Toward stationary solutions for cubic NLS with linear band crossing of Dirac type

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Introduction

The nonlinear Dirac equation has been widely used to build *relativistic* models of extended particles

- Existence and properties of solutions: Cazenave-Vazquez '86, Balabane-Cazenave-Vazquez '90, Coti Zelati-Nolasco '25, Esteban-Séré '95,...
- Dirac-Maxwell,Dirac-Maxwell-Einstein eqs: Esteban-Georgiev-Séré '96, Nolasco '21, Paturel '00, Rota Nodari '10, Comech-Stuart '18,...
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Recently, nonlinear Dirac equations appeared as effective models in solid state physics/optical systems, where particles have *linear* dispersion relation, in some regime.

Introduction

We consider cubic focusing NLS

$$i\partial_t v = (-\Delta + V)v - |v|^2 v\,, \quad \text{on } \mathbb{R}_t \times \mathbb{R}_x^d,\ d = 1, 2\,, \tag{1}$$

where $V \in C^\infty(\mathbb{R}^d; \mathbb{R})$ is periodic w.r.t. some d-dimensional lattice Λ, describing an underlying periodic structure.

The spectrum of $-\Delta + V$

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The spectrum $\sigma(-\Delta + V) \subset \mathbb{R}$ is given by a union of spectral bands. They can be obtained by solving, for each $k \in \mathcal{B}$ the Floquet-Bloch eigenvalue problem:

$$\begin{cases} (-\Delta + V)\Phi(x,k) = \mu(k)\Phi(x,k), & x \in \mathbb{R}^d \\ \Phi(x+v,k) = e^{ik\cdot v}\Phi(x,k), & v \in \Lambda \end{cases} \tag{2}$$

This yields a countable sequence of real-valued eigenvalues which are ordered, including multiplicity, such that

$$\mu_0(k) \leq \mu_1(k) \leq \mu_2(k) \leq \dots$$

Floquet-Bloch decomposition

The corresponding eigenfunctions $\Phi_j(\cdot, k)$ are called Bloch waves. There holds

$$\sigma(-\Delta + V) = \bigcup_{j \in \mathbb{N}} \{\mu_j(k) : k \in \mathcal{B}\}$$

and the *Bloch functions* $(\Phi_j(*, k))_{j \in \mathbb{N}, k \in \mathcal{B}}$ form a complete set in $L^2(\mathbb{R}^d)$:

$$f(x) = \sum_{j \in \mathbb{N}} \int_{\mathcal{B}} \langle \Phi_j(\cdot, k), f(\cdot) \rangle_{L^2(\mathbb{R}^2)} \Phi_j(x, k) \, dk \,,$$

$$e^{-it(-\Delta+V)} f(x) = \sum_{j \in \mathbb{N}} \int_{\mathcal{B}} e^{-i\mu_j(k)t} \langle \Phi_j(\cdot, k), f(\cdot) \rangle_{L^2(\mathbb{R}^2)} \Phi_j(x, k) \, dk \,.$$

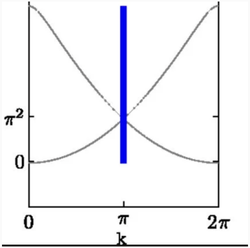
Dirac points

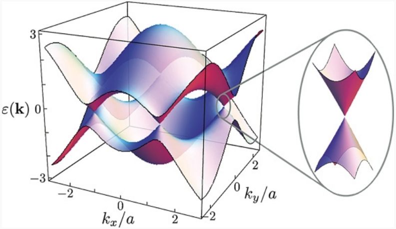
We say that $-\Delta + V$ has a linear/conical crossing at a Dirac point $k_* \in \mathcal{B}$, if there are two bands $\mu_n(k), \mu_{n+1}(k)$ such that

