Emergent Dimensions:

Geometry and Topology

from Parameter Spaces

Chia-Yi Ju

@AAMP XXII

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Non-Hermitian Inspired Work

$$i\partial_t |\Psi\rangle = H |\Psi\rangle$$
$$H = H^\dagger$$

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Richer Insights
More Tools

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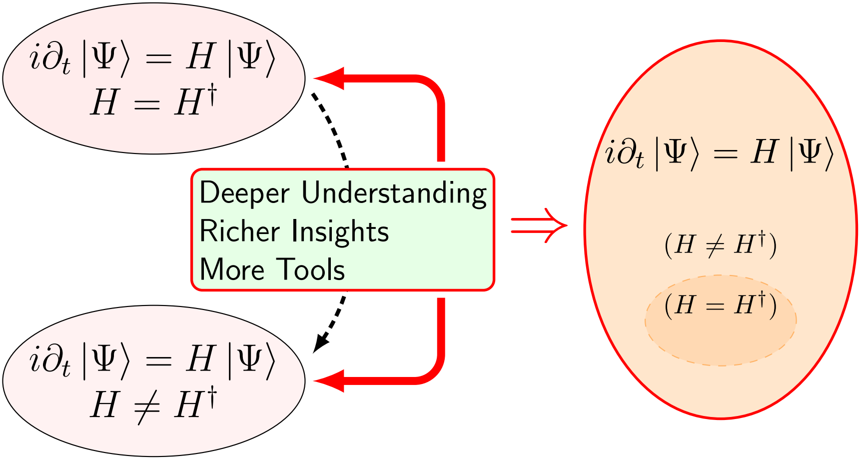

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$$\Rightarrow$$

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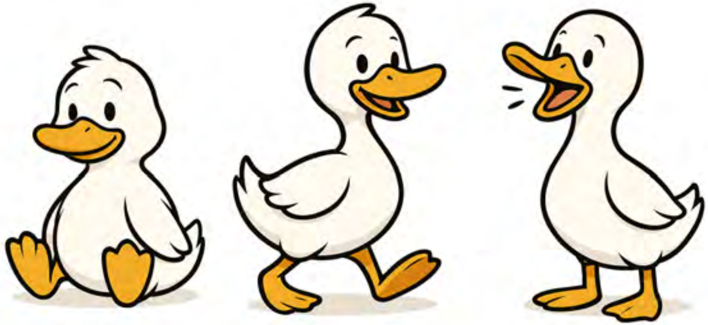
Duck Test



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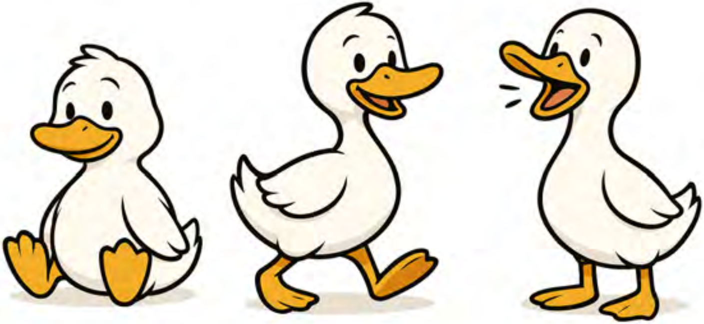

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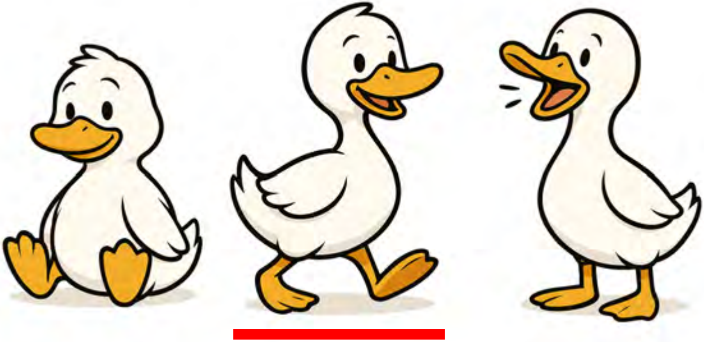

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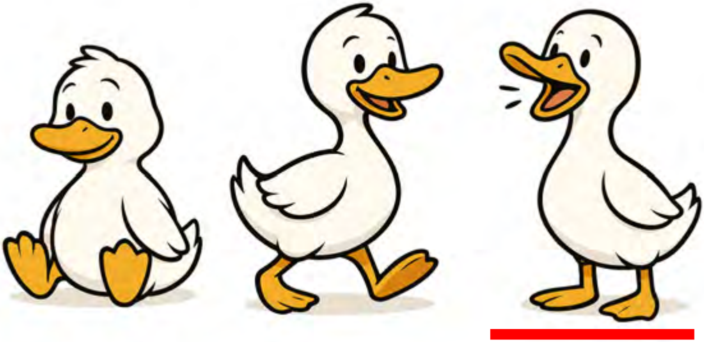

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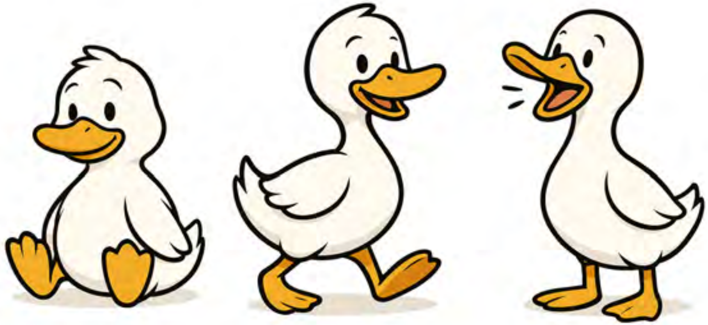

Probably a **duck**

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The Hilbert Space Metric

Relax the quantum state inner product:

$$\langle \Psi_1 | \Psi_2 \rangle \to \langle\langle \Psi_1 \,|\, \Psi_2 \rangle\rangle \equiv \langle \Psi_1 |\, G \,| \Psi_2 \rangle \,,$$

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metric operator

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i) $G = G^\dagger$.

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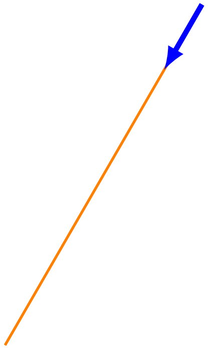

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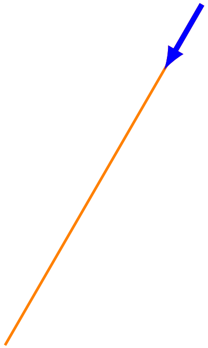

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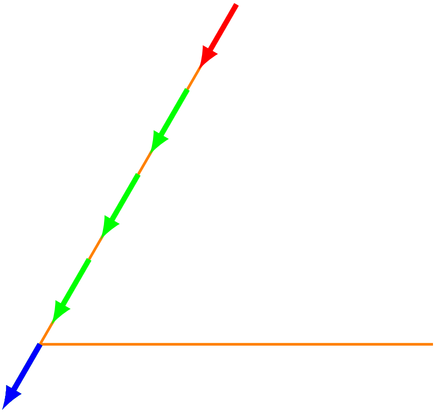

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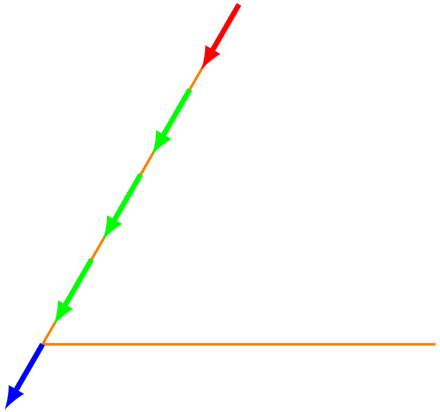

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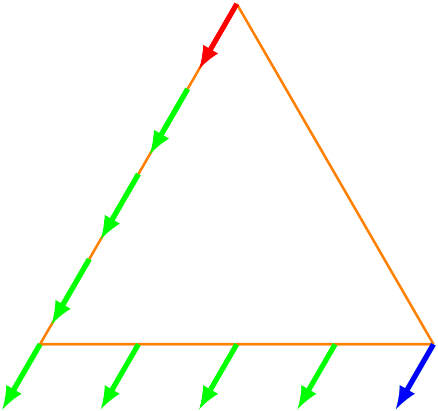

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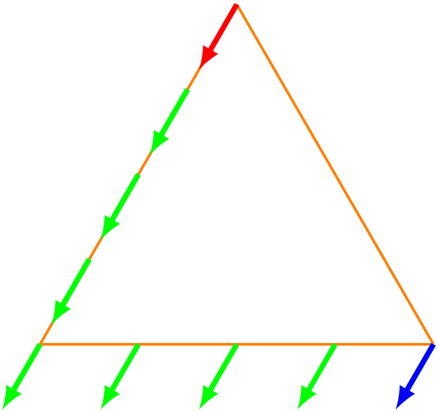

