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Unitary quantum systems with maximally non-Hermitian potentials $V(x)$

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BASIC REFERENCE

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[1] MZ,
Construction of maximally non-Hermitian potentials
under unbroken $\mathcal{PT}-$symmetry constraint.
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arXiv 2507.09567

• popular benchmark model that failed

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= imaginary cubic oscillator (ICO) with

$$H = -\frac{d^2}{dx^2} + V(x)\,, \qquad V(x) = V^{(ICO)}(x) = \mathrm{i}x^3$$

= 2012: Siegl with Krejčiřík:

= H is singular ("intrinsic exceptional point", IEP)

= non-Rieszian behavior of the eigenbasis

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= 2019: Günther with Stefani:

"what is still lacking is a simple physical explanation scheme
for the non-Rieszian behavior of the eigenfunction sets"

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= REMEDY: $\boxed{\text{boundedness}}$ needed, a <u>regularization by discretization</u>

• AMENDED models living on lattices:

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• Scholtz et al (1992) recommended to consider just bounded operators H

• established their "normal quantum-mechanical interpretation"

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• one of implementations: the trick of DISCRETIZATION

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= equidistant grid-point lattice of coordinates

$$x_k = x_0 + k\,\delta\,, \qquad k = 0, 1, \ldots, N+1$$

with, say, $\delta = 1$;

= difference Schrödinger equation

$$-\psi_n(x_{k-1}) + 2\,\psi_n(x_k) - \psi_n(x_{k+1}) + V(x_k)\,\psi_n(x_k) = E_n\,\psi_n(x_k)\,, \qquad n = 1, 2, \ldots\,, N\,.$$

• PROJECT: finite lattices

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= study N by N matrix Hamiltonians

$$H^{(N)}(A,B,\ldots) = \triangle^{(N)} + V^{(N)}(A,B,\ldots)$$

with one-dimensional discrete Laplacean

$$\triangle^{(N)} = \left[\begin{array}{ccccc} 0 & -1 & 0 & \ldots & 0 \\ -1 & 0 & -1 & \ddots & \vdots \\ 0 & -1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & -1 \\ 0 & \ldots & 0 & -1 & 0 \end{array} \right] .$$

and any suitable $\mathcal{PT}-$symmetric interaction potential.

• CHOICE of toy-model $V(x)$

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• discrete $\mathcal{PT}-$ symmetric imaginary local interaction potentials

$$V^{(N)}(A,B,\ldots) = \left[\begin{array}{ccccc} -i\,A & 0 & 0 & \ldots & 0 \\ 0 & -i\,B & 0 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & i\,B & 0 \\ 0 & \ldots & 0 & 0 & i\,A \end{array} \right].$$

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• AIM: make all "quasi-Hermitian" again (MAQA)

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• Belzebub's recommendation: make all as non-Hermitian as possible,

FIND "parameters of collapse" $A = A^{(EPN)}, B = B^{(EPN)}, C = C^{(EPN)}, \ldots$

$=$ "maximum" at which one loses the unitarity

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- **DEFENCE against doubts:**

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- essence: observables in non-Hermitian operator representation $\Lambda_j \neq \Lambda_j^\dagger$, $j = 0, 1, \ldots$

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- required quasi-Hermitian, $\Lambda_j^\dagger \, \Theta = \Theta \, \Lambda_j$; a conventional interpretation achieved

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- tool: replace $\langle \psi_a | \psi_b \rangle \ \to \ \langle \psi_a | \Theta | \psi_b \rangle \ \equiv \ \langle\!\langle \psi_a | \psi_b \rangle$

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- consequence: unphysical Hilbert space $\mathcal{H} = \mathcal{H}_{mathematical}$
  replaced by $\Theta-$dependent $\mathcal{H} = \mathcal{H}_{physical} = \mathcal{H}_\Theta$.

• ILLUSTRATION: two-level Bose-Hubbard system

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$$H^{(2)}(A) = \begin{bmatrix} -iA & -1 \\ -1 & iA \end{bmatrix}, \qquad A \in \mathbb{R}$$

• secular equation $E^2 + A^2 - 1 = 0$

• pair of eigenvalues

$$E_{\pm}^{(2)}(A) = \pm\sqrt{1 - A^2}$$

• real if and only if $-1 \leq A \leq 1$.

