INHOMOGENEOUS CONDENSATES IN A LIFSHITZ-DEFORMED GROSS-NEVEU MODEL IN 3+1 DIMENSIONS

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Analytic & Algebraic Methods in Physics XXII
Czech Technical University, Prague
26 August 2025

Work in progress…

The Gross-Neveu (GN) Model

Originally formulated in 1+1 dimensions: a renormalizable, asymptotically free four-Fermi interaction.

$$S = \int d^2x \left[\sum_{a=1}^{N} \bar{\psi}_a i \partial\!\!\!/ \psi_a + \frac{g^2}{2} \left(\sum_{a=1}^{N} \bar{\psi}_a \psi_a \right)^2 \right]$$

N species of Dirac fermions $\psi_a \,, \quad N := 1, 2, \ldots, N$

For $N \geq 2$ mimicking QCD phenomenology in 3+1 dim: asymptotic freedom, dynamical (discrete chiral) symmetry breaking, dynamical mass generation (At *N=1* it is just the Thirring model, which is conformal (no mass gap).)

Obvious *SU(N)* symmetry, but actually has a much higher *O(2N)* symmetry (break each Dirac fermion into a sum of two real Majorana fermions)

Exactly solvable (completely integrable) for any value of *N*. For $N \geq 2$ factorisable S-matrix. Rich spectrum of bound states and topological kinks.

An important application in condensed matter physics: it is the continuum limit of the Lagrangian describing electron dynamics in the SSH (Heeger-Su-Scrieffer) model of conducting polymer chains (polyacetylene) (after integrating phonon fluctuations out).

A related model is Nambu-Jona-Lasinio's (NJL) in 1+1 dimensions, defined by the action

$$S = \int d^2x \left\{ \sum_{a=1}^{N} \overline{\psi_a} i \partial\!\!\!/ \psi_a + \frac{g^2}{2} \left[\left(\sum_{a=1}^{N} \overline{\psi_a} \psi_a \right)^2 - \left(\sum_{a=1}^{N} \overline{\psi_a} \gamma_5 \psi_a \right)^2 \right] \right\}$$

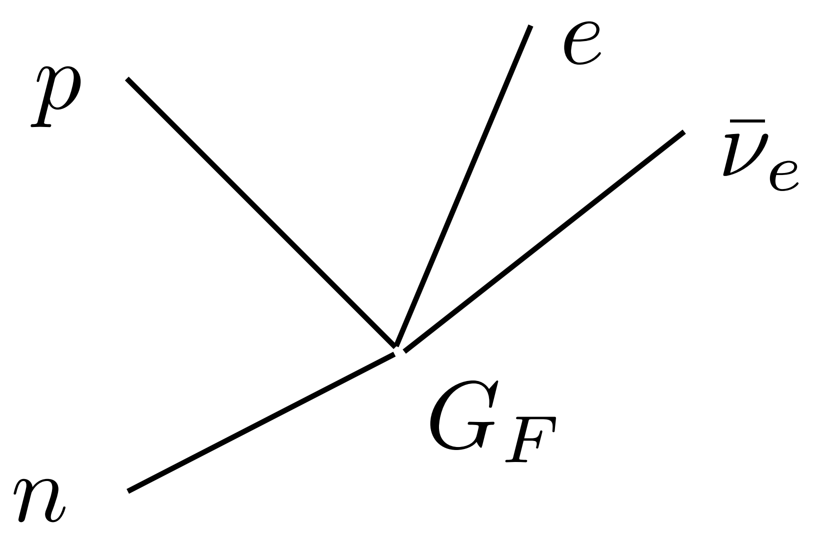
The real world is 3+1 dimensional

Does a four-fermi model like GN (or NJL) make sense in 3+1 dimensions?