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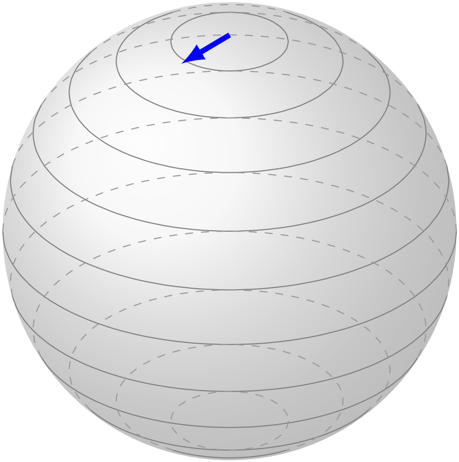

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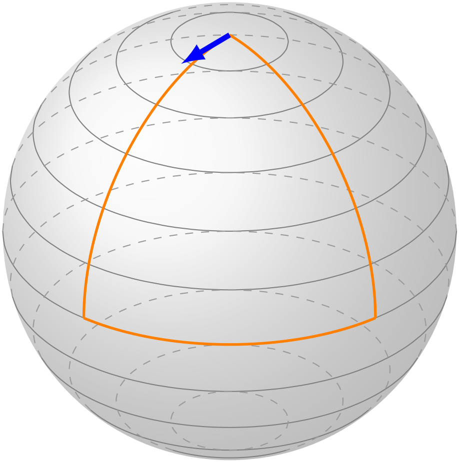

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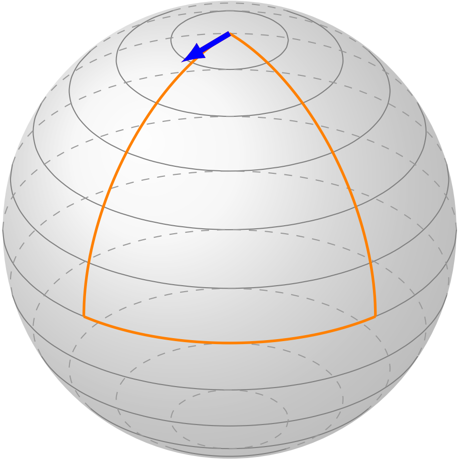

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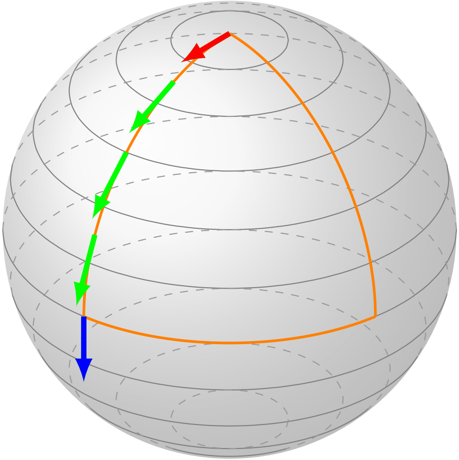

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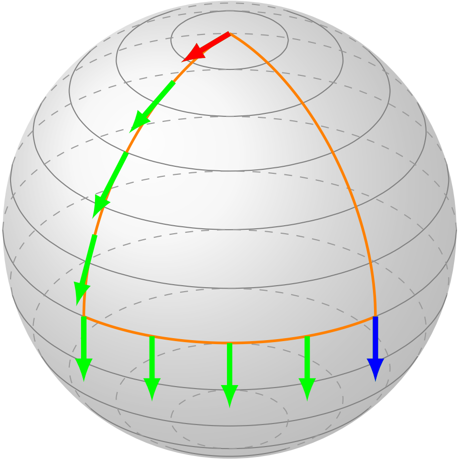

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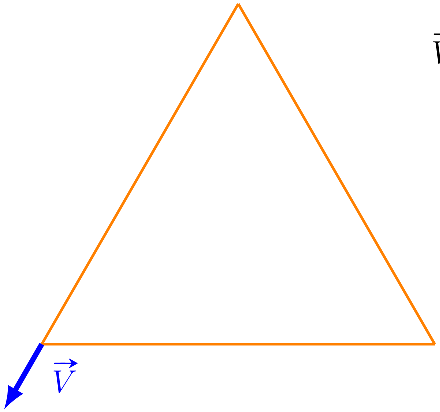

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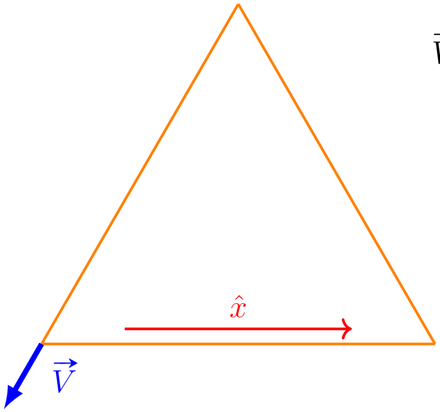

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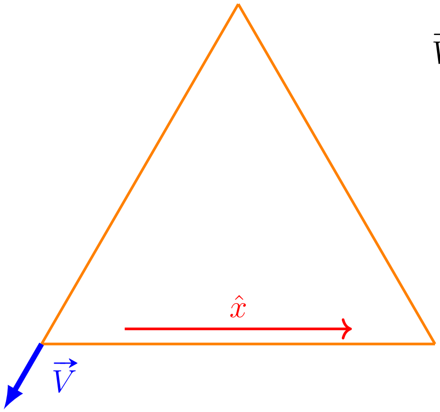

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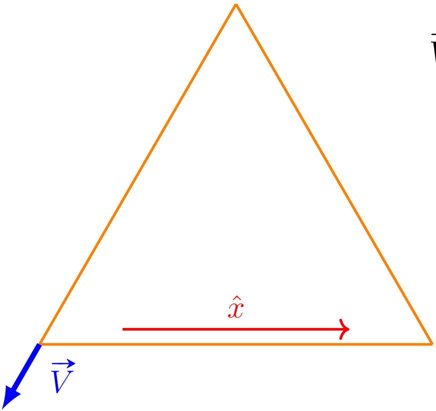

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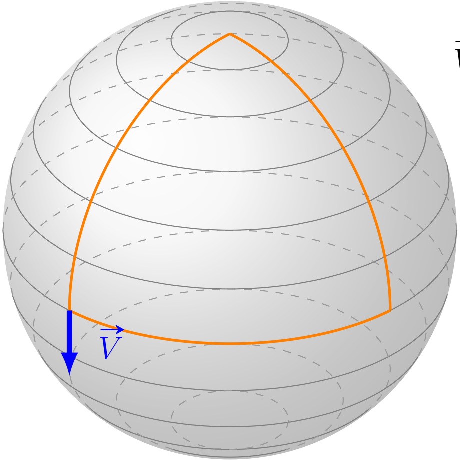

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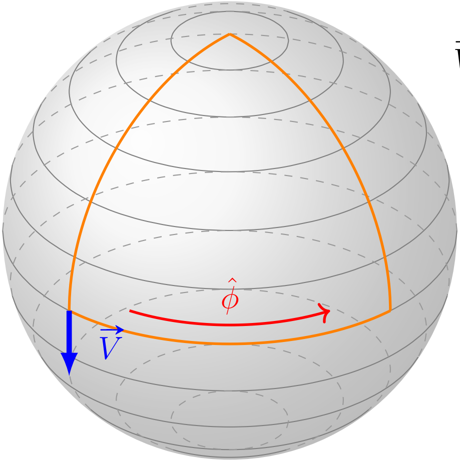

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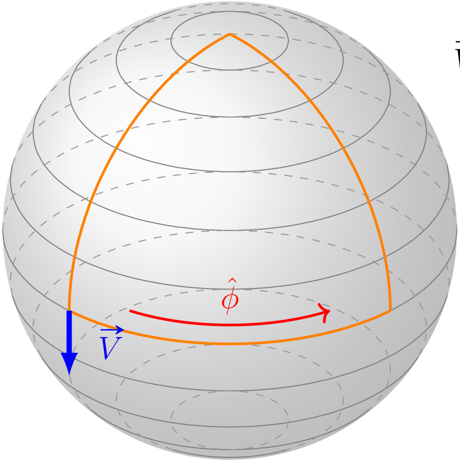

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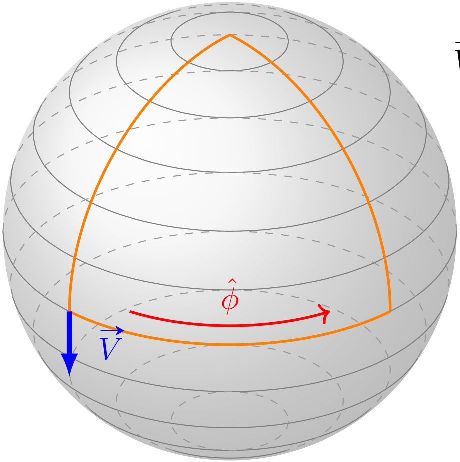

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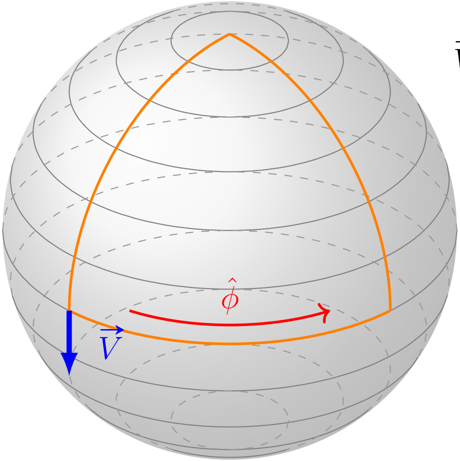

