Local perturbations of potential well arrays

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Our question here: can *much weaker* local shifts (of finitely many wells) already produce eigenvalues at the band bottom?

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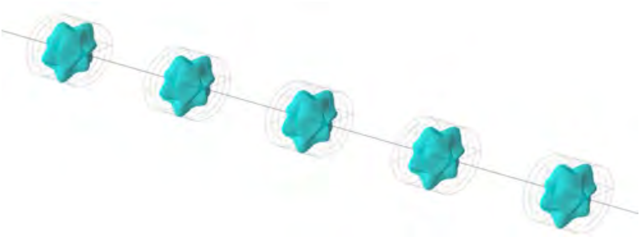

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The array is constructed by shifting V , i.e.

$$V_j : x \mapsto V(x - y_j), \text{ with } \Sigma_{\rho,R}(y_i) := \Sigma_{\rho,R}(0) + y_i$$

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In the negative part there may or may not be gaps, their number is finite and does not exceed $\#\sigma_{disc}(H_V)$.

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Proposition (3.1.)

$H_{V,Y}$ is self-adjoint on $H^2(\mathbb{R}^\nu)$ for any local perturbation and $\sigma_{ess}(H_{V,Y}) = \sigma_{ess}(H_{V,Y_0})$.

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$H_{V,Y}$ is self-adjoint on $H^2(\mathbb{R}^\nu)$ for any local perturbation and $\sigma_{ess}(H_{V,Y}) = \sigma_{ess}(H_{V,Y_0})$.

- The perturbation is well-defined
- Its essential spectrum is stable

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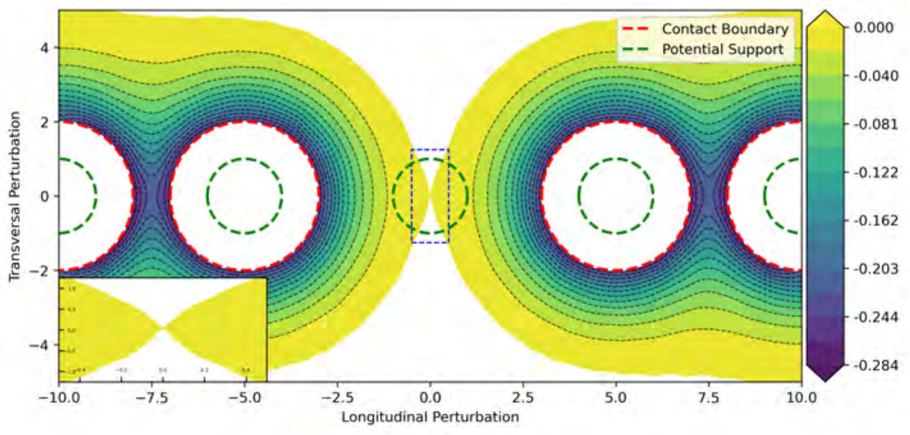

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$$(\phi, K_{V,Y}(-\kappa_0^2)\phi) > \|\phi\|^2$$

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Mollifier: $h_n(x) = \sum_i h_{n,i}\, \chi_{\Sigma_{\rho,R}(y_i)}(x)$, $\;h_{n,i} = \frac{n^2}{n^2+i^2}$.

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Lemma

$\langle h_n\phi_0, K_{V,Y_0}(-\kappa_0^2)h_n\phi_0\rangle - \|h_n\phi_0\|^2 = O(n^{-2}) \to 0.$

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Due to the local character of the perturbation and some technicalities we have

$$\sum_{i,j\in\mathbb{Z}}\left(\phi_0, \left[K_{V,Y}^{(i,j)}(-\kappa_0^2) - K_{V,Y_0}^{(i,j)}(-\kappa_0^2)\right]\phi_0\right).$$

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This suggests to employ Jensen inequality! But, we cannot use the convexity of the resolvent directly, since the perturbation enters its argument through Euclidian distance.

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We can thus use, again after some technicalities, the Jensen inequality and by assumption for at least some i, j, $\eta_{ij} \neq 0$, thus we have positivity.

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- This condition is *sufficient but not necessary* (likely proof artifact; estimates are rough).
- Possible goals are to relax aspect-ratio bound(find optimal bound for convexity); allow decaying (non-compact) shifts

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