Four-Fermi interaction in 3+1 Dimensions

Fermi's historic phenomenological model for beta decay (1933)

$$\mathcal{H}_{\text{Fermi}} = \frac{G_F}{\sqrt{2}} \left(\bar{\psi}_p \gamma^\mu \psi_n\right) \left(\bar{\psi}_e \gamma_\mu \psi_\nu\right)$$



effective point-wise interaction

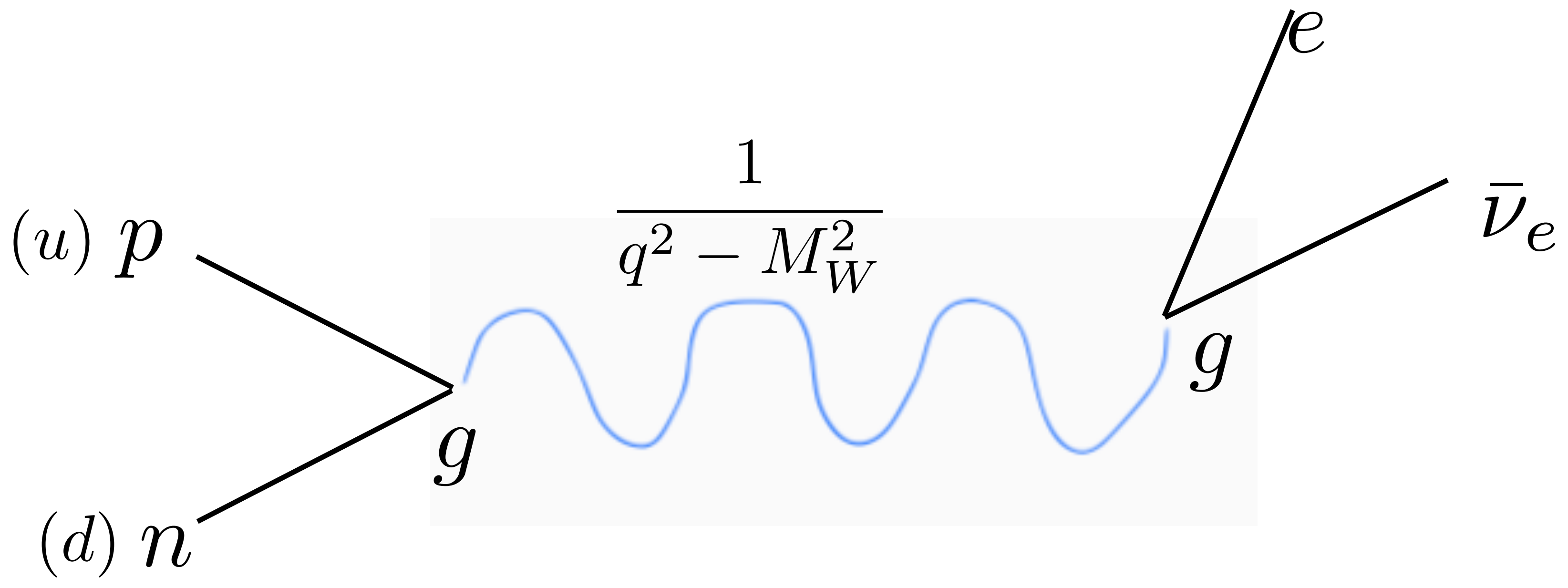
Only an effective hamiltonian, describing correctly low-energy $E \ll M_W \sim 80_{\text{GeV}}$ physics

mass-scale dimensions: $[\psi] = \frac{3}{2} \quad \Rightarrow [G_F] = 4 - 4 \times \frac{3}{2} = -2 \quad \Rightarrow$ non-renormalizable perturbatively

a low-energy effective field theory, useless at high energies (no predictive power)

UV completion: The UV completion of Fermi's effective field theory of the weak interactions is the Electroweak sector of the Standard Model of particle physics: it is a local, consistent (anomaly free) renormalizable gauge theory with accurate predictive power at high energy

$$G_F \sim \frac{g^2}{M_W^2} = \frac{\text{dimensionless EW coupling}}{M_W^2}$$



4-Fermi interaction appears point-like on length scales >> $\lambda_C = \frac{\hbar}{M_W c} = \frac{\hbar c}{M_W c^2} \sim \frac{0.2\text{GeV Fermi}}{80\text{GeV}} = 0.0025\text{Fermi} \sim 0.0025\text{proton size}$

The EW gauge theory is not the only possible UV completion of four-Fermi interactions in 3+1d

