Planar Dirac equation with radial contact potentials

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August 28, 2025

Financiado por la Unión Europea NextGenerationEU
Plan de Recuperación, Transformación y Resiliencia

Dirac Materials

Whose lattice structure allows us to treat electrons as Dirac particles

$$(i\gamma^{\mu}\partial_{\mu} - m)\,\psi\,(x) = \Omega[\psi]$$

- Effective mass
- Confining Properties
- Graphene
- Qbits

General 2D Dirac Equation:

Consider a $2D$ Dirac equation in polar coordinates:

$$\left(\gamma^0 E + i\gamma^r \partial_r + \frac{i}{r}\gamma^\theta \partial_\theta - m\right)\psi(r,\theta) = \mathcal{I}(r,\theta)\psi(r,\theta)$$

Where the general structure of a central potential is:

$$\mathcal{I}(r,\theta) = \left[B(r)\mathbb{1} + A_0(r)\gamma^0 + A_r(r)\gamma^r + A_\theta(r)\gamma^\theta\right]$$

$$\psi(r,\theta) = \beta(\theta)\Phi(r); \qquad \beta(\theta) = \begin{pmatrix} e^{il\theta} & 0 \\ 0 & e^{i(l+1)\theta} \end{pmatrix} \quad \Phi(r) = \begin{pmatrix} \phi_1(r) \\ i\phi_2(r) \end{pmatrix}$$

Distributional Approach

Effective Radial Problem:

$$\left[\gamma^0 E + i\gamma^1\partial_r + \frac{i}{2r}\gamma^1 - \frac{1}{r}\left(l + \frac{1}{2}\right)\gamma^2 - m\right]\Phi(r) = D[\Phi](r)$$

Distributional approach considerations:

- Point interaction:

  $$D[\Phi] = \left(B\mathbb{1} + A_0\gamma^0 + A_r\gamma^1 + A_\theta\gamma^2\right)\frac{\Phi(R^-)+\Phi(R^+)}{2}\delta(r-R)$$

- Distributional state structure:

  $$\Phi(r) = \Phi_i(r)H(R-r) + \Phi_o(r)H(r-R)$$

- Current conservation $j^r(R^-) = j^r(R^+)$

Bound State Equations

$$
\begin{aligned}
-\frac{d\phi_{1,\alpha}}{dr} + \frac{l}{r}\,\phi_{1,\alpha} &= (E+m)\,\phi_{2,\alpha}, \\
\frac{d\phi_{2,\alpha}}{dr} + \frac{l+1}{r}\,\phi_{2,\alpha} &= (E-m)\,\phi_{1,\alpha},
\end{aligned}
\qquad \alpha = i, o
$$

Inside Region

- $0 \leq r < R$

$$\Phi_i(r) \propto \begin{pmatrix} \sqrt{m+E}\, I_l(qr) \\ -i\sqrt{m-E}\, I_{l+1}(qr) \end{pmatrix}$$

Outside Region

- $R < r < \infty$

$$\Phi_o(r) \propto \begin{pmatrix} \sqrt{m+E}\, K_l(qR) \\ i\sqrt{m-E}\, K_{l+1}(qR) \end{pmatrix}$$

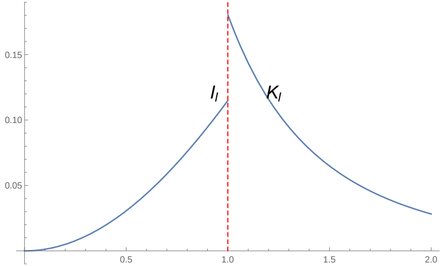
Current conservation

$$\Phi_o\left(R^+\right) = \Lambda \Phi_i\left(R^-\right), \quad \Lambda = e^{i\varphi} \begin{pmatrix} a & ib \\ -ic & d \end{pmatrix}, \quad ad - bc = 1$$