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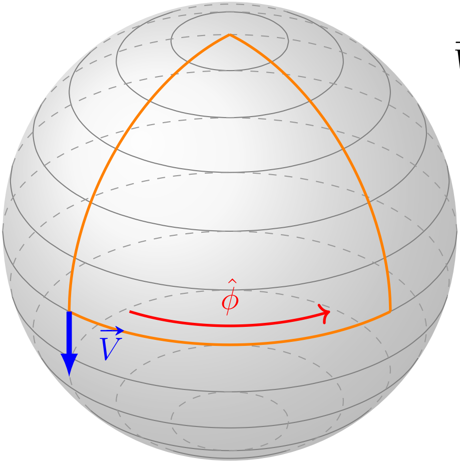

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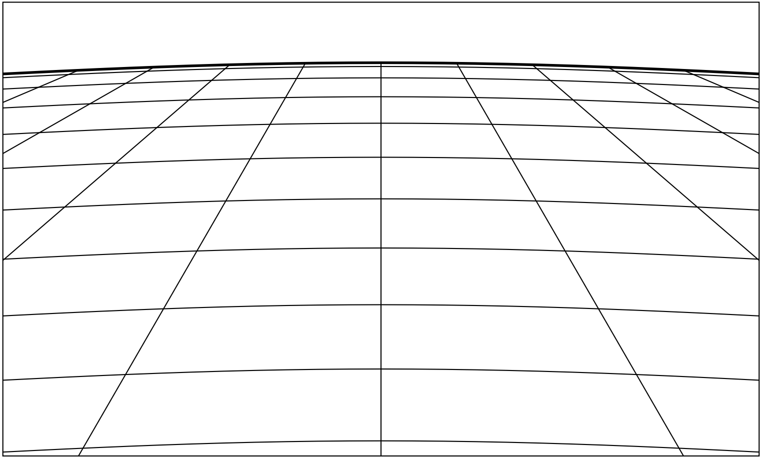
$\Rightarrow$ Determines the geometry of the Hilbert space bundle.

The Inner Product



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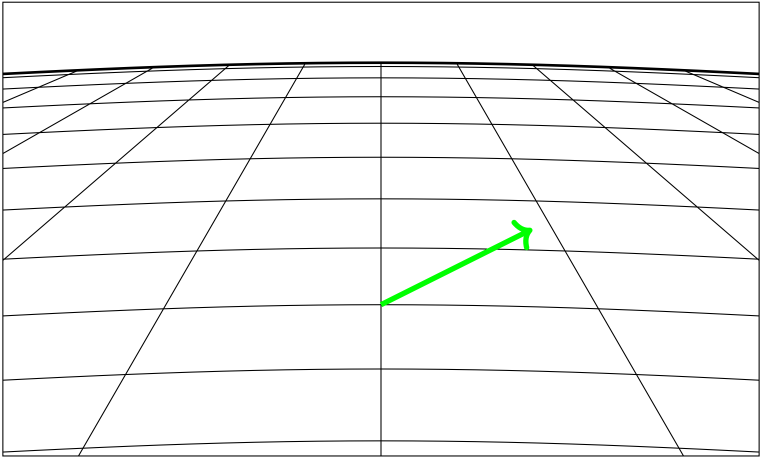

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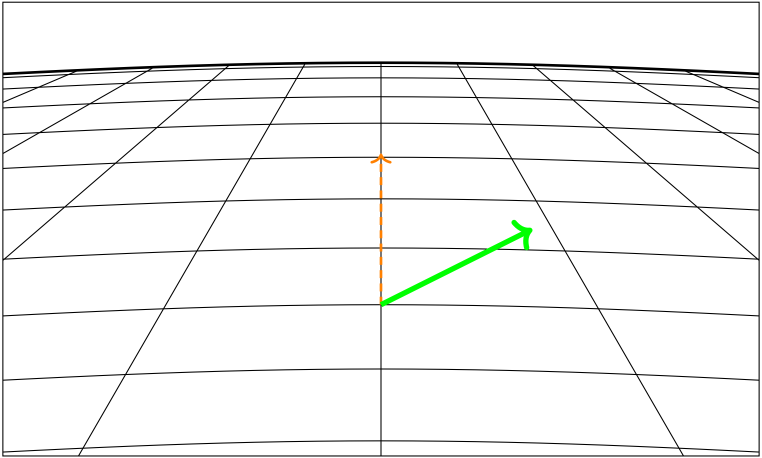

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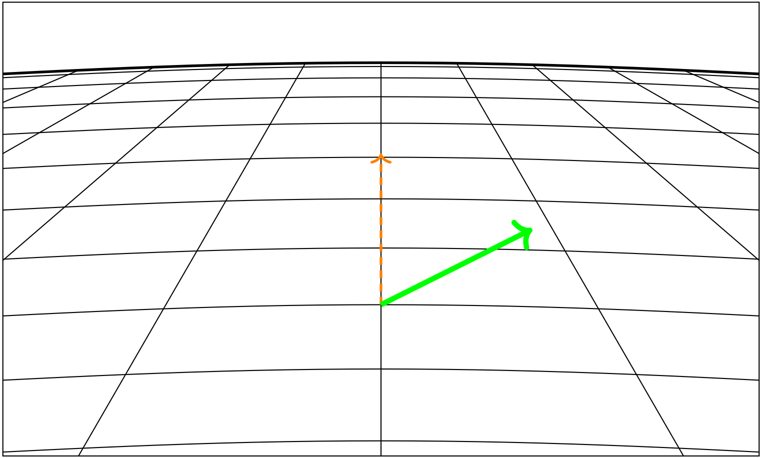

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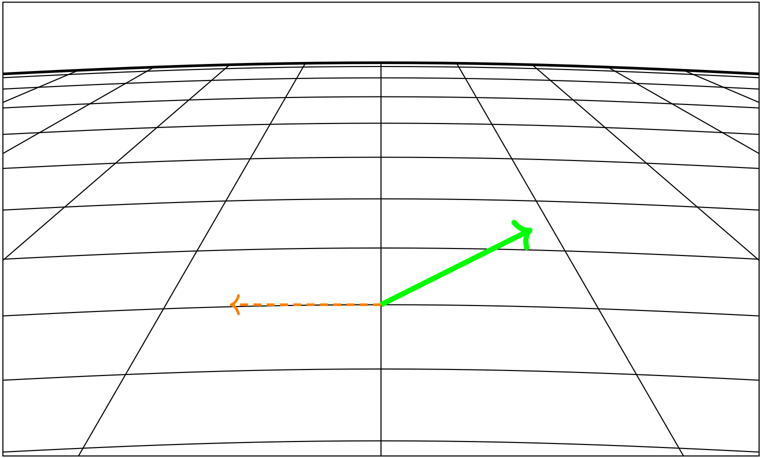

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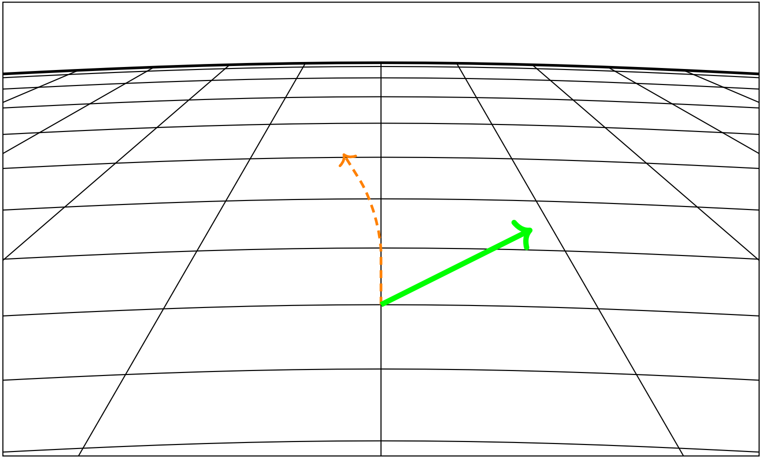

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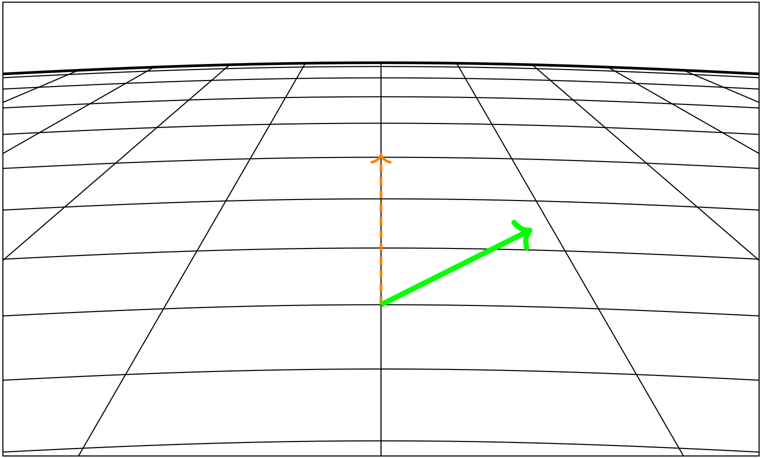