In $2 \leq d < 4 \quad (d = 4 - \epsilon)$ The Gross-Neveu model at large-N is renormalizable non-perturbatively.

e.g., B. Rosenstein, B.J. Warr & S.H. Park 1991 large-N non-perturbative renormalizability

J. Zinn-Justin 1991 sensitivity to fine details of the bare hamiltonian (the particular manner in which the UV cutoff is set)

H. Gies, J. Braun & D.D. Scherer, 2010 There is a non-perturbative UV fixed point (asymptotic safety)

Want a more robust UV completion, valid also for finite N and not sensitive to details of the bare hamiltonian

Lifshitz-type deformation offers such UV completion. This is achieved by adding higher **spatial** derivatives to the kinetic term, thus improving the UV behavior of propagators.

This comes at a price: breaking Lorentz invariance at short distances. However, Lorentz symmetry emerges effectively at low energies.

(Insisting on preserving Lorentz invariance would require also higher temporal derivatives, which would introduce unwanted ghosts.)

A clear and thorough paper to this effect is DMW:

PHYSICAL REVIEW D **80,** 105018 (2009)

Asymptotically free four-Fermi theory in 4 dimensions at the $z = 3$ Lifshitz-like fixed point

Avinash Dhar,[1,*] Gautam Mandal,[1,†] and Spenta R. Wadia[1,2,‡]

DMW construct a renormalizable Lifshitz deformation of the 3+1 dimensional Nambu-Jona-Lasinio (NJL) model.

It is an asymptotically free UV completion of the original Lorentz-invariant 3+1d NJL effective field theory, describing the phenomenology of chiral symmetry breaking in terms of the composite scalar field $\sigma \sim \bar{\psi}\psi$ and the pseudo scalar “pion” field $\pi \sim \bar{\psi}\gamma_5\psi$.

The σ field develops a non-vanishing VEV $\langle\sigma\rangle \neq 0$, indicating the dynamical breakdown of the continuous $U(1)_A$ axial symmetry, and the massless pion appears as the associated Goldstone boson.

In this talk: a renormalizable Lifshitz deformation of the (simpler) 3+1 dimensional Gross-Neveu model.

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_4$$

kinetic term: $$\mathcal{L}_2 = (\psi^\dagger_{1i}, \psi^\dagger_{2i}) \left[\mathbb{1}_4 i\partial_t - M^2 (i\vec{\alpha}\cdot\nabla) + (i\vec{\alpha}\cdot\nabla)\nabla^2 \right] \begin{pmatrix} \psi_{1i} \\ \psi_{2i} \end{pmatrix}$$

spinors: ψ_{1i}, ψ_{2i} are each 2-component *SU(2)* spinors (*SU(2)* here being the double-cover of the spatial rotation group *SO(3)*)

In a Lorentz invariant theory they would comprise together a four-component Dirac spinor.

i:=1,2,… N is a *U(N)* index

The usual *4x4* Dirac matrices in 3+1 dimensions

$$\gamma^0 = \beta = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} \qquad \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \qquad \vec{\gamma} = \beta\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$$

Pauli $\vec{\sigma}$

M a bare coupling of mass-dimension 1 (a bare mass scale)

it is a relevant coupling, needed for having emergent Lorentz invariance at low energies

Lifshitz scaling- by comparing the first and last terms in the kinetic term (inverse propagator) we see that space and time scale differently:

$x \to bx\,, t \to b^3 t$ (Lifshitz scaling exponent *z=3*)

The bare model is manifestly **Lorentz non-invariant**!

pack together the two two-component spinors into a 4-component spinor

$$\psi_i = \begin{pmatrix} \psi_{1i} \\ \psi_{2i} \end{pmatrix}$$

(it would become a Dirac spinor in the low-energy Lorentz invariant limit)

kinetic term: $\mathcal{L}_2 = \bar{\psi}_i \left[\gamma^0 i\partial_t + (i\vec{\gamma}\cdot\nabla)(\nabla^2 - M^2)\right]\psi_i$

$$\bar{\psi}_i = (\psi^\dagger_{1i}, \psi^\dagger_{2i})\gamma^0 = \psi^\dagger_i\gamma^0$$