• Jordan block

$$J^{(2)} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

• transition matrix

$$Q^{(2)} = \begin{bmatrix} -i & 1 \\ -1 & 0 \end{bmatrix}.$$

• inner-product metric

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$$\Theta = \Theta^{(2)}(A,\xi) = \left[\begin{array}{cc} 1 & \xi - iA \\ \xi + iA & 1 \end{array} \right]$$

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• free parameter ξ

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• $\xi-$dependent eigenvalues $\theta_\pm = 1 \pm \sqrt{A^2+\xi^2}$

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• must be positive:

$$\xi \in (-\sqrt{1-A^2}, \sqrt{1-A^2})\,.$$

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CONTENTS of the talk

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- popular model: IMAGINARY CUBIC OSCILLATOR

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- the notion of HIDDEN HERMITICTY

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- systems living on a discrete lattice

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- note on PT SYMMETRY alias Krein-space self-adjointness

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- CONSTRUCTIONS with $N = 2, 3, 4, 5$ and 6

• Krein-space self-adjointness

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a.k.a. PT SYMMETRY of H: relation $H\mathcal{PT} = \mathcal{PT}H$: recent history

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- 1992: Bessis with Zinn-Justin: $V(x) \sim ix^3$ (ICO, real spectrum conjecture)
- 1993: Buslaev with Grecchi: $V(x) \sim -x^4$ (isospectrality with DW)
- 1998: Bender with Boettcher: $V(x) \sim ix^3 \cdot (ix)^\delta$ (ICO made popular)
- 2012: Siegl with Krejčiřík: THE END of the story:

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"there is no quantum mechanical Hamiltonian associated with ICO"

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- 2025: challenging OPEN QUESTIONS survive

• math. once more: HIDDEN HERMITICTY

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- 1907 - 1989: M. G. Krein: pseudo-Hermiticity ($H^\dagger\eta = \eta H$)
- 1956: F. Dyson: ferromagnets & non-Hermitian H with real spectrum
- 1960: J.-P. Dieudonné: the first unbounded-operator counterexamples
- 1970s: the Dyson's non-unitary isospectrality $H \leftrightarrow \mathfrak{h} = \Omega\, H\, \Omega^{-1}$ in nuclear physics
- 1992: Scholtz, Geyer and Hahne: the first consistent $\boxed{\text{quasi-Hermitian quantum theory:}}$

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$$H^\dagger\Theta = \Theta H \quad \text{with} \quad \Theta = \Omega^\dagger\Omega \quad \text{and with } \underline{\text{bounded}}\ H$$

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- 1998 - 2012: studies of unitary (called "closed") quantum systems
  using non-Hermitian but $\mathcal{PT}-$symmetric Hamiltonians
- after 2012: studies of unitary (called "closed") quantum systems
  using bounded non-Hermitian Hamiltonians

• back to EXTREME PHYSICS:

$$H^{(EPN)} = \triangle^{(N)} + V^{(EPN)}(A^{(EPN)}, B^{(EPN)}, \ldots)$$

- canonically represented by the N by N Jordan matrix

$$J^{(N)} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & 0 & 1 \\ 0 & 0 & \ldots & 0 & 0 \end{bmatrix}$$

- in such a way that

$$J^{(N)} = [Q^{(N)}]^{-1} \cdot H^{(EPN)} \cdot Q^{(N)} .$$

- map $Q^{(N)}$ is called transition matrix.

• THE FIRST NONTRIVIAL CASE: $N = 3$

$$H^{(3)} = \triangle^{(3)} + V^{(3)}(A) = \left[\begin{array}{ccc} -iA & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & iA \end{array} \right] .$$

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• energies $E_0 = 0$ and $E_\pm = \pm\sqrt{2 - A^2}$

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• two Kato's EP3 singularities at $A_\pm^{(EP3)} = \pm\sqrt{2}$

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• at $A_+^{(EP3)} = \sqrt{2}$, transition matrix

$$Q^{(3)} = \left[\begin{array}{ccc} -1 & -i\sqrt{2} & 1 \\ i\sqrt{2} & -1 & 0 \\ 1 & 0 & 0 \end{array} \right]$$

• constructions of metric - version A.

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• two parameters: choose the real parts of $\Theta^{(3)}_{12}$ and $\Theta^{(3)}_{13}$

• metric then forms the two-parametric family

$$\Theta^{(3)}(A,\xi,\eta) = \left[\begin{array}{ccc} 1 & \eta - iA & \xi - iA\eta \\ \eta + iA & \xi + 1 + A^2 & \eta - iA \\ \xi + iA\eta & \eta + iA & 1 \end{array} \right] .$$

• the choice of $\xi = \eta = 0$ minimizes the anisotropy and yields

$$\theta_0 = 1 \quad \text{and} \quad \theta_{\pm} = 1 + \frac{1}{2}\left[A^2 \pm \sqrt{8A^2 + A^4}\right]$$