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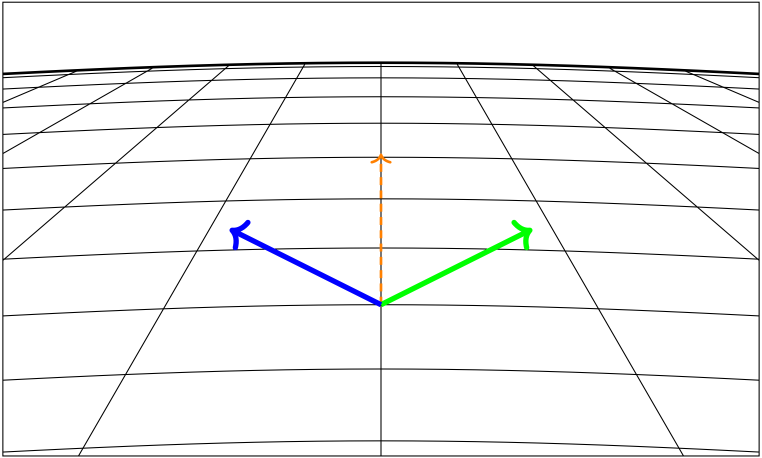

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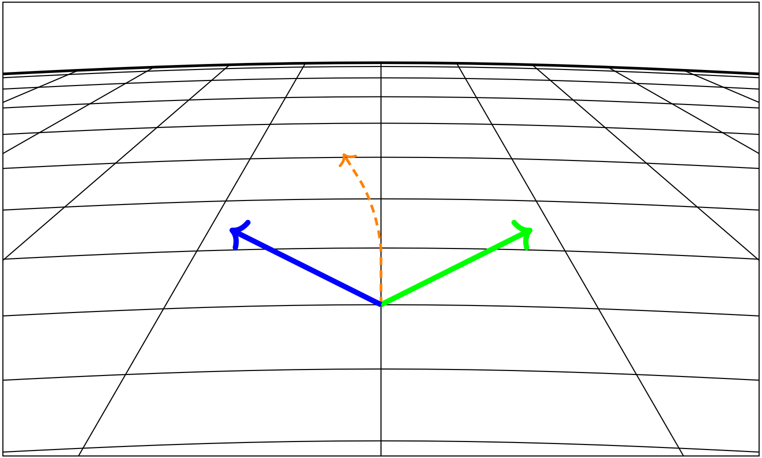

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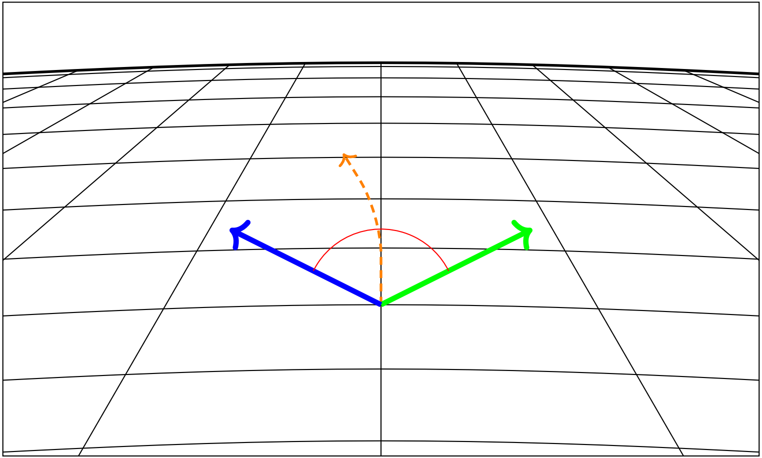

Connection-Compatible-Metric

$$\nabla_t \left|\Psi\right\rangle \equiv \left(\partial_t + iH\right)\left|\Psi\right\rangle$$

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connection/covariant derivative

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Connection-compatible-metric:

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Non-Hermitian Caused Non-triviality

From the metric equation $\partial_t G = i\left(GH - H^\dagger G\right)$:

1. The metric is trivial only if the system is Hermitian, i.e.,

$$G = \mathbb{1} \Rightarrow H = H^\dagger.$$

2. Non-Hermitian systems implies non-trivial metric, i.e.,

$$H \neq H^\dagger \Rightarrow G \neq \mathbb{1}.$$

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$\boxed{\text{Most questionable “predictions” do not appear.}}$

Uncomfor“table”

U n c o m f o r T a b l e

Parameter	Hamiltonian	State Evolution
Independent	$H(t)$	$i\partial_t \lvert\Psi(t)\rangle = H(t) \lvert\Psi(t)\rangle$
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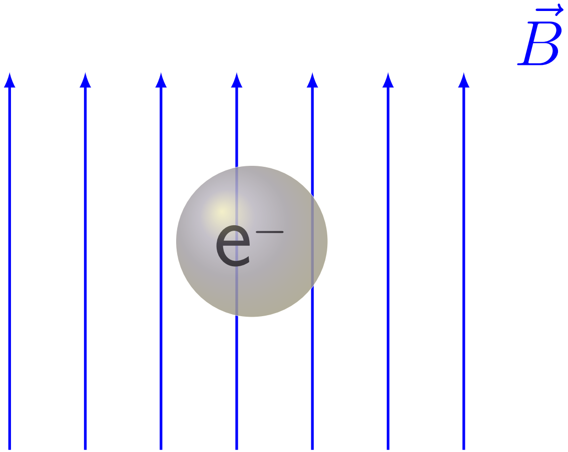

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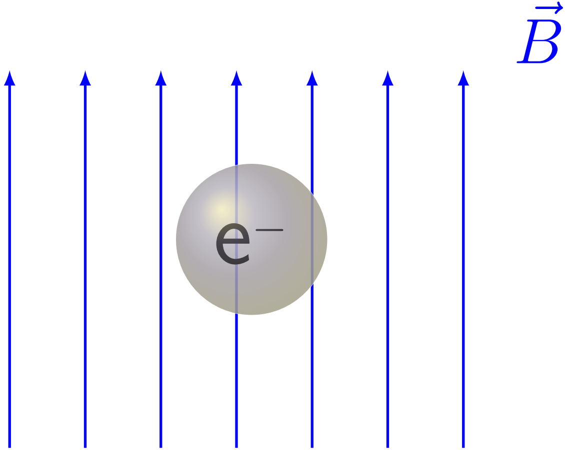

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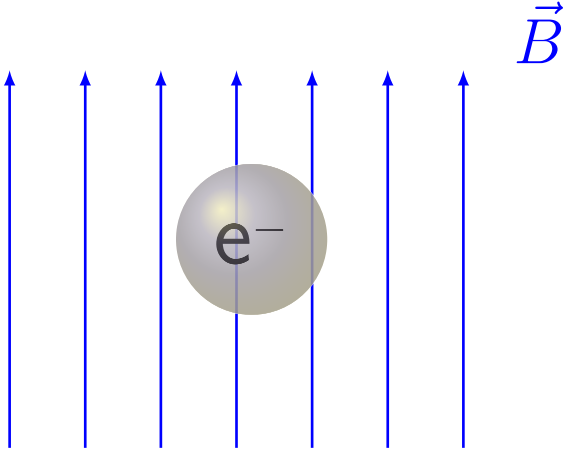

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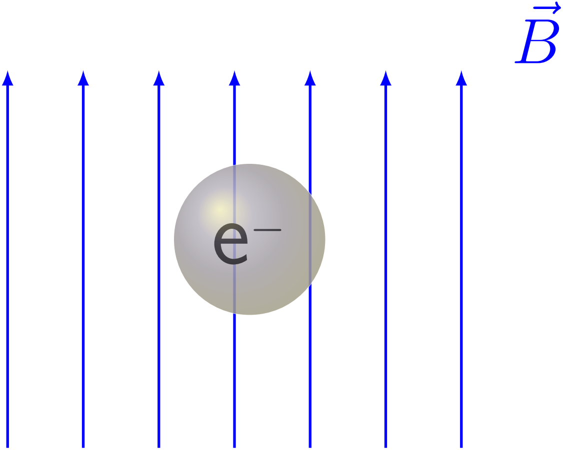

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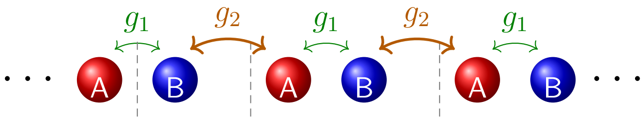

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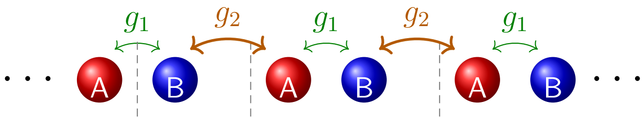

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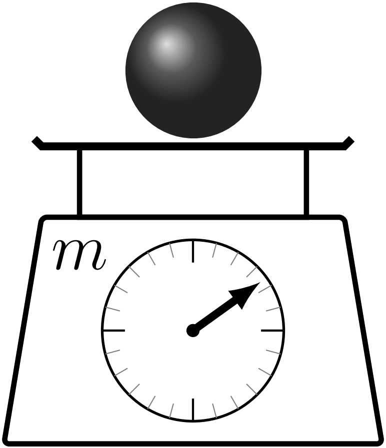

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Berry connection

For $H(q)\lvert\phi_n(q)\rangle = h_n(q)\lvert\phi_n(q)\rangle$,

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where $\mathcal{A}_n(q) = i\langle\phi_n(q)\rvert\,\partial_q\lvert\phi_n(q)\rangle$.
[Left-right eigenstates needed for $H^\dagger(q) \neq H(q)$.]