interaction term $\mathcal{L}_4 = \frac{g^2}{2}(\psi^\dagger_{1i}\psi_{1i} - \psi^\dagger_{2i}\psi_{2i})^2 = \frac{g^2}{2}(\bar{\psi}_i\psi_i)^2$

g^2 bare 4-Fermi coupling

z=3 Lifshitz scaling and power counting:

mass dimensions

$$[\partial_x] = 1, \quad [\partial_t] = [(\vec{\gamma}\cdot\nabla)\nabla^2] = 3, \quad [M^2] = 2, \quad [dtd^3x] = -6 \quad \Rightarrow [\psi] = \frac{6-3}{2} = \frac{3}{2}$$

$$[g^2] = 6 - 4\times\frac{3}{2} = 0$$

Marginal!

in fact, this coupling flows to zero under the RG - the model is asymptotically free and thus provides a UV completion of the GN model in 3+1 dimensions

Symmetries:

rotational invariance $\psi_{1i} \to V\psi_{1i}, \quad \psi_{2i} \to V\psi_{2i}, \quad V \in SU(2)$ (same V)

unitary symmetry $\psi_{1i} \to U_{ij}\psi_{1j}, \quad \psi_{2i} \to U_{ij}\psi_{2j}, \quad U \in U(N) = SU(N) \otimes U(1)$ (same U)

includes both special unitary rotations and abelian phase

these transformations leave the bilinear combinations $\psi^\dagger_{1i}\psi_{1i}, \quad \psi^\dagger_{2i}\psi_{2i}, \quad \psi^\dagger_{1i}\psi_{2i}$ invariant

rendering $\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_4$ invariant as a whole

Discrete $\mathbb{Z}_2$ chiral γ_5-symmetry: $\psi_i \to \gamma_5\psi_i, \quad \bar{\psi}_i \to -\bar{\psi}_i\gamma_5$

$$\mathcal{L}_2 \to (-\bar{\psi}_i\gamma_5)\left[\gamma^0 i\partial_t + (i\vec{\gamma}\cdot\nabla)(\nabla^2 - M^2)\right]\gamma_5\psi_i = (-1)^2\bar{\psi}_i\left[\gamma^0 i\partial_t + (i\vec{\gamma}\cdot\nabla)(\nabla^2 - M^2)\right]\gamma_5^2\psi_i = \mathcal{L}_2$$

$$\mathcal{L}_4 \to (-\bar{\psi}_i\gamma_5\gamma_5\psi_i)^2 = (\bar{\psi}_i\psi_i)^2 = \mathcal{L}_4$$

this symmetry is broken dynamically at the quantum level, a fact underlining the physics of the model

Effective potential and asymptotic freedom

$$\mathcal{L} = \bar{\psi}_i \left[\gamma^0 i\partial_t + (i\vec{\gamma}\cdot\nabla)(\nabla^2 - M^2)\right]\psi_i + \frac{g^2}{2}(\bar{\psi}_i\psi_i)^2$$

Hubbard-Stratanovich auxiliary field $\sigma(t,\vec{x})$

$$\tilde{\mathcal{L}} = \bar{\psi}_i \left[\gamma^0 i\partial_t + (i\vec{\gamma}\cdot\nabla)(\nabla^2 - M^2) - \sigma\right]\psi_i - \frac{1}{2g^2}\sigma^2$$

EOM for $\sigma(t,\vec{x})$ $\qquad \sigma = -g^2\bar{\psi}_i\psi_i$

it is the chiral condensate. under the discrete chiral γ_5 symmetry

$\sigma \to -\sigma \quad (\bar{\psi}_i\psi_i \to -\bar{\psi}_i\psi_i)$

thus, σ is the order parameter the discrete γ_5 symmetry breaking

mass dimensions $\quad [\sigma] = [\partial_t] = [\nabla^3] = 3$

Path integral quantization

inverse propagator $\mathcal{D}[\sigma] = \gamma^0 i\partial_t + (i\vec{\gamma}\cdot\nabla)(\nabla^2 - M^2) - \sigma$ (the would-be Dirac operator in the low energy Lorentz limit)

note for later reference that $\gamma_5 \mathcal{D}[\sigma]\gamma_5 = -\mathcal{D}[-\sigma]$