= metric only positive for $0 \le A < 1$,

= i.e., for non-Hermiticity far from maximal

• constructions of metric - version B, for $1 \leq A < \sqrt{2}$

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factorize

$$\Theta = \Omega^\dagger \, \Omega$$

where

$$\Omega^\dagger = \left[\ |\psi_1\rangle\!\rangle \, , \ \ |\psi_2\rangle\!\rangle \, , \ \ \ldots \ \ |\psi_N\rangle\!\rangle \ \right]$$

where

$$H^\dagger \, |\psi_n\rangle\!\rangle = E_n \, |\psi_n\rangle\!\rangle \, , \qquad n = 1, 2, \ldots , N \, ;$$

reparametrization $A = A(t) = \sqrt{2 - 2t^2}$ and normalization

$$\langle N|\psi_n\rangle\!\rangle = 1$$

yielding "sample" metric $\Theta_S^{(3)}(t)$ of non-minimal anisotropy (PTO)

Lemma 1 *Eigenvalues are ordered,* $0 < \theta_1(t) \ll \theta_2(t) \ll \theta_3(t)\,,\ 0 < t^2 \ll 1\,,$ *and we have*

$$\theta_1 = -3\,t^2 + 6 - \sqrt{t^4 - 36\,t^2 + 36} \approx \frac{2}{3}t^4 + \frac{1}{3}t^6 + \frac{11}{54}t^8 + \ldots\,, \qquad \theta_2 = 4\,t^2\,,$$

$$\theta_3 = -3\,t^2 + 6 + \sqrt{t^4 - 36\,t^2 + 36} \approx 12 - 6\,t^2 - \frac{2}{3}t^4 - \frac{1}{3}t^6 - \frac{11}{54}t^8 + \ldots\,.$$

- the decrease of $|t|$ marks a non-empty corridor to extreme non-Hermiticity
- *alias* a trajectory of a smooth access to the EP3 singularity
- in the EP3 limit the metric degenerates, with eigenvalues 0, 0 and 12,

$$\lim_{t\to 0^+} \Theta_S^{(3)}(t) = \left[\begin{array}{ccc} 3 & -3\,i\sqrt{2} & -3 \\ 3\,i\sqrt{2} & 6 & -3\,i\sqrt{2} \\ -3 & 3\,i\sqrt{2} & 3 \end{array}\right]\,.$$

= a “double-Hermiticity” paradox:

$$\lim_{t\to 1}\Theta_S^{(3)}(t)=\left[\begin{array}{ccc} 3 & 0 & 1\\ 0 & 4 & 0\\ 1 & 0 & 3 \end{array}\right]$$

with eigenvalues 4, 4 and 2

• the metric remains acceptable for

$$t \in (-t_{\max}, t_{\max})\,, \qquad t_{\max} \approx 1.014611872\,.$$

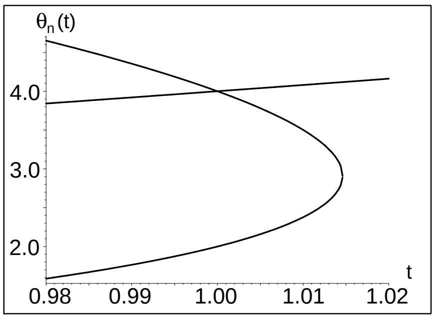

Figure 1: The triplet of eigenvalues of our special metric $\Theta_S^{(3)}(t)$ in the vicinity of the instant $t = 1$, beyond which the Hamiltonian is Hermitian matrix.

• a detour to math. of EXCEPTIONAL POINTS

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- the spectral reality of the observables need not be robust

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- IN PARTICULAR, ∃ Kato's exceptional points of maximal order (EPN)

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- TO BE LOCALIZED, more easily, due to $\mathcal{PT}-$symmetry

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- EXISTING at the related degenerate bound-state energy $E = E^{(EPN)} = 0$

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- APPEAL: maximalization of the non-Hermiticity near EPN

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- ADMITTING access to a quantum phase transition *alias* "quantum catastrophe"

• take NEXT: $N = 4$

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- AIM: constructive proof of existence of EP for

$$H^{(4)} = \triangle^{(4)} + V^{(4)}(A,B) = \left[\begin{array}{cccc} -iA & -1 & 0 & 0 \\ -1 & -iB & -1 & 0 \\ 0 & -1 & iB & -1 \\ 0 & 0 & -1 & iA \end{array}\right]$$

- secular equation

$$E^4 + b\,E^2 + c = 0\,, \quad b = b(A,B) = -3 + A^2 + B^2\,, \quad c = c(A,B) = (1 + A\,B)^2 - A^2$$