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Emergent Evolution Dimension

EOMs for quantum states and metric operator:

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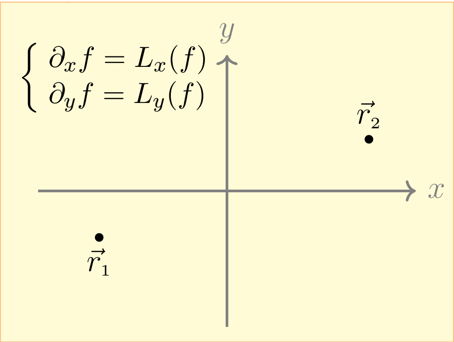

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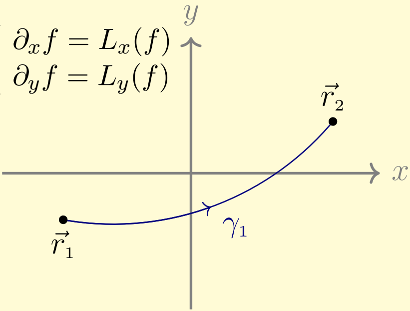

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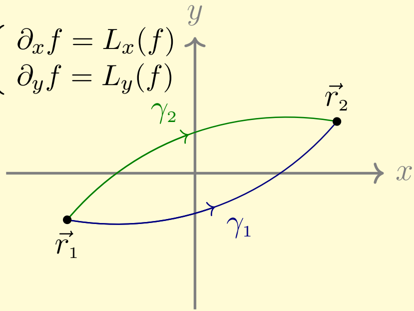

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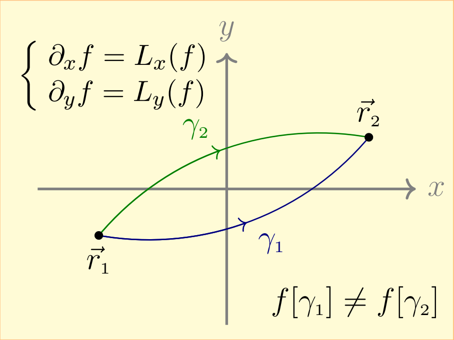

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Assumption:

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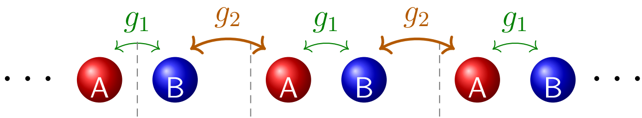

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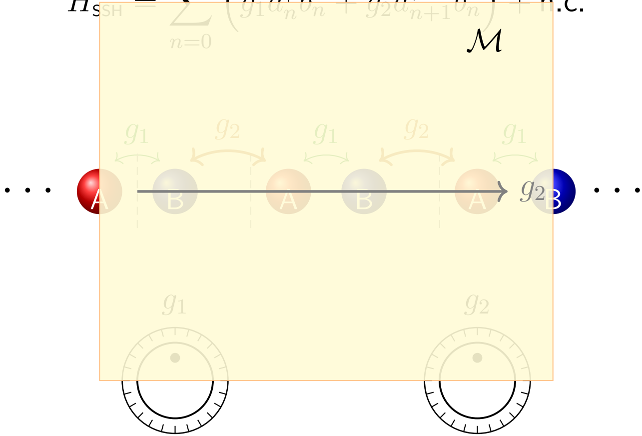

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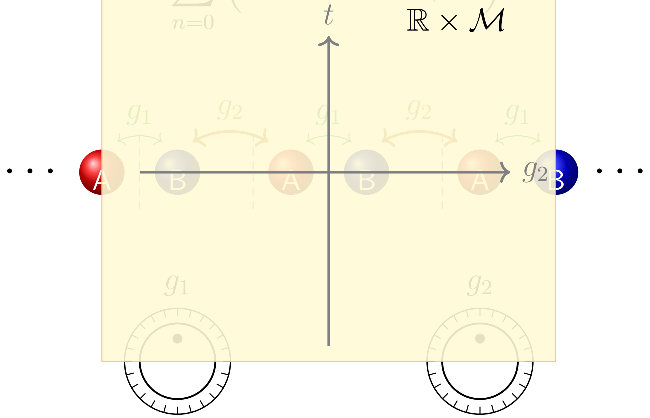

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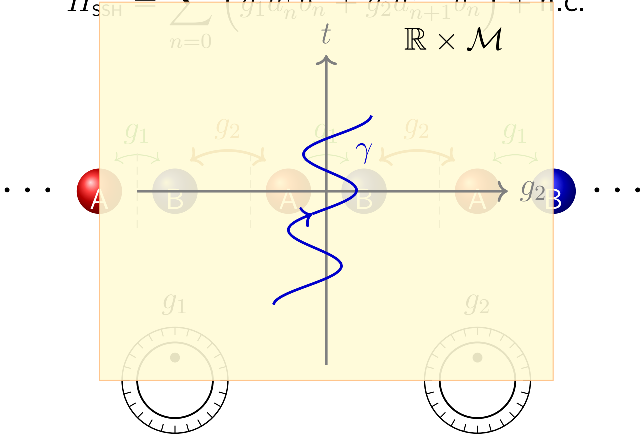

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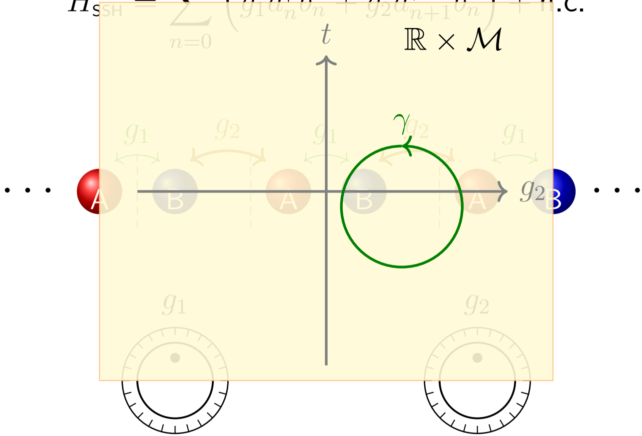

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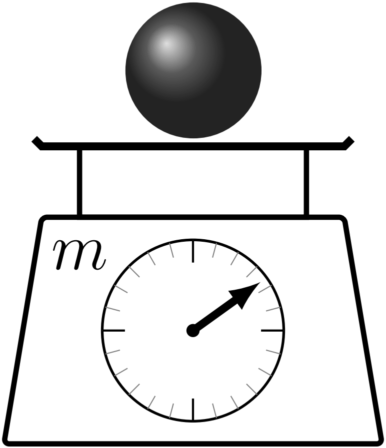

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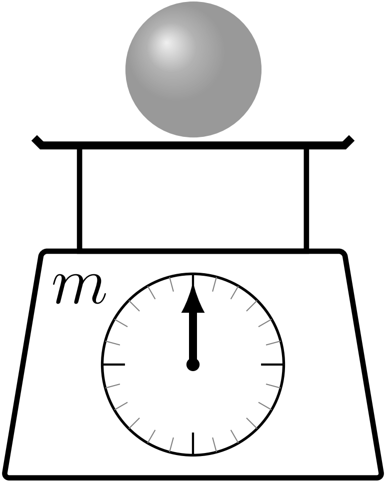

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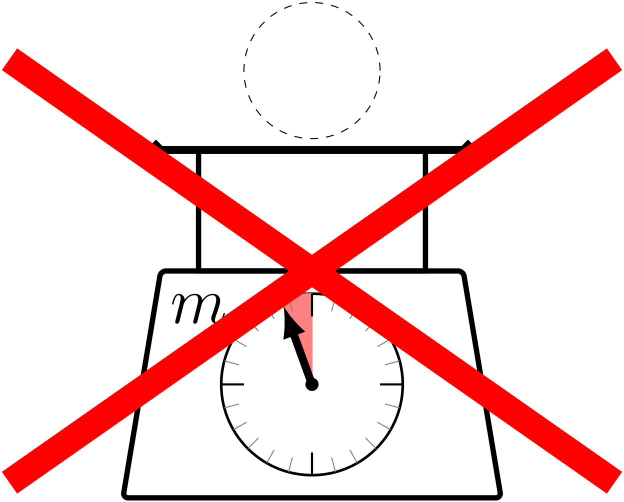

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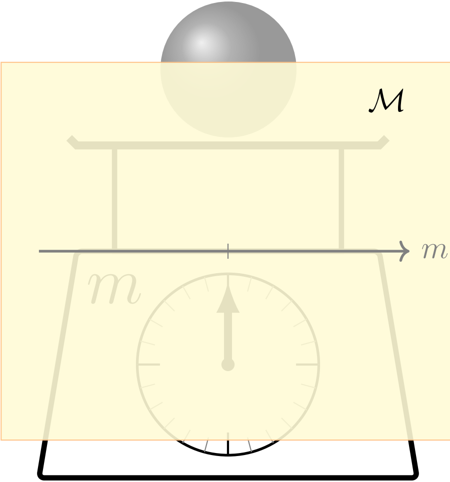