- $\mu(k_*) = \mu_n(k_*) = \mu_{n+1}(k_*)$ is a double eigenvalue of (2) ;
- there exist $E_\pm : B_\delta \to \mathbb{R}$, Lipschitz, s.t. if $|k - k_*| < \delta$

$$\begin{aligned} \mu_{n+1}(k) - \mu(k_*) &= |\lambda|\,|k - k_*|\,(1 + E_+(k - k_*)) \\ \mu_n(k) - \mu(k_*) &= -|\lambda|\,|k - k_*|\,(1 + E_-(k - k_*)) \end{aligned} \tag{3}$$

where $E_\pm(\kappa) = \mathcal{O}(\kappa)$, as $\kappa \to 0$, and $\lambda \in \mathbb{C}$ depends on V.





Main goal

- We assume that the linear part has a *linear/conical band crossing*.
- This leads to some effective Dirac dynamics.
- The latter can be justified for special initial data, on suitable timescales.
- We are interested in the corresponding stationary NLD.
- Our aim is to construct *stationary solutions* to (1), using Dirac solitons.

Linear Dirac dynamics in $2d$

Consider a wave packet spectrally concentrated around k_*:

$$u_0^\delta(x) = \delta(\psi_{1,0}(\delta x)\Phi_1(x) + \psi_{2,0}(\delta x)\Phi_2(x)), \quad \delta > 0\,, \tag{4}$$

the $\Phi_{1,2}$ are the Bloch functions at k_*.

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the $\Phi_{1,2}$ are the Bloch functions at k_*. The problem

$$\begin{cases} i\partial_t u = (-\Delta + V)u \\ u^\delta(0,x) = u_0^\delta(x) \end{cases}$$

has a unique solution of the form

$$u^\delta(t,x) = e^{-i\mu_* t}\left(\sum_{j=1}^{2} \delta\psi_j(\delta t, \delta x)\Phi_j(x) + \eta^\delta(t,x)\right) \tag{5}$$

with $\eta^\delta(0,x) = 0$ and $\sup_{0\le t\le \delta^{-2+}} \|\eta^\delta(t,\cdot)\|_{H_x^s} = \mathcal{O}(\delta^\tau)$ for any $s > 0$, for some $\tau > 0$ [Fefferman-Weinstein '13].

Linear Dirac dynamics in $2d$

The coefficients ψ_j form a global-in-time solution to the following Dirac equation

$$\partial_t \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} 0 & -\overline{\lambda}(\partial_1 + i\partial_2) \\ -\lambda(\partial_1 - i\partial_2) & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad 0 \neq \lambda \in \mathbb{C}$$

with initial data $\begin{pmatrix} \psi_1(0,x) \\ \psi_2(0,x) \end{pmatrix} = \begin{pmatrix} \psi_{1,0}(x) \\ \psi_{2,0}(x) \end{pmatrix} \in \left[\mathcal{S}(\mathbb{R}^2)\right]^2$.

The parameter $\lambda \in \mathbb{C} \setminus \{0\}$ depends on the potential V.

Nonlinear Dirac dynamics in $2d$

Consider the following *focusing* NLS:

$$i\partial_t v = (-\Delta + V)v - |v|^2 v$$

with V as before.

The effective equation around a Dirac point is ([Fefferman-Weinstein'12], formal derivation):

$$\begin{cases} \partial_t \psi_1 + \overline{\lambda}(\partial_1 + i\partial_2)\psi_2 = i(\beta_1|\psi_1|^2 + 2\beta_2|\psi_2|^2)\psi_1 \\ \partial_t \psi_2 + \lambda(\partial_1 - i\partial_2)\psi_1 = i(\beta_1|\psi_2|^2 + 2\beta_2|\psi_1|^2)\psi_2 \end{cases} \tag{6}$$

with $0 \neq \lambda \in \mathbb{C}$, $0 < \beta_2(V) \leq \beta_1(V)$ and $\psi = (\psi_1, \psi_2)^T$ is a $\mathbb{C}^2$-spinor.