then $\tilde{\mathcal{L}} = \bar{\psi}_i \mathcal{D}[\sigma]\psi_i - \frac{1}{2g^2}\sigma^2$

thus $\mathcal{Z} = \int \mathcal{D}\bar{\psi}_i \mathcal{D}\psi_i \mathcal{D}\sigma e^{i\int \tilde{\mathcal{L}} d^3x dt}$

perform Gaussian integration over the *N* complex quadratic Fermi fields to obtain $\mathcal{Z} = \int \mathcal{D}\sigma\, (\det \mathcal{D})^N e^{-i\int \frac{\sigma^2}{2g^2} d^3x dt}$

Thus $\mathcal{Z} = \int \mathcal{D}\sigma\, e^{-i\int \frac{\sigma^2}{2g^2} d^3x dt + N \mathrm{Tr}\log \mathcal{D}[\sigma]} = \int \mathcal{D}\sigma\, e^{iN S_{\mathrm{eff}}[\sigma]}$

where $S_{\mathrm{eff}}[\sigma] = -\int \frac{\sigma^2}{2Ng^2} d^3x dt - i\mathrm{Tr}\log \mathcal{D}[\sigma]$ ***Tr*** over both Dirac and Hilbert space indices

Study the model in the 't Hooft Limit: $N \to \infty, \quad g^2 \sim \frac{1}{N} \to 0, \quad \lambda = Ng^2 = \text{finite}$

Ground State: translational invariance in $t, \vec{x}$

$\sigma = \text{const.}$

$$\mathrm{Tr}\log \mathcal{D}[\sigma] = \frac{L^3 T}{(2\pi)^4} \int d\omega d\vec{k}\, \mathrm{tr}(\gamma^0\omega - (\vec{k}^2 + M^2)\vec{k}\cdot\vec{\gamma} - \sigma)$$

tr over Dirac indices only

$$\frac{\partial}{\partial\sigma}\mathrm{Tr}\log \mathcal{D}[\sigma] = -\frac{L^3 T}{(2\pi)^4} \int d\omega d\vec{k}\, \mathrm{tr}\frac{1}{\gamma^0\omega - (\vec{k}^2 + M^2)\vec{k}\cdot\vec{\gamma} - \sigma}$$

$$= -\frac{L^3 T}{(2\pi)^4} \int d\omega d\vec{k}\, \mathrm{tr}\frac{\gamma^0\omega - (\vec{k}^2 + M^2)\vec{k}\cdot\vec{\gamma} + \sigma}{[\gamma^0\omega - (\vec{k}^2 + M^2)\vec{k}\cdot\vec{\gamma} - \sigma][\gamma^0\omega - (\vec{k}^2 + M^2)\vec{k}\cdot\vec{\gamma} + \sigma]}$$

$$= -\frac{L^3 T}{(2\pi)^4} \int d\omega d\vec{k}\, \frac{4\sigma}{\omega^2 + i\epsilon - (\vec{k}^2 + M^2)^2\vec{k}^2 - \sigma^2}$$

where the Feynman $i\epsilon$ prescription is displayed explicitly

Wick rotation $$\int_{-\infty}^{\infty} d\omega = i\int_{-\infty}^{\infty} d\omega_E, \quad \omega = i\omega_E$$

Thus obtain

$$\frac{\partial}{\partial\sigma}\mathrm{Tr}\log\mathcal{D}[\sigma] = \frac{iL^3T}{\pi^3}\int_{-\infty}^{\infty} d\omega_E \int_0^\infty k^2 dk\, \frac{\sigma}{\omega_E^2 + (k^2+M^2)^2k^2 + \sigma^2}$$

where we have also performed the angular integration over $\hat{\mathbf{k}}$

with $M \neq 0$ we could carry the integration through, obtaining a rather complicated incomplete elliptic integral, which would clutter the discussion and would require going beyond the limited time frame of this talk. In order to demonstrate asymptotic freedom it is enough to set $M = 0$ in which case we obtain

