens always, and furthermore, this bending generated spectrum has a classical analogue. In layers, however, we have again examples of a *purely quantum transport*

An intermezzo

Consider a flat layer (now called Ω) as in the beginning and perturb it by changing the Dirichlet boundary condition at a strip of width $2a$ at one of the boundaries to *Neumann*:



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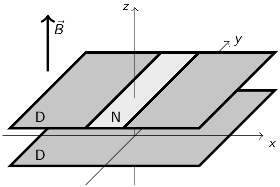

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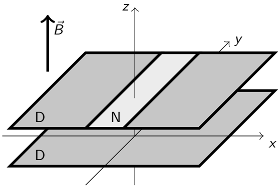

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$$H = -\partial_x^2 + (-i\partial_y + Bx)^2 - \partial_z^2$$

with the derivatives understood in the distributional sense and the domain

$$D(H) = \{\psi \in H_A^2(\Omega) :\ H\psi \in L^2(\Omega),\ \psi(\vec{x}) = 0 \ \text{ if } \ x \in \partial\Omega \setminus W,\ \partial_z\psi(\vec{x}) = 0 \ \text{ if } \ x \in W\}$$

The direct integral decomposition

As before, by *translational symmetry* H is unitarily equivalent to

$$H = H_a = \int_{\mathbb{R}}^{\oplus} H(p)\,\mathrm{d}p, \quad \text{where} \quad H(p) = -\partial_x^2 + (p+Bx)^2 - \partial_z^2$$

with the fiber operators acting on $L^2(\Sigma)$ and the domain

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If $a=0$, the eigenvalues of the 'free' fiber operator $H_0(p)$ are

$$\lambda_{n,m} = B(2n+1) + \left(\frac{\pi m}{d}\right)^2, \quad n\in\mathbb{N}_0,\ m\in\mathbb{N},$$

independent of the momentum p and associated with the eigenfunctions

$$\phi_{n,m}(x,z) = \sqrt{\frac{2}{d}}\, h_n\!\left(x+\frac{p}{B}\right) \sin\frac{\pi m z}{d},$$

where, of course, $h_n(u) = \frac{1}{\sqrt{2^n n!}}\left(\frac{B}{\pi}\right)^{1/4} \mathrm{e}^{-Bu^2/4} H_n(\sqrt{B}u)$ are oscillator ef's.

Properties of dispersion curves

The eigenvalues are simple provided $\frac{Bd^2}{\pi^2} \notin \mathbb{Q}$, otherwise they may have multiplicity two. We arrange them into *ascending sequence* $\{\lambda_k :\, k \in \mathbb{N}\}$ the terms of which are indexed by $k = k(n, m)$.

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To proceed, we must establish the following properties of functions $\lambda_k(\cdot)$:

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- the family $\{H_a(p) \mid p \in \mathbb{R}\}$ is *analytic* in the sense of Kato for any fixed $a \geq 0$ (by a relative boundedness argument), and consequently, each $\lambda_k(\cdot)$ is *real analytic*

Band spectrum

Theorem

Spectrum of H_a is for any positive a, d, and B purely absolutely continuous and has the band-and-gap structure. In addition, we have:

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P.E.: Magnetic transport in laterally coupled layers, *Physica Scripta* **97** (2022), 104004

Remarks

- Continuity of $\lambda_k(p;\cdot)$ at the $B=0$ makes no sense because the *spectral character changes* as $B\to 0$. The transverse part in the (x,z)-plane has then the essential spectrum covering $\left(\left(\frac{\pi}{d}\right)^2,\infty\right)$ and a *finite number of positive discrete eigenvalues* below the threshold. Consequently, $\sigma(H_a)$ is absolutely continuous and the states with the energy support below $\left(\frac{\pi}{d}\right)^2$ can only propagate being localized in the vicinity of the window.

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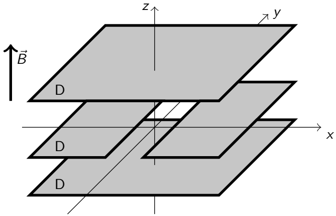
- We know that a fixed gap is open for a small enough but it does not tell us whether the number of open gaps may be infinite; to this aim one has to work out an *asymptotic expansion* of $\lambda_k(a)$ at $a=0$.

The example of laterally coupled layers

Consider now a charged particle confined to a pair of adjacent layers Ω_j, $j = 1, 2$, of widths d_1, d_2 coupled laterally through a straight window in the form of an infinite strip of width $2a$. The field is again supposed to be homogeneous, perpendicular to the layers, and pointing upwards.



We denote $\Omega := \Sigma \times \mathbb{R}$ with $\Sigma := \mathbb{R} \times (-d_2, d_1) \setminus \{(x, 0) : |x| \geq a\}$; the Dirichlet condition is imposed at the 'outer' boundary, $z = -d_2, d_1$, and at $z = 0$ except the window, $W := \{\vec{x} = (x, y, 0) : x \in (-a, a), y \in \mathbb{R}\}$.