Localized states

Bound states properties

- Highly Localized at R
- Qbit candidate
- $q = \sqrt{m^2 - E^2}$
- $\sqrt{m - E}\, I_l(qr)$
- Allowed energy values



Secular Equation:

$$\Phi_o\left(R^+\right) = \Lambda \Phi_i\left(R^-\right), \quad \Lambda = e^{i\varphi}\begin{pmatrix} a & ib \\ -ic & d \end{pmatrix}, \quad ad - bc = 1$$

Next Steps

Hidden Potential dependence $\Lambda = \Lambda(B, A_0, A_r, A_\theta)$

$$\Phi_o\left(R^+\right) = \Lambda \Phi_i\left(R^-\right), \quad \Lambda = e^{i\varphi}\begin{pmatrix} a & ib \\ -ic & d \end{pmatrix}, \quad ad - bc = 1$$

- Bound states

  Real energy values $-m < E < m$
- Scattering states

  Real energy values $|E| > m$
  Unbounded states with associated phase shift
  $J_\mu(p\,r) - (\tan\delta_\mu)\; Y_\mu(p\,r)$
- Resonance values

  Complex energy values $|Re(E)| > m$

δ Potential

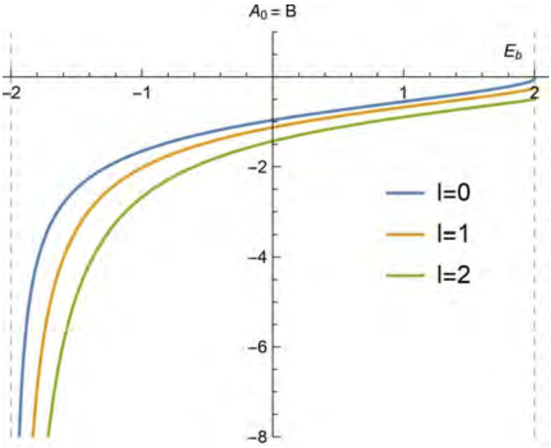
$A_0 = B; \;\; A_r = A_\theta = 0$

Scalar and Electrostatic equal mixing

$$\Lambda_\delta = \begin{pmatrix} 1 & 0 \\ -2iA_0 & 1 \end{pmatrix}$$

$A_0 \in (-\infty, \frac{-l}{2mR}]$

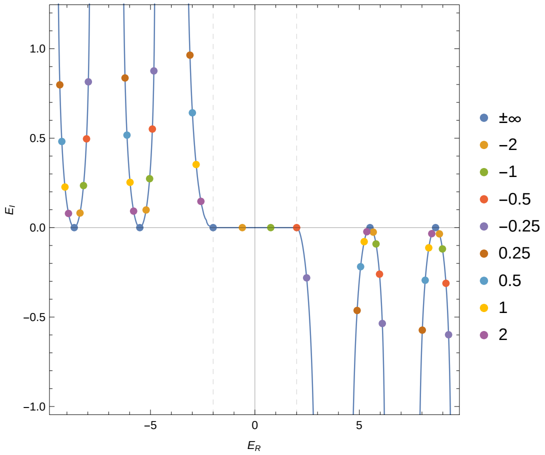
Single state



$$A_0 = \frac{-1}{2R(E+m)I_l\,(qR)\,K_l\,(qR)}$$

δ Case Resonances

- $\frac{1}{A_0}$ Dependence
- Impermeable barrier $A_0 = \pm\infty$
- Connection with δ' system
- Energy in terms of A_0