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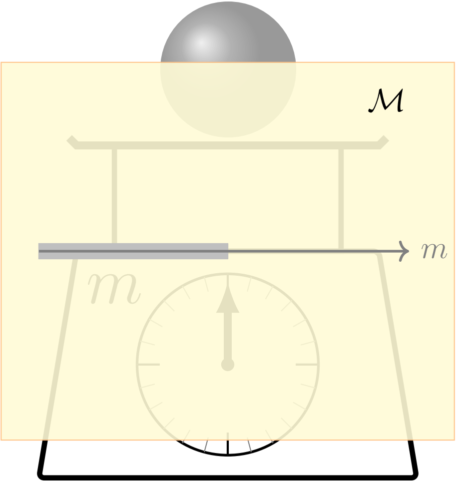

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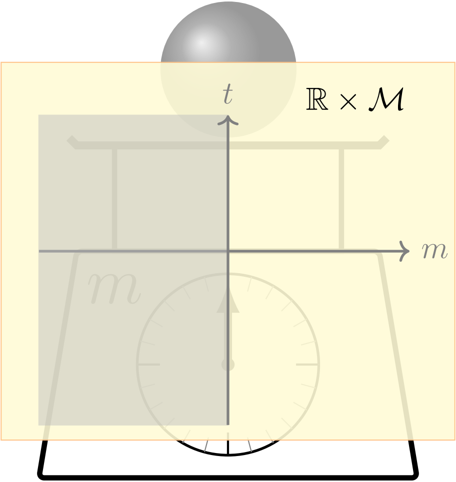

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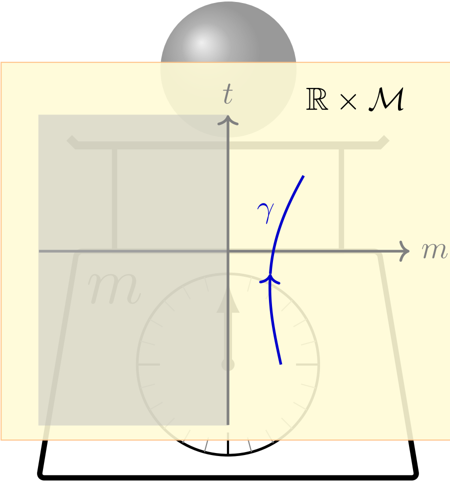

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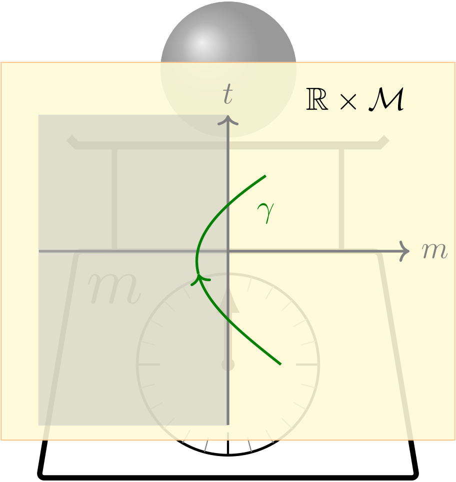

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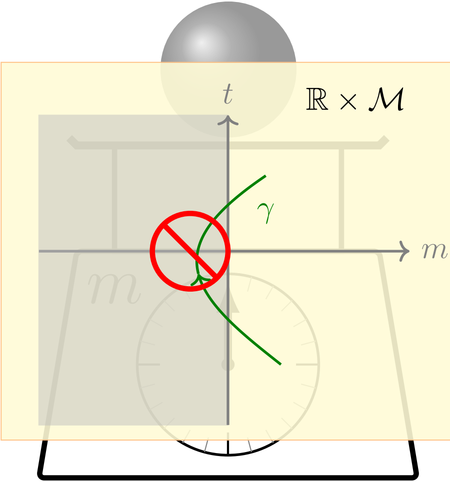

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Assumption:

$|\Psi[\gamma]\rangle$ is physical if every point on the path γ is

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The local curvature two-form:

$$\mathcal{F} = \frac{1}{2}\left(F_{tq} dt \wedge dq + F_{qt} dq \wedge dt\right),$$

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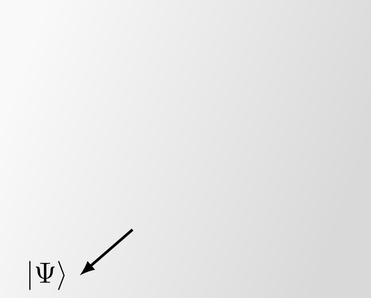

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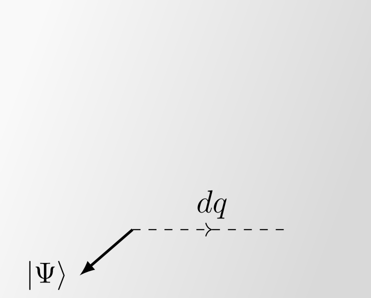

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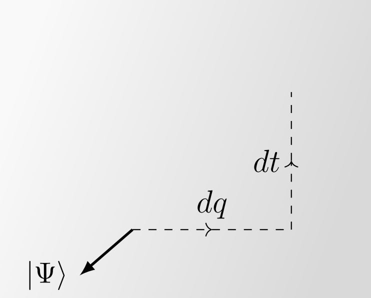

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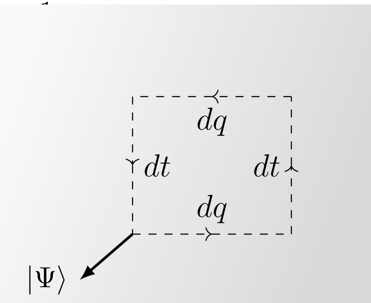

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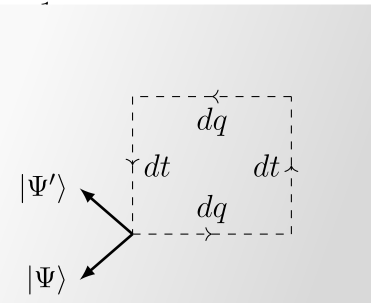

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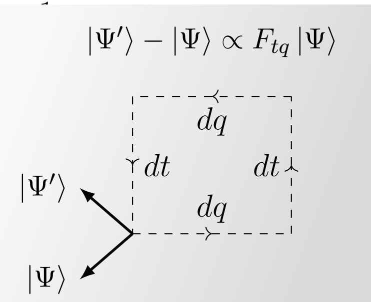

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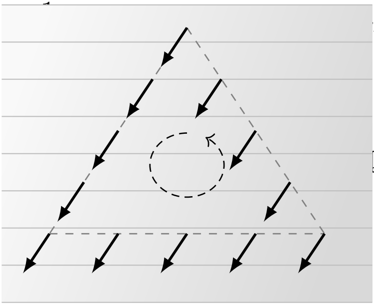

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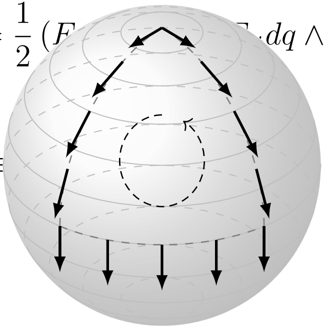

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Given $H = H(t, q^1, q^2, \cdots, q^d)$,

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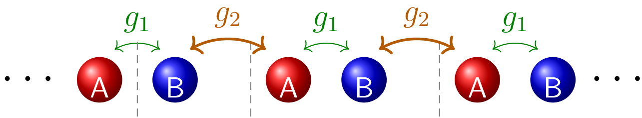
$$K_0 = \sum_{n=0}^{N-1} \left(g_1 a_n^\dagger b_n + g_2 a_{n+1}^\dagger b_n \right) + \text{h.c.}$$



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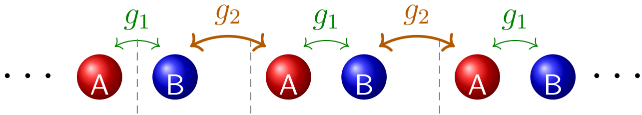

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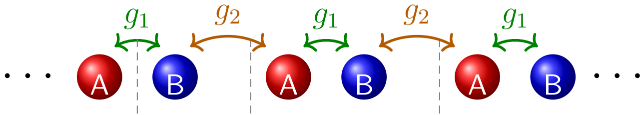

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Berry Potential and Curvature from Projection

For $H(\{q\})\,|\phi_n(\{q\})\rangle = h_n(\{q\})\,|\phi_n(\{q\})\rangle$,

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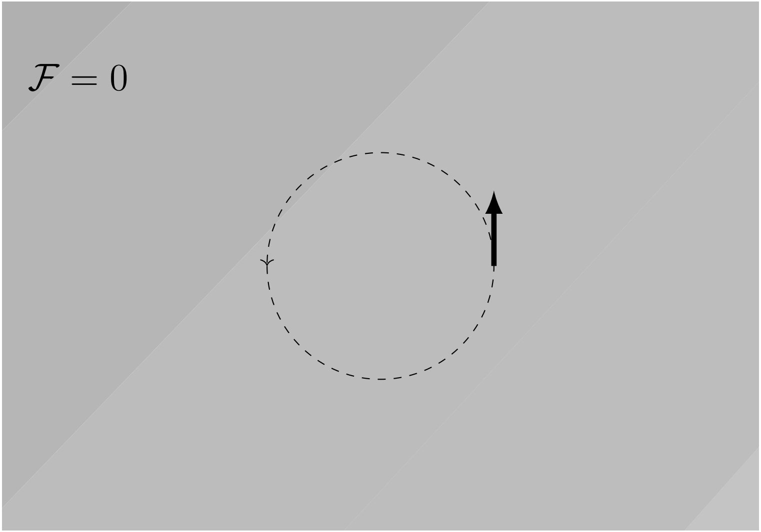

