On the C^*-algebraic rigidity of Heisenberg groups

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Analytic and Algebraic Methods in Physics XXII

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References

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$H_n = (\mathfrak{h}_n, \cdot)$, $x \cdot y := x + y + \frac{1}{2}[x,y]$

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$$\pi_\lambda(q,p,t)f(x) = \mathrm{e}^{\mathrm{i}\lambda(\langle p,x\rangle + \frac{1}{2}\langle p,q\rangle + t)} f(q+x)$$

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Problem on the Canonical Commutation Relations (CCR)

Is the "canonical host group" in some sense unique?

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• $\widehat{H_n} = \{[\pi_\lambda] \mid \lambda \in \mathbb{R}\setminus\{0\}\} \cup \{\chi_{a,b} \mid (a,b) \in \mathbb{R}^n \times \mathbb{R}^n\} \simeq (\mathbb{R}\setminus\{0\}) \sqcup \mathbb{R}^{2n}$ endowed with the (non-separated!) topology given by the condition:

$$\boxed{\text{If } \lambda_n \xrightarrow[n\to\infty]{} 0 \text{ in } \mathbb{R}\setminus\{0\}, \text{ then } [\pi_{\lambda_n}] \xrightarrow[n\to\infty]{} \chi_{a,b} \text{ in } \widehat{H_n} \text{ for all } (a,b) \in \mathbb{R}^{2n}}$$

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• Topology of $\widehat{G}$ is defined by convergence of representation coefficients $c_{vu} := \langle v, \pi(\cdot)u\rangle \in \mathcal{C}(G, \mathbb{C})$, $u, v \in \mathcal{H}_\pi$, on the compact subsets of G.

E.g., if $\mathcal{H}_\pi = \mathbb{C}^n$, then $\pi(\cdot) = (c_{jk}(\cdot))_{j,k}\colon G \to \mathrm{U}(n)$.

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Inverse Problem ($G \leftsquigarrow \widehat{G}$). Can G be recovered from $\widehat{G}$?

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This suggests that **IP** for noncommutative Lie groups may be approached via noncommutative topology, that is, C^*-algebras.

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Equivalently: subalgebra $\mathcal{A} = \overline{\mathcal{A}}^{\|\cdot\|} \subseteq \mathcal{B}(\mathcal{H})$ with $a \in \mathcal{A} \Leftrightarrow a^* \in \mathcal{A}$

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Here $\pi(f)\pi(h) = \pi(f * h)$ and $\pi(f)^* = \pi(f^*)$ with $f * h, f^* \in \mathcal{C}_c(G)$ independent on π.

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Fact: $\boxed{\widehat{G} \simeq \widehat{C^*(G)}}$ so the unitary dual $\widehat{G}$ can be recovered from $C^*(G)$

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Step 2: $\mathcal{C}_0(\widehat{G_0}) \simeq \mathcal{C}_0(\widehat{G}) \implies G_0 \simeq G$ as above, by Brouwer's theorem

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Example 2 (1-connected solvable Lie groups). For $D \in M_n(\mathbb{R})$ let
$\boxed{G_D := \mathbb{R}^n \rtimes_D \mathbb{R}}$ with $(b_1, t_1) \cdot (b_2, t_2) = (b_1 + \mathrm{e}^{t_1 D} b_2, t_1 + t_2)$.
Also let $n_\pm^D$ be the number of eigenvalues $z \in \sigma(D)$ with $\pm \mathrm{Re}\, z > 0$.

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If $D_j \in M_n(\mathbb{R})$ and $n_+^{D_j} + n_-^{D_j} = n$ for $j = 1, 2$, then

- $G_{D_1} \simeq G_{D_2}$ as Lie groups iff $\sigma(D_1) = r \cdot \sigma(D_2)$ for some $r \in \mathbb{R} \setminus \{0\}$

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- $G_{D_1} \simeq G_{D_2}$ as Lie groups iff $\sigma(D_1) = r \cdot \sigma(D_2)$ for some $r \in \mathbb{R} \setminus \{0\}$
- $C^*(G_{D_1}) \simeq C^*(G_{D_2})$ iff $\{n_+^{D_1}, n_-^{D_1}\} = \{n_+^{D_2}, n_-^{D_2}\}$.

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That is, for every 1-connected Lie group,

$$\boxed{H_n \simeq G \iff C^*(H_n) \simeq C^*(G)}$$

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Thus $G = S$ is a solvable Lie group.

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Step 3: H_n is C^*-rigid within the class of nilpotent Lie groups (earlier result).

Q.E.D.

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E.g., the compact Lie groups $\mathrm{SU}(2)$ and $\mathrm{SU}(3)$ are 1-connected and $\widehat{\mathrm{SU}(2)} \simeq \widehat{\mathrm{SU}(3)}$ but $C^*(\mathrm{SU}(2)) \not\simeq C^*(\mathrm{SU}(3))$.