Convergence of first order operators on thick graphs (joint work with Pavel Exner)

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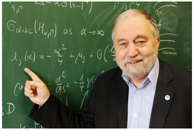

Metric graphs and natural first order operators

- Metric star graph $X_\infty = \dot{\bigcup}_{e \in E} X_{\infty,e} \subset \mathbb{R}^2$ with $E = \{1, \dots, r\}$ edges,
- $\bigcap_{e \in E} X_{\infty,e} = \{0\}$ central vertex, $X_{\infty,e} \cong [0,\infty[$
- function $(\tilde{\cdot})$ and vector field $(\vec{\cdot})$ ($\circ$-form)
  $\mathring{f} = (\mathring{f}_e)_{e \in E} \in \mathring{\mathscr{H}}_\infty := \mathsf{L}_2(X_\infty) := \bigoplus_{e \in E} \mathsf{L}_2(X_{\infty,e})$,
- gradient (exterior derivative) $d_\infty \colon \mathsf{H}^1(X_\varepsilon) \longrightarrow \mathsf{L}_2(X_\infty)$, $d_\infty \tilde{f} = \tilde{f}'$
$$\mathsf{H}^1(X_\infty) = \Big\{ \tilde{f} \in \bigoplus_{e \in E} \mathsf{H}^1(X_{\infty,e}) \Bigm| \tilde{f}_e(0) =: \tilde{f}(0) \text{ indep. of } e \Big\}$$
- adjoint (divergence) $d_\infty^* \colon \operatorname{dom} d_\infty^* \longrightarrow \mathsf{L}_2(X_\infty)$, $d_\infty^* \vec{f} = -\vec{f}'$, Kirchhoff flux condition for vector fields:
$$\operatorname{dom} d_\infty^* = \Big\{ \vec{f} \in \bigoplus_{e \in E} \mathsf{H}^1(X_{\infty,e}) \Bigm| \sum_{e \in E} \vec{f}_e(0) = 0 \Big\}$$
- first order ("Dirac") operator
$$D_\infty := \begin{pmatrix} 0 & d_\infty^* \\ d_\infty & 0 \end{pmatrix}, \quad \text{i.e.,} \quad D_\infty \begin{pmatrix} \tilde{f} \\ \vec{f} \end{pmatrix} = \begin{pmatrix} d_\infty^* \vec{f} \\ d_\infty \tilde{f} \end{pmatrix}$$

Metric graphs and natural first order operators II

- gradient (exterior derivative) $d_\infty \tilde{f} = \tilde{f}'$, $\quad \tilde{f}$ continuous
- minus divergence $d_\infty^* \vec{f} = -\vec{f}'$, $\quad \sum_{e \in E} \vec{f}_e'(0) = 0$
- first order ("Dirac") operator (self-adjoint)

$$D_\infty := \begin{pmatrix} 0 & d_\infty^* \\ d_\infty & 0 \end{pmatrix}, \quad \text{i.e.,} \quad D_\infty \begin{pmatrix} \tilde{f} \\ \vec{f} \end{pmatrix} = \begin{pmatrix} d_\infty^* \vec{f} \\ d_\infty \tilde{f} \end{pmatrix}$$

  for $f = (\tilde{f}, \vec{f}) \in \operatorname{dom} D_\infty \subset \mathscr{H}_\infty = \tilde{\mathscr{H}}_\infty \oplus \vec{\mathscr{H}}_\infty$.
-
$$D_\infty^2 = \begin{pmatrix} d_\infty^* d_\infty & 0 \\ 0 & d_\infty d_\infty^* \end{pmatrix},$$

  function component $d_\infty^* d_\infty \tilde{f} = -\tilde{f}''$ for $\tilde{f}$ continous, $\sum_e \tilde{f}_e'(0) = 0$, Kirchhoff or standard metric graph Laplacian.
- justfies name "Dirac" (square is Laplacian)