$$A_0 = \frac{-1}{2R(E+m)I_l\,(qR)\,K_l\,(qR)}$$

δ' Potential

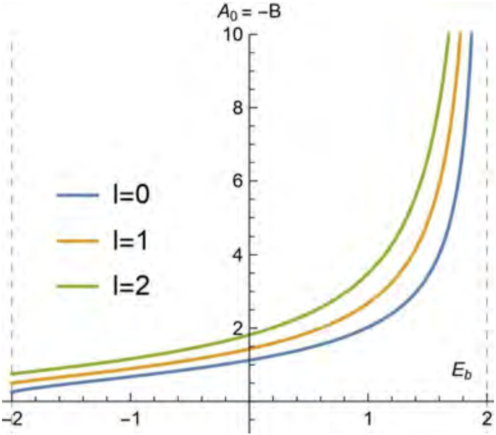
$A_0 = -B; \;\; A_r = A_\theta = 0$

Scalar and Electrostatic inverse mixing

$$\Lambda_{\delta'} = \begin{pmatrix} 1 & -2iA_0 \\ 0 & 1 \end{pmatrix}$$

$A_0 \in [-\frac{l+1}{2mR}, \infty)$

Single state



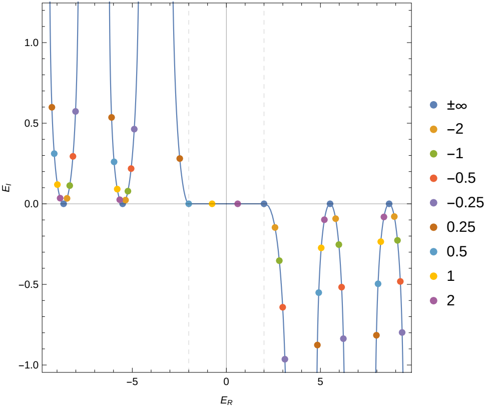
$$A_0 = \frac{1}{2R(m-E) I_{l+1}(qR)\, K_{l+1}(qR)}$$

δ' Resonances

$\frac{1}{A_0}$ Dependence

Connection with δ

A_0, E, J Inversion



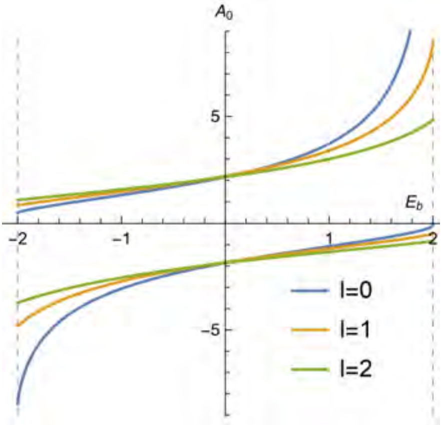
Pure Electrostatic Potential

$$B = A_r = A_\theta = 0$$

$$\Lambda_{A_0} = \begin{pmatrix} \cos\beta & i\sin\beta \\ i\sin\beta & \cos\beta \end{pmatrix}$$

$$\cos\beta = \frac{A_0^2-4}{A_0^2+4} \qquad \sin\beta = \frac{4A_0}{A_0^2+4}$$

$$\tan\beta = \frac{l}{mR} = -\frac{l+1}{mR}$$



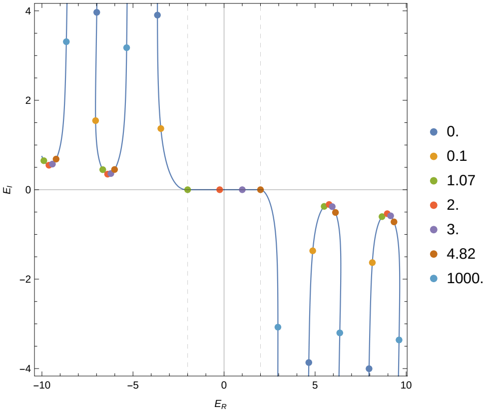
$$\frac{A_0^2-4}{4A_0} = R(E-m)I_{l+1}(qR)\,K_{l+1}(qR) + R(E+m)I_l(qR)\,K_l(qR)$$