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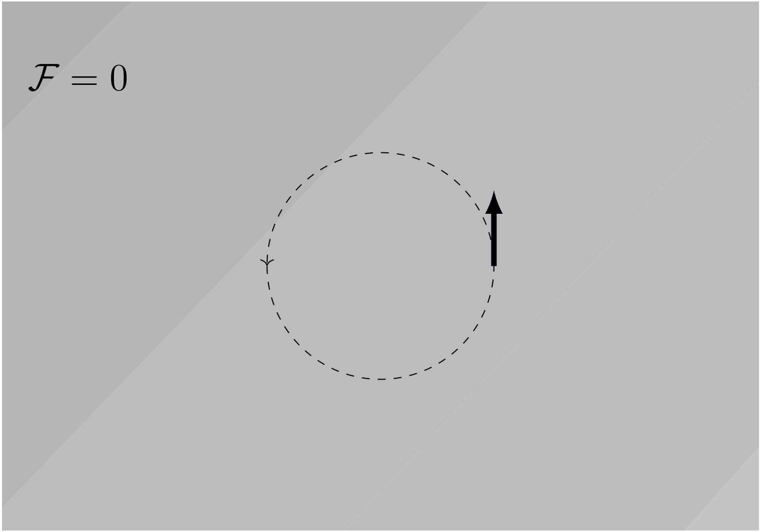

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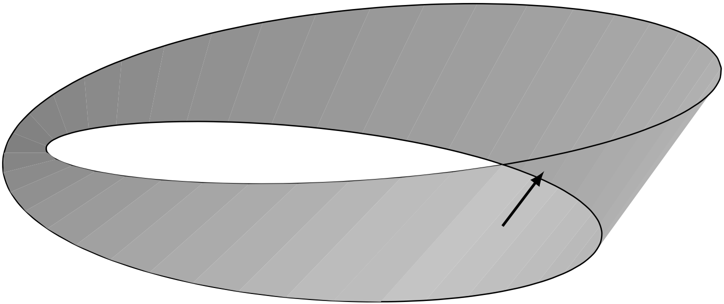

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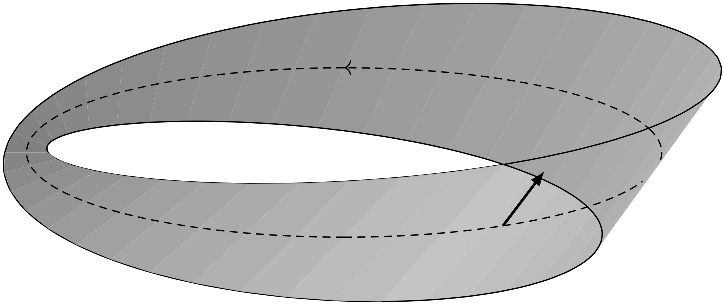

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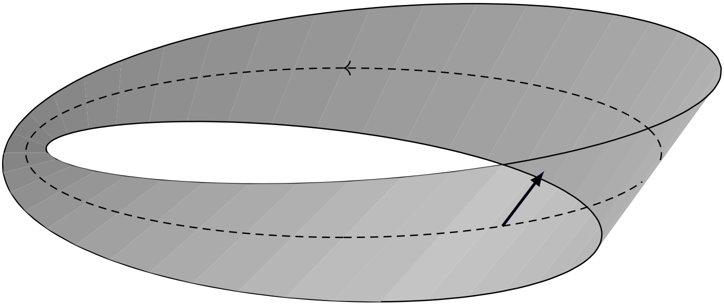

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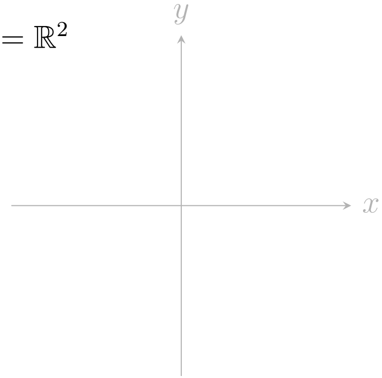

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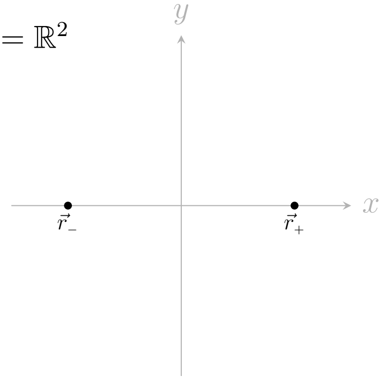

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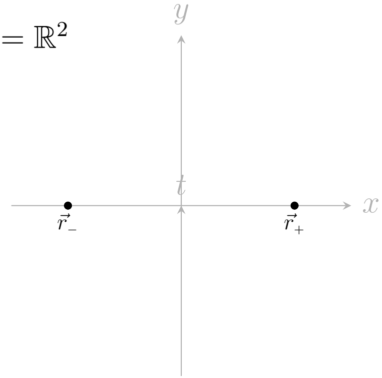

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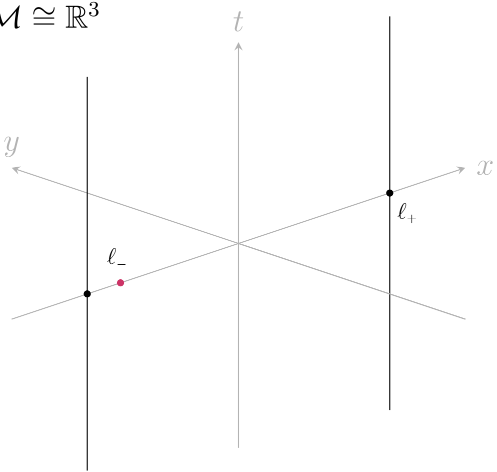

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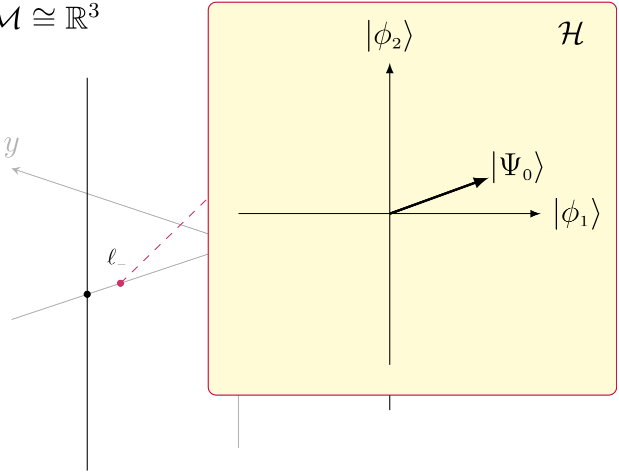

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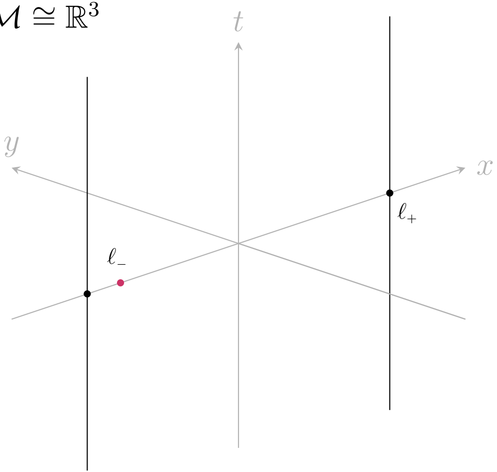

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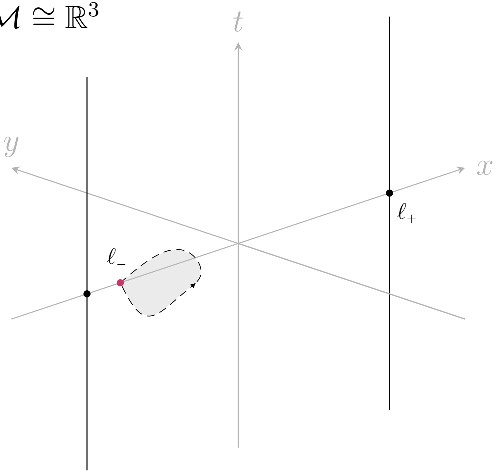

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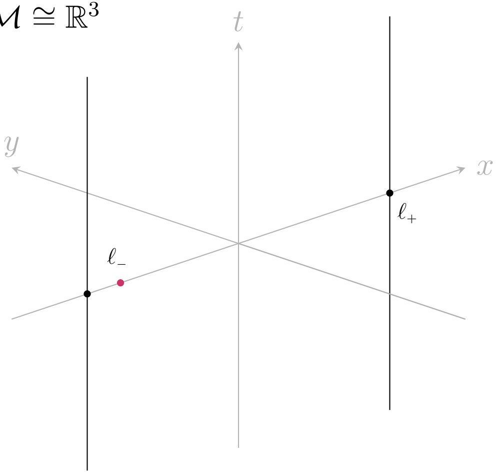

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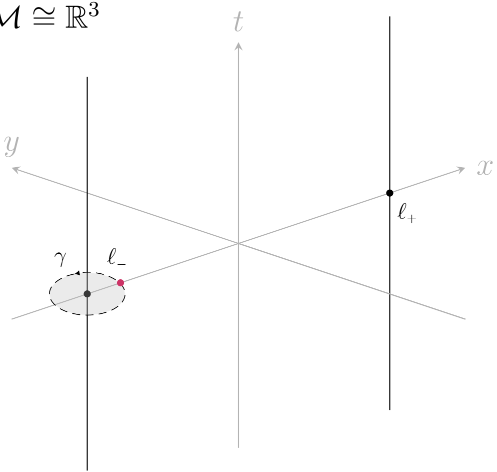

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Fact: $\mathcal{I}^2 = -\mathbb{1} \quad (\Rightarrow \mathcal{I}^4 = \mathbb{1})$.