$$\frac{\partial}{\partial\sigma}\mathrm{Tr}\log\mathcal{D}[\sigma] = \frac{iL^3T}{\pi^3}\int_{-\infty}^{\infty} d\omega_E \int_0^\infty k^2 dk\, \frac{\sigma}{\omega_E^2 + (k^2)^3 + \sigma^2} = \frac{iL^3T}{3\pi^3}\int_{-\infty}^{\infty} d\omega_E \int_0^\infty du\, \frac{\sigma}{\omega_E^2 + u^2 + \sigma^2}$$

$$= \frac{iL^3T}{6\pi^3}\int_{-\infty}^{\infty} d\omega_E \int_{-\infty}^{\infty} du\, \frac{\sigma}{\omega_E^2 + u^2 + \sigma^2}$$

where we have introduced $u = k^3$ and extended integration over *u* symmetrically throughout the entire real axis. The last integral can be done immediately, by transforming to polar coordinates in the $\omega_E - u$ plane

It diverges ***logarithmically,*** so we introduce a UV cutoff Λ (recall $[\sigma^2] = [\Lambda^6] = \mathrm{mass}^6$)

We thus obtain $$\frac{\partial}{\partial\sigma}\mathrm{Tr}\log\mathcal{D}[\sigma] = \frac{iL^3T}{6\pi^2}\log\left(\frac{\sigma^2+\Lambda^6}{\sigma^2}\right) \simeq \frac{iL^3T}{6\pi^2}\log\left(\frac{\Lambda^6}{\sigma^2}\right)$$

We can integrate this expression over σ and obtain

$$\mathrm{Tr}\log\mathcal{D}[\sigma] = \frac{iL^3T}{12\pi^2}\left[\sigma^2 - \sigma^2\log\left(\frac{\sigma^2}{\Lambda^6}\right)\right] + \mathrm{const.}$$

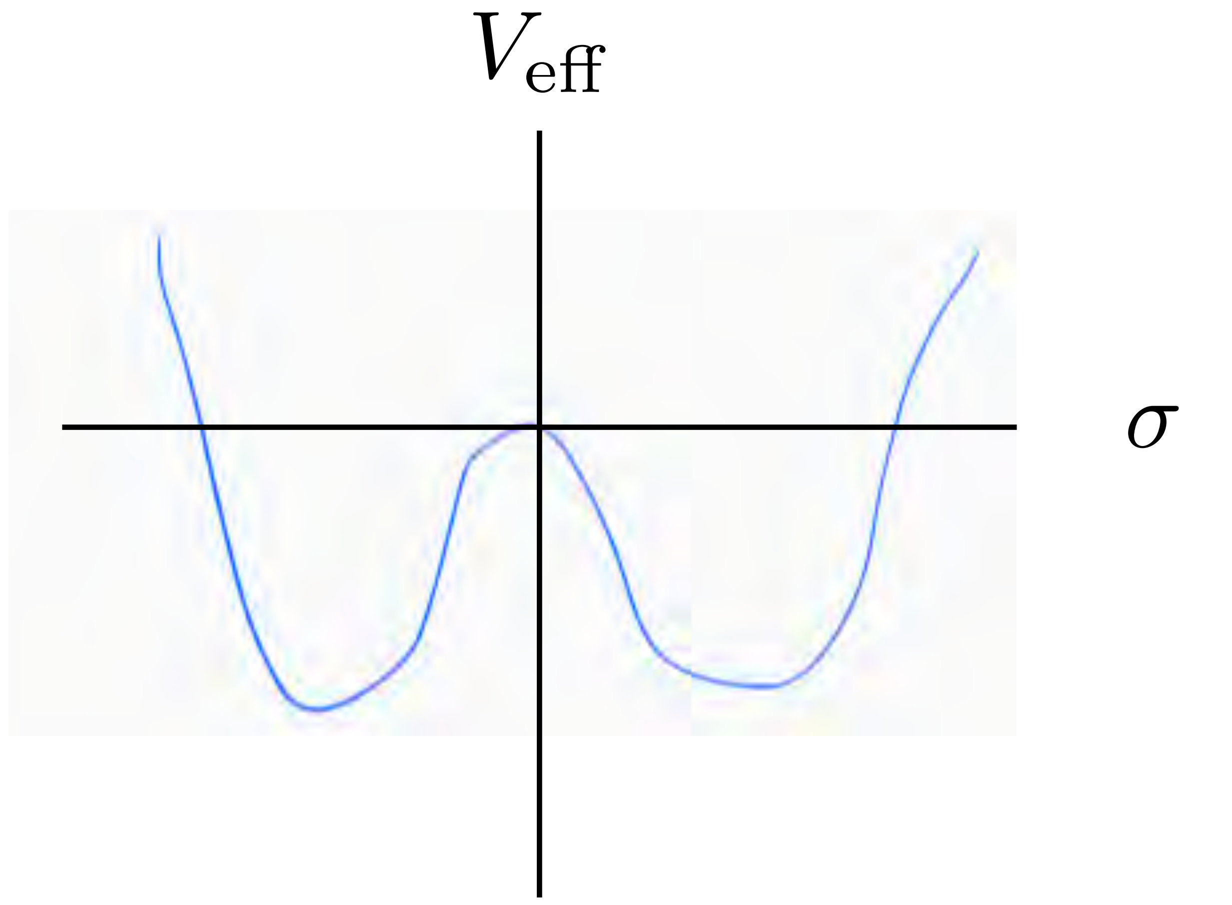
Plugging this back to the effective action we thus obtain