- its roots,

$$E_{\pm,\pm} = \pm 1/2\,\sqrt{6 \pm 2\,\sqrt{5 - 2\,A^2 - 6\,B^2 + A^4 - 2\,B^2A^2 + B^4 - 8\,BA} - 2\,A^2 - 2\,B^2}\,.$$

• EP4 singularity

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• at $E = 0$, iff $b = c = 0$,

$$-3 + A^2 + B^2 = 0\,, \qquad 1 + 2\,BA - A^2 + B^2 A^2 = 0\,.$$

• roots

$$A^{(EP4)} = 1.683771565\,, \qquad B^{(EP4)} = 0.4060952085\,,$$

• $B^{(EP4)}$ is a (unique) positive and real root of polynomial $B^4 - 2\,B^3 - 2\,B^2 + 6\,B - 2$.

• monotony mimics the shape of ICO

• transition matrix

$$Q^{(4)} = \begin{bmatrix} i & -1.835086683 & -1.683771565\,i & 1 \\ 1.683771562 & 2.089866772\,i & -1 & 0 \\ -\,1.683771565\,i & 1 & 0 & 0 \\ -\,1 & 0 & 0 & 0 \end{bmatrix}.$$

• take $N = 5$

$$H^{(5)} = \triangle^{(5)} + V^{(5)}(A,B) = \begin{bmatrix} -iA & -1 & 0 & 0 & 0 \\ -1 & -iB & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & iB & -1 \\ 0 & 0 & 0 & -1 & iA \end{bmatrix}$$

secular equation

$$E^5 - \left(4 - A^2 - B^2\right) E^3 - \left(-3 - 2\,BA + 2\,A^2 - B^2 A^2\right) E = 0$$

yielding $E_0 = 0$ and

$$E_{\pm,\pm} = \pm 1/2 \sqrt{8 \pm 2\sqrt{4 - 8\,B^2 + A^4 - 2\,B^2 A^2 + B^4 - 8\,BA} - 2\,A^2 - 2\,B^2}\,.$$

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• search for EP5:

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Groebner-basis elimination technique

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• $\Longrightarrow$ $B^{(EP5)}$ = root of polynomial $B^4 - 2\,B^3 - 4\,B^2 + 6\,B + 5$

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• so that $B^{(EP5)} = 0.6683178062$ and $A^{(EP)} = 1.885033504$

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etc.

• **take** $N = 6$

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• energies = Cardano formulae (truly ugly)

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• in contrast, localization of $A^{(EP6)}, B^{(EP6)}$ and $C^{(EP6)}$ still straightforward:

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• determined by the coupled triplet of polynomial algebraic equations

$$-5 + A^2 + B^2 + C^2 = 0\,,$$

$$6 - B^2 + 2\,BA - 2\,C^2 - 3\,A^2 + B^2A^2 + C^2A^2 + 2\,CB + C^2B^2 = 0\,,$$

$$-1 + C^2 - 2\,BA + 2\,CA + 2\,C^2BA + A^2 + 2\,CBA^2 - B^2A^2 + B^2C^2A^2 = 0\,.$$

• Groebneriana

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• the real and positive value of $B^{(EP6)}$ is a root of $P^{(N)}(B) =$

$$P^{(6)}(B) = B^{23} - 2\,B^{22} - 20\,B^{21} + 32\,B^{20} + 188\,B^{19} - 216\,B^{18} - 1060\,B^{17} + 768\,B^{16} +$$

$$+3782\,B^{15} - 1308\,B^{14} - 8492\,B^{13} + 16\,B^{12} + 11164\,B^{11} + 4008\,B^{10} - 6668\,B^{9} - 7072\,B^{8} - 703\,B^{7} +$$

$$+5678\,B^{6} + 2320\,B^{5} - 1200\,B^{4} - 1248\,B^{3} - 96\,B^{2} + 160\,B + 32\,.$$

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• providing $B^{(EP6)}_{expected} = 0.8635733388$ and $B^{(EP6)}_{unexpected} = 0.4333101655$.

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• preferring monotonous to oscillatory we get the ultimate numerical result

$$A^{(EP6)} = 2.046061191\,, \quad B^{(EP6)} = 0.8635733388\,, \quad C^{(EP6)} = 0.2605285271\,.$$

Table 1: Real and positive EPN-supporting parameters at the first few N.

N	2	3	4	5	6
$A^{(EP6)}$	1.000	1.414	1.684	1.885	2.046
$B^{(EP6)}$	-	-	0.406	0.608	0.864
$C^{(EP6)}$	-	-	-	-	0.261

• what happens far from EPNs?