Pure Electrostatic Resonances

$\frac{A_0^2-4}{4A_0}$ Dependence

Hidden symmetry
$A_0 \to \frac{-4}{A_0}$

Always permeable barrier



Purely Magnetic Potential

$$A_0 = B = 0$$

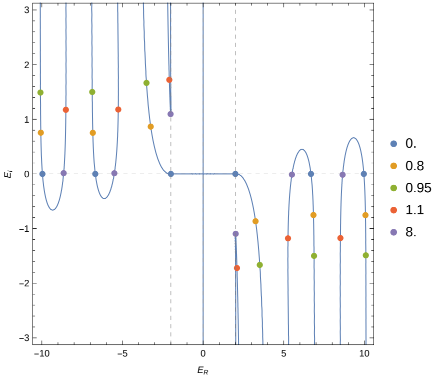
$$\Lambda = \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix}$$

Permeable barrier

$A_r = 0 \;\; A_\theta = \pm 2$

$$a(A_r, A_\theta) = a(0, A_{\theta'})$$

No bound states



$$a^2 = \frac{A_r^2 + (A_\theta + 2)^2}{A_r^2 + (A_\theta - 2)^2} = -\frac{I_{l+1}(qR)\; K_l(qR)}{I_l(qR)\; K_{l+1}(qR)}$$

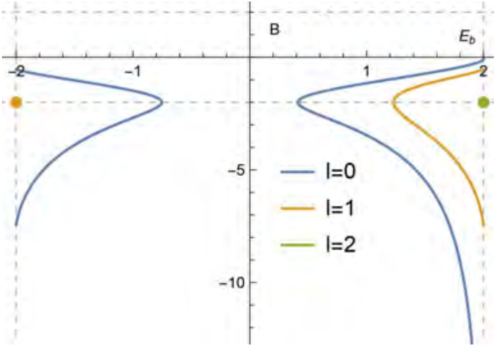
Scalar Potential

$$A_0 = A_r = A_\theta = 0$$

$$\Lambda_B = \begin{pmatrix} \cosh\beta & i\sinh\beta \\ -i\sinh\beta & \cosh\beta \end{pmatrix}$$

$$\tan\beta = \frac{-4B}{4+B^2} = \frac{|l|}{mR} = \frac{l+1}{mR}$$

Up to 2 bound states



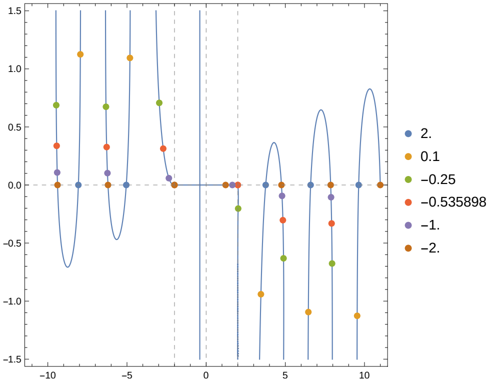
$$\frac{4+B^2}{4B} = R(E-m)I_{l+1}(qR)\,K_{l+1}(qR) - R(E+m)I_l(qR)\,K_l(qR)$$

Scalar Resonances

$\frac{4+B^2}{4B}$ Dependence

Hidden symmetry
$B \to \frac{4}{B}$

Impermeable barrier for
$B = \pm 2$



Conclusions

The following table summarizes the confining properties of the problem:

Potential	State Capture	Requirement
Pure Magnetic		
δ shell $A_0 = B$	✓	$A_0 \leq 0$
δ' shell $A_0 = -B$	✓	$A_0 \geq 0$
Pure Electrostatic	✓	$\frac{A_0^2-4}{4A_0} \geq 0$
Pure Scalar	✓	$\frac{B^2+4}{B} \geq 0$

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Acknowledgments

- This research was supported by Spanish MCIN with funding from European Union NextGenerationEU (PRTRC17.I1) and Consejeria de Educacion from JCyL through the QCAYLE project.

- Financial support of the Department of Education, Junta de Castilla y León, and FEDER Funds is gratefully acknowledged (Reference: CLU-2023-1-05)