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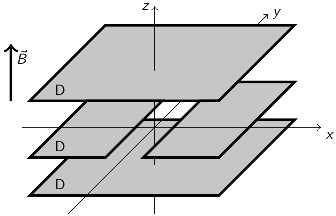

We denote $\Omega := \Sigma \times \mathbb{R}$ with $\Sigma := \mathbb{R} \times (-d_2, d_1) \setminus \{(x, 0) : |x| \geq a\}$; the Dirichlet condition is imposed at the 'outer' boundary, $z = -d_2, d_1$, and at $z = 0$ except the window, $W := \{\vec{x} = (x, y, 0) : x \in (-a, a),\ y \in \mathbb{R}\}$. The Hamiltonian is then an operator on $L^2(\Omega)$ acting as

$$H = -\partial_x^2 + (-i\partial_y + Bx)^2 - \partial_z^2$$

with the domain $D(H) = \{\psi \in H_A^2(\Omega) \cap H_{A,0}^1(\Omega) :\ H\psi \in L^2(\Omega)\}$.

Direct integral decomposition

Using the translational invariance as before, we get the decomposition

$$H = \int_{\mathbb{R}}^{\oplus} H(p)\,\mathrm{d}p,$$

where the fiber operators (if needed, indexed by a, d_1, d_2) act on $L^2(\Sigma)$ as

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If the layers are decoupled, $\sigma(H_0(p))$ consists of the eigenvalues

$$\lambda_{n,m_1,m_2} = B(2n+1) + \left(\frac{\pi m_1}{d_1}\right)^2 + \left(\frac{\pi m_2}{d_2}\right)^2, \quad n \in \mathbb{N}_0,\ m_1, m_2 \in \mathbb{N},$$

independent of the momentum p and associated with the eigenfunctions

$$\phi_{n,m}(x,z) = \sqrt{\frac{2}{d}}\, h_n\!\left(x + \frac{p}{B}\right) \sin \frac{\pi m z}{d_j}, \quad (-1)^{j-1} z \in (0, d_j),\ j = 1, 2,$$

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The maximum multiplicity is now three depending on commensurability of B and $\left(\frac{\pi}{d_j}\right)^2$; we again arrange the eigenvalues into an ascending sequence $\{\lambda_k :\; k \in \mathbb{N}\}$ indexed by $k = k(n, m_1, m_2)$.

Symmetric layers

The case $d_1 = d_2 =: d$ is easy to deal with using the symmetry: the even part refers to the above discussion and the odd one is trivial.

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4. *The eigenfunctions satisfy $\phi_k(x,z;p) = \phi_k(-x,z;-p)$ so that the probability current $p|\phi_k(x,z;p)|^2$ changes sign under the mirror transformation with respect to the $x = 0$ plane.*

Asymmetric layers

The case when d_1, d_2 need not coincide is a little more complicated:

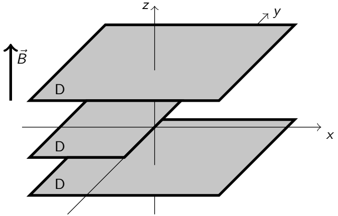
Theorem (E'22)

$\sigma(H)$ has a band-and-gap structure with bands labeled by $k(n, m_1, m_2)$, plus possibly an additional label specified in (i) below, and we have:

- (i) *The spectrum is absolutely continuous if d_1, d_2 are incommensurate. In the opposite case there is an infinite number of flat bands corresponding to pairs (m_1, m_2) satisfying $\frac{m_1}{m_2} = \frac{d_1}{d_2}$; to each of them there is an absolutely continuous band adjacent to it from below.*
- (ii) *In the limit $a \to 0$ the spectrum shrinks to the family of flat bands.*
- (iii) *The eigenvalues $\lambda_k(p)$ determining the absolutely continuous bands depend continuously on the (positive values of) p, B, a, and d.*
- (iv) *For fixed $d, B > 0$ and fixed band index $k(n, m_1, m_2)$, the gap below the corresponding band(s) is open for all a small enough.*
- (v) *For ac spectral bands we have again $\phi_k(x, z; p) = \phi_k(-x, z; -p)$ so that the probability current $p|\phi_k(x, z; p)|^2$ changes sign under the mirror transformation with respect to the $x = 0$ plane.*

One-sided barrier

If the window width $2a$ is large the transport at an edge is practically independent of the other one. It is thus useful to fix the position of the edge and to look what happens if the other disappears.



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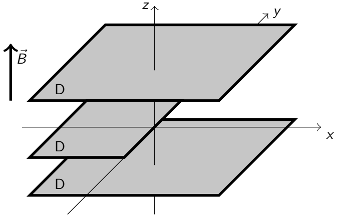

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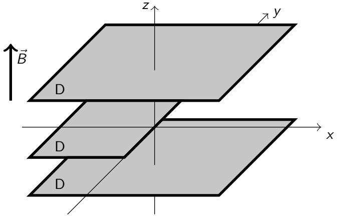

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$$\lambda_{n,m}^{\mathrm{free}} = B(2n+1) + \left(\frac{\pi m}{d_1+d_2}\right)^2, \quad n \in \mathbb{N}_0,\ m \in \mathbb{N},$$

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   in particular, $\inf \sigma(H) = B + \Big(\frac{\pi}{d_1+d_2}\Big)^2$ and the spectrum in the vicinity of the threshold is absolutely continuous.
3. *If the layer widths d_1, d_2 are unequal and $B \ge 3\Big(\frac{\pi}{d_1+d_2}\Big)^2$, the spectrum contains an open gap.*

This is not the end of the story

What I have discussed here does not exhaust by far all questions one can ask. One may add, for instance, the following ones:

- *Weak coupling:* We know that the bands shrink to points as $a \to 0$, *asymptotic expansion* of the functions $\lambda_k(\cdot)$ in the vicinity of this point would allow us to understand better the dependence of the band widths on the parameters and to make conclusions about the number of open gaps in the spectrum.

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Thank you for your attention!