$$S_{\mathrm{eff}}[\sigma] = -L^3T\left[\frac{\sigma^2}{2Ng^2} - \frac{1}{12\pi^2}(\sigma^2 - \sigma^2\log\frac{\sigma^2}{\Lambda^6})\right] \equiv -L^3TV_{\mathrm{eff}}[\sigma]$$

with effective potential

$$V_{\mathrm{eff}}[\sigma] = \frac{\sigma^2}{2Ng^2} - \frac{1}{12\pi^2}(\sigma^2 - \sigma^2\log\frac{\sigma^2}{\Lambda^6})$$

symmetric double-well shape



two degenerate minima $\pm\langle\sigma\rangle = \pm m^3 \neq 0$ $[m] = \mathrm{mass}$

Discrete chiral symmetry is broken dynamically, leading to generation of a fermion mass gap

Asymptotic Freedom

$V_{\text{eff}}[\sigma]$ must be RG invariant: its values at two different energy scales must be the same.

Thus, comparing its values at the scale of the bare cutoff and at some lower energy scale μ

we conclude that $\dfrac{1}{Ng^2(\Lambda)} - \dfrac{1}{Ng^2(\mu)} = \dfrac{1}{6\pi^2}\log\dfrac{\Lambda^6}{\mu^6} = \dfrac{1}{\pi^2}\log\dfrac{\Lambda}{\mu}$ which is asymptotic freedom!

Under the change of the cutoff we find $\Lambda\dfrac{\partial g(\Lambda)}{\partial\Lambda} = -\dfrac{N}{2\pi^2}g^3(\Lambda) = \beta(g) < 0$

The gap equation determines the RG invariant dynamically generated mass *m*

$$\frac{\partial V_{\text{eff}}}{\partial\sigma} = \frac{\sigma}{Ng^2} + \frac{\sigma}{3\pi^2}\log\frac{\sigma}{\Lambda^3} = 0 \Rightarrow \langle\sigma\rangle = m^3 = \Lambda^3 e^{-\frac{3\pi^2}{Ng^2(\Lambda)}}$$ = RG invariant

Emergent Lorentz invariance at low energies

it can be shown that the dispersion relation of fermion fluctuations around the ground state σ condensate when $M \neq 0$ is $\omega^2 - k^2(k^2 + M^2)^2 - m^6 = 0$

thus, in the momentum range $k \ll M$ we obtain $\omega^2 - M^4(k^2 + m_*^2) = 0, \quad m_* = \frac{m^3}{M^2}$

which after proper rescaling of energy and momentum becomes the usual Lorentz invariant dispersion relation.

Domain Wall kinks

The symmetric double well shape of the effective potential, like in the 1+1 dimensional case, indicates the existence of topological solitons: domain wall kinks, which interpolate from one vacuum on one side of the universe, to the other vacuum on the other side, as one travels along one of the coordinate axes.

This is work in progress…

Thanks for your attention!