Nonlinear Dirac dynamics in $2d$

Idea of the derivation: make a multiscale ansatz

$$v(t,x) \approx \sqrt{\delta} e^{-it\mu_*} \sum_{n=0}^{N} \delta^n v_n(\delta t, \delta x, x). \tag{7}$$

The system (6) describes the leading order term. In this case the approximation is valid in H^s-sense, $s > 1$, on a time-scale $T_\delta \sim \delta^{-1}$ [Arbunich-Sparber '18].

Towards Dirac solitons for honeycomb NLS

We are interested in stationary solutions of the focusing NLS

$$v(t,x) = e^{-i\mu_* t} u(x)$$

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In particular, we may look for solutions of the form

$$u^\delta(x) = \sqrt{\delta} \sum_{j=1}^{2} \psi_j(\delta x)\Phi_j(x) + \sqrt{\delta}\, \eta^\delta(x), \qquad \delta > 0.$$

The $1d$ case

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Consider the lattice $\Lambda = \mathbb{Z}$, with Brillouin zone $\mathcal{B} = [0, 2\pi]$, so $V \in C^\infty(\mathbb{R})$ is 1-periodic.

- If V is even (plus an additional translational symmetry) it has a Dirac point at $k_* = \pi$ [FLW '12].
- Breaking parity with a perturbation δW one can open a gap of size $\mathcal{O}(\delta)$: if $0 < \delta \ll 1$ one may look for stationary solutions with frequency in the gap.

Dirac solitons in $1d$ NLS

We consider the following $1d$ NLS

$$\left(-\frac{d^2}{dx^2}+V(x)+\delta W(x)-\mu_\delta\right)u=|u|^2u\,, \tag{8}$$

where V and W are smooth 1-periodic potentials as described before, and $\mu_\delta \in \mathbb{R}$.

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- Assume there exists a spectral gap $I_\delta := (\mu_* - c\delta, \mu_* + c\delta)$ around the Dirac energy μ_*.
- Taking $\mu_\delta \in I_\delta$ it should be possibile to prove the existence of stationary (localized) solutions (of arbitrary form), for instance by variational methods.
- So it makes sense to also look for them with the “Dirac ansatz”.

The effective Dirac equation in $1d$

As in the $2d$ case, the effective equation can be found *via* a multiscale expansion.

$$u(x) = \sqrt{\delta}\,\mathcal{U}_\delta(x, y) = \sqrt{\delta}\Big(U_0(x, y) + \delta U_1(x, y) + \dots\Big),$$

with $y = \delta x$, $\mathcal{U}_\delta(x + 1, y) = e^{ik_*}\mathcal{U}_\delta(x, y)$ and

$$\mu_\delta = \mu_* + \delta\mu_1 + \delta^2\mu_2 + \dots .$$

Treating x and y as independent variables and substituting into the NLS:

$$\Big[-(\partial_x + \delta\partial_y)^2 + V(x) + \delta W(x) - \lambda_* - \delta\mu_1 - \dots \Big]\mathcal{U}_\delta = \delta|\mathcal{U}_\delta|^2\mathcal{U}_\delta.$$

We now solve the above equation order by order in powers of δ. One finds

$$U_0(x,y) = \psi(y)\Phi_1(x) + \psi_2(y)\Phi_2(x)\,,$$

where Φ_j are the Bloch waves at (k_*, μ_*) and $\psi = (\psi_1, \psi_-)^T$ solves a NLD.

The effective Dirac equation in $1d$

The effective Dirac equation is given by

$$(i\sigma_3\partial_y + \sigma_1 - \mu_1)\psi = \mathcal{G}_{\beta_1,\beta_2,\beta_3}(\psi) \tag{9}$$

where $\psi = (\psi_1, \psi_2)^T$, and $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are Pauli matrices and the nonlinearity $\mathcal{G}_{\beta_1,\beta_2,\beta_3}$ is given by

$$\mathcal{G}_{\beta_1,\beta_2,\beta_3}(\psi) = \begin{pmatrix} \beta_1|\psi_1|^2 + 2\beta_2|\psi_2|^2 & \beta_3\, \bar{\psi}_1\psi_2 \\ \beta_3\, \psi_1\bar{\psi}_2 & \beta_1|\psi_2|^2 + 2\beta_2|\psi_1|^2 \end{pmatrix}\psi$$