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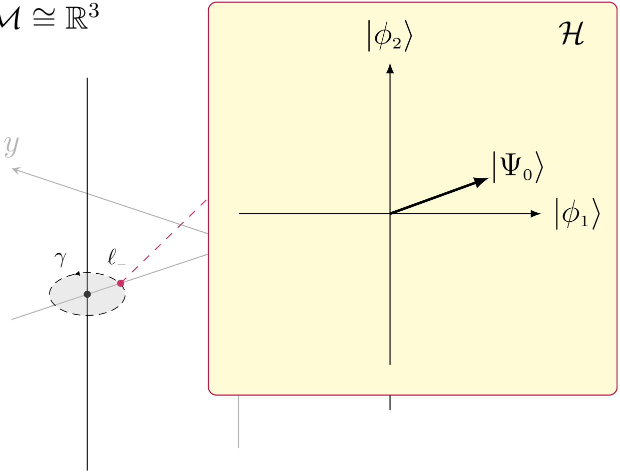
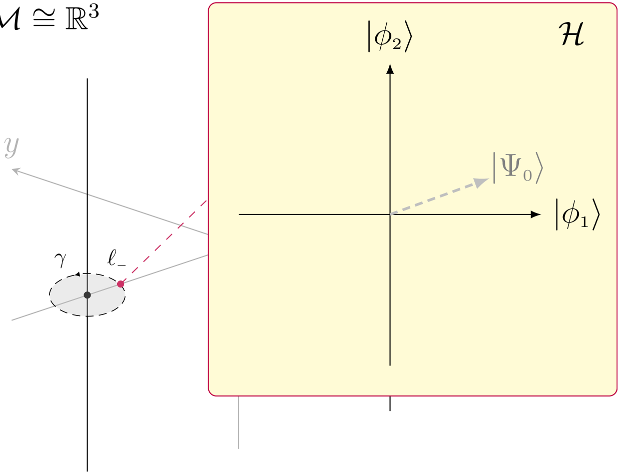


Topological Defects

$\mathbb{R} \times \mathcal{M} \cong \mathbb{R}^3$



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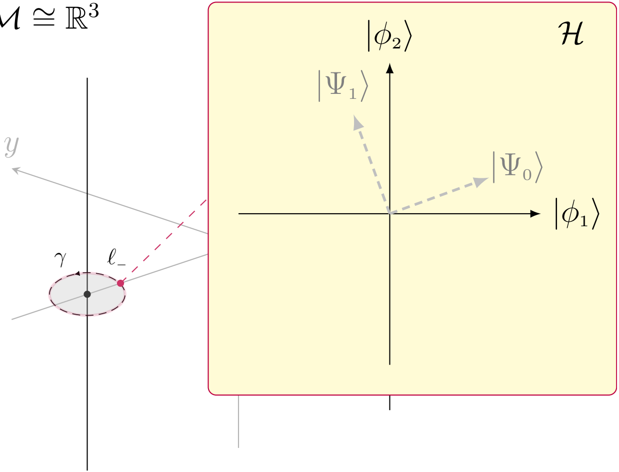

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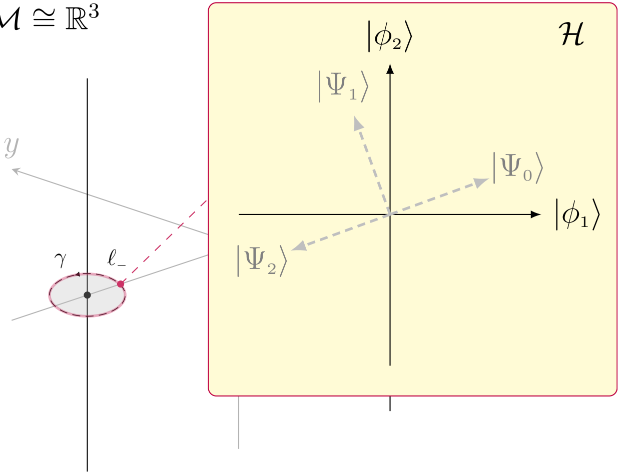

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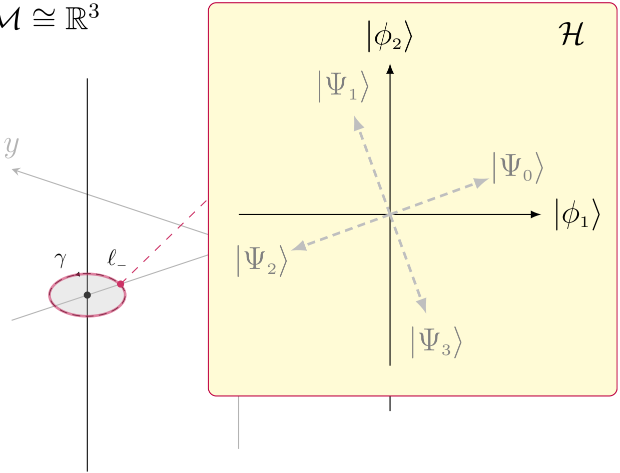

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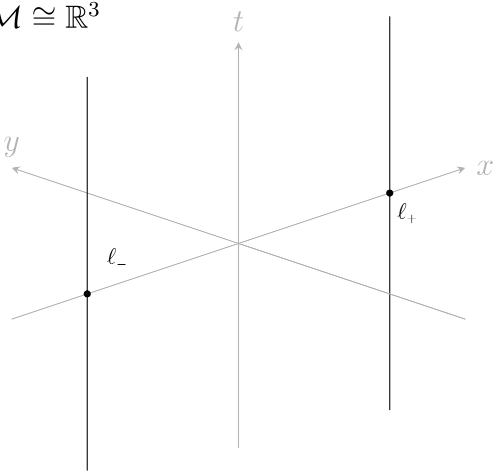

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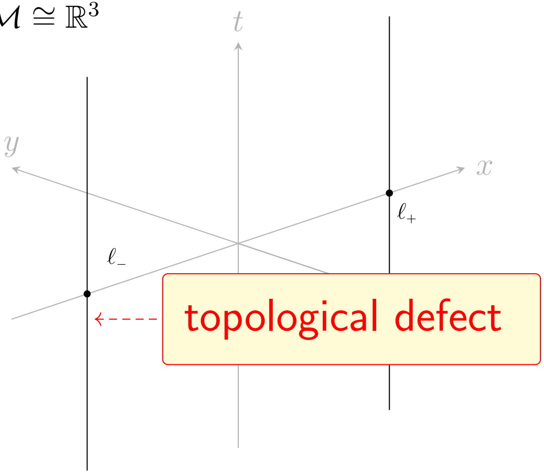

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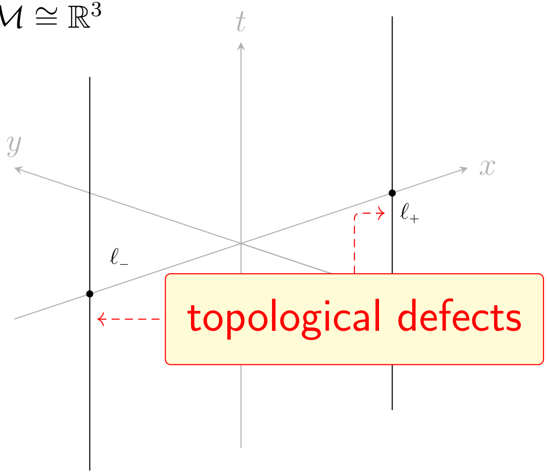

Summary

- ✯ Quantum explained through (differential) geometry
- ✯ Parameter space induces dimensions
- ✯ Locally flat but not necessarily globally trivial
- ✯ Results apply in the Hermitian regime

References:

[1] C.-Y. Ju and S.-M. Chen, arXiv:2412.06548 [quant-ph] (2025).

[2] C.-Y. Ju, A. Miranowicz, Y.-N. Chen, G.-Y. Chen, F. Nori, Quantum **8**, 1277 (2024).

[3] C.-Y. Ju, A. Miranowicz, G.-Y. Chen, F. Nori, Phys. Rev. A **100**, 062118 (2019).

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