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• numerical answer at $N = 4$:

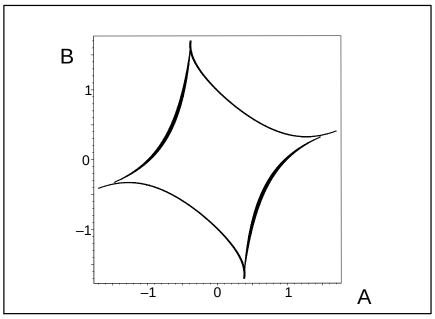

Figure 2: Two-dimensional physical domain of parameters $\mathcal{D}$ and its four spike-shaped maximal-non-Hermiticity extremes.

• analytic construction:

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"unitary-access" corridor:

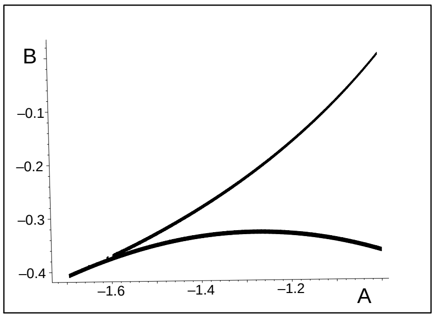

Figure 3: Spiked form of the boundary of the physical star-shaped domain $\mathcal{D}$ of unbroken $\mathcal{PT}$ symmetry in the vicinity of one of its four maximal-non-Hermiticity extremes ($N = 4$).

• details:

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• inequalities define the corridor locally

$$A \gtrapprox A_-^{(EP4)} \approx -1.683771565\,, \qquad B \gtrapprox B_-^{(EP4)} \approx -0.4060952085$$

• providing unitary ACCESS to the EP horizon;
• PHYSICS: processes of quantum phase transition;
• MATH.: the locality of $V(x)$ is what makes the problem nontrivial.

Lemma 2 *The pairs of values $A^{(EP4)}$ and $B^{(EP4)}$ exist and lie on the boundary of a non-empty physical domain $\mathcal{D}$ defined by the requirement that the related spectrum of energies is real, discrete and non-degenerate.*

Proof. . The triplet of conditions

$$b(A,B) < 0\,, \quad c(A,B) > 0\,, \quad b^2(A,B) > 4\,c(A,B)\,.$$

The interior of $\mathcal{D}$ can be reparametrized in terms of three new real, positive and not too large auxiliary parameters α, β and γ,

$$A^2 + B^2 = 3\,\cos^2\alpha\,, \quad 1 + A\,B = A\,\cosh\beta\,, \quad 3\,\sin^2\alpha = 2\,A\,\sinh\beta\,\cos\gamma\,.$$

The elimination of $B = \cosh\beta - 1/A$ and $3\,\sin^2\alpha$ yields

$$P(A,\beta,\gamma) = A^4 + (A\,\cosh\beta - 1)^2 - 3 + 2\,A\,\sinh\beta\cos\gamma = 0\,.$$

In the limit $\beta \to 0$ we get the EP4 root $A^{(EP4)} = 1.683771565$. The curve $P(A,0,0)$ then grows as a function of A. An inclusion of $\mathcal{O}(\beta^2)$ and $\mathcal{O}(\beta\,\gamma^2)$ only shifts the curve slightly upwards. This forces the root to move slightly to the left, and the value of A only gets slightly smaller and stays real. Also B remains real. Result: $(\beta,\gamma)-$parametrized unitarity-preserving corridor in which the EP4 degeneracy unfolds. □

• SUMMARY: existence confirmed

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• of exceptional points of maximal order N (EPN)

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• of open domain $\mathcal{D}$ containing the unitarity-compatible "physical" parameters $A, B, \ldots$

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• of a unitary-evolution accessibility of the EPN extremes

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• of several mathematical tools specifying the boundaries of $\mathcal{D}$

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• of anomalous physics in the maximally non-Hermitian dynamical regime

- **THANKS FOR YOUR ATTENTION**

- **ABSTRACT**

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A family of discrete Schrödinger equations with imaginary and maximally non-Hermitian multiparametric potentials $V(x, A, B, \ldots)$ is studied. In the domain of unitarity-compatible parameters $\mathcal{D}$ (known as the domain of spontaneously unbroken $\mathcal{PT}-$symmetry) the reality of all of the bound-state energies $E_n(A, B, \ldots)$ survives up to the maximally non-Hermitian "exceptional-point" (EP) extreme with parameters $A^{(EP)}, B^{(EP)}, \ldots$. The computer-assisted proof of existence and the symbolic-manipulation localization of such a spectral-degeneracy limit are sampled showing that their complexity grows quickly with the number of parameters.