Here $\beta_1 = \int_0^1 |\Phi_1|^4 dx = \int_0^1 |\Phi_2|^4 dx$, $\beta_2 = \int_0^1 |\Phi_1|^2|\Phi_2|^2 dx$, $\beta_3 := \int_0^1 \bar{\Phi}_1\bar{\Phi}_1\Phi_2\Phi_2\, dx$.

The $1d$ Dirac operator $\mathcal{D} = i\sigma_3\partial_y + \sigma_1$ has spectrum $(-\infty, -1] \cup [1, +\infty)$, so that we take $|\mu_1| < 1$.

The main result in 1d

Theorem [Borrelli-D.-Dovetta-Tentarelli]

Taking $|\mu_1| < 1$, the NLD admits a smooth solution $\psi = (\psi_1, \psi_2)^T$, such that $|\psi(x)| \lesssim e^{-\sqrt{1-\mu^2}|x|}$ for $|x| \gg 1$.

Given $0 < \delta \ll 1$, for any $\mu_\delta \in I_\delta$ in the linear spectral gap, the NLS

$$\Big(- \frac{d^2}{dx^2} + V(x) + \delta W(x) - \mu_\delta \Big) u = |u|^2 u$$

admits a smooth solution in $H^1(\mathbb{R})$, of the form

$$u^\delta(x) = \sqrt{\delta} \sum_{j=1}^{2} \psi_j(\delta x) \Phi_j(x) + \eta^\delta(x) ,$$

with $\|\eta^\delta\|_{H^1} = \mathcal{O}(\delta)$.

Some ideas on the proof

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First of all, one first proves the existence of a solution to the NLD and study its properties.
Then, one uses the solution of the effective NLD to construct a solution of the NLS:

- Plugging the ansatz

$$u^\delta(x) = \sqrt{\delta}\sum_{j=1}^{2}\psi_j(\delta x)\Phi_j(x) + \sqrt{\delta}\eta^\delta(x)$$

  into the NLS one gets an equation for η^δ.
- One gets a nonlinear equation

$$\left(-\frac{d^2}{dx^2} + V(x) + \delta W(x) - \mu_*\right)\eta^\delta = \mathcal{F}(\psi, \eta^\delta, \delta)\,. \tag{10}$$

Some ideas on the proof

Idea: the dominant contribution should come from $(k, \mu) \approx (k_*, \mu_*)$.

- One decomposes $\eta^\delta = \eta^\delta_{\mathrm{near}} + \eta^\delta_{\mathrm{far}}$ according to momentum/energy near and far from (k_*, μ_*).
- Projecting onto Bloch waves one gets an equation for $\eta^\delta_{\mathrm{far}} = \eta^\delta_{\mathrm{far}}[\eta^\delta_{\mathrm{near}}, \delta]$ and then a closed equation for $\eta^\delta_{\mathrm{near}}$.
- The nonlinear term mixes up the near/far part of the splitting.

The $2d$ case is conceptually analogous, but technically more involved.

- As in the $1d$ case, one can open a gap in the linear spectrum adding a perturbation δW *breaking parity*.
- The effective equation is now a *massive* Dirac equation, for which smooth and exponentially localized solutions are known to exist [Borrelli '17,'22].
- Then we may consider an analogous "Dirac ansatz" for the NLS.

The $2d$ case

However

- when projecting onto Bloch waves we use the fact that in $1d$ we can parametrize crossing bands *in a smooth way*. Not possible in $2d$.
- In $2d$ Bloch bands are only Lipschitz continuous and Bloch waves are discontinuous at the conical point.
- At some point in $1d$ we use ODE arguments.

Thank you for your attention!