Thick graphs and natural first order operators

- Thick star graph $X_n = \bigcup_{e \in E} X_{n,e} \cup X_{n,0}$ (smooth boundary)
- $X_{n,e} \cong [0,\infty[\times Y_n,\ Y_n = [0,\varepsilon_n],\ \varepsilon_n \to 0$
- $x = (s,y) \in X_{n,e}, \begin{cases} s \in X_{\infty,e} & \text{longitudinal,} \\ y \in Y_n & \text{cross-sectional/transversal} \end{cases}$ direction
- vertex neighbourhood $X_{n,0} \cong \varepsilon_n X_0$ ε_n-homothetic
- functions $\tilde u = (\tilde u_e)_{e \in E \cup \{0\}} \in \tilde{\mathscr{H}}_n = \bigoplus_{e \in E \cup \{0\}} \mathsf{L}_2(X_{n,e}),\ \tilde u_e = \tilde u{\restriction}_{X_{n,e}}$
- gradient (exterior derivative) $d_n \colon \mathsf{H}^1(X_n) \longrightarrow \mathsf{L}_2(X_n, \mathbb{C}^2),\ d_n \tilde u = \nabla \tilde u$
- restrict to irrotational vector fields (closed 1-forms) (no equivalent in one dimension)

$$\vec{\mathscr{H}}_n := \{\, \vec u \in \mathsf{L}_2(X_n, \mathbb{C}^2) \,|\, \operatorname{rot} \vec u = \partial_2 \vec u_1 - \partial_1 \vec u_2 = 0 \,\}$$

- adjoint (minus divergence) $d_n^* \colon \operatorname{dom} d_n^* \longrightarrow \mathsf{L}_2(X_n), \qquad d_n^* \vec u = -\operatorname{div} \vec u,$

$$\operatorname{dom} d_n^* = \Big\{ \vec u \in \vec{\mathscr{H}}_n \,\Big|\, \operatorname{div} \vec u \in \mathsf{L}_2(X_n),\ \vec u \cdot \vec n{\restriction}_{\partial X_n} = 0 \Big\} \quad \text{tang. vct. fields}$$

Thick graphs and natural first order operators II

- first order ("Dirac") operator (self-adjoint)

$$D_n := \begin{pmatrix} 0 & d_n^* \\ d_n & 0 \end{pmatrix}, \quad \text{i.e.,} \quad D_n \begin{pmatrix} \tilde{u} \\ \vec{u} \end{pmatrix} = \begin{pmatrix} d_n^* \vec{u} \\ d_n \tilde{u} \end{pmatrix}$$

  for $u = (\tilde{u}, \vec{u}) \in \operatorname{dom} D_n \subset \mathscr{H}_n = \tilde{\mathscr{H}}_n \oplus \vec{\mathscr{H}}_n$.
- Decompose according to graph (functions and vector fields)

$$\mathscr{H}_n = \bigoplus_{e \in E \cup \{0\}} \mathscr{H}_{n,e}, \qquad \mathscr{H}_{n,e} = \tilde{\mathscr{H}}_{n,e} \oplus \vec{\mathscr{H}}_{n,e}$$

- again

$$D_n^2 = \begin{pmatrix} d_n^* d_n & 0 \\ 0 & d_n d_n^* \end{pmatrix},$$

  function comp. $d_n^* d_n \tilde{u} = \nabla^* \nabla \tilde{u}$ with Neumann boundary cond.
- justfies again name "Dirac" (square is Laplacian)

Some norm equalities for Dirac-type operators

- Recall

$$D_n := \begin{pmatrix} 0 & d_n^* \\ d_n & 0 \end{pmatrix}, \quad \text{i.e.,} \quad D_n \begin{pmatrix} \tilde{u} \\ \vec{u} \end{pmatrix} = \begin{pmatrix} d_n^* \vec{u} \\ d_n \tilde{u} \end{pmatrix}$$

for $u = (\tilde{u}, \vec{u}) \in \operatorname{dom} D_n \subset \mathscr{H}_n = \tilde{\mathscr{H}}_n \oplus \vec{\mathscr{H}}_n$

- we have

$$\begin{aligned} \|D_n u\|^2 &= \|d_n \tilde{u}\|^2 + \|d_n^* \vec{u}\|^2 \\ \|(D_n \mp \mathrm{i})u\|^2 &= \|D_n u\|^2 + \|u\|^2 = \|d_n \tilde{u}\|^2 + \|d_n^* \vec{u}\|^2 + \|\tilde{u}\|^2 + \|\vec{u}\|^2 \end{aligned}$$

- later we need resolvents $R_n^{\pm} := (D_n \mp \mathrm{i})^{-1}$ for $n \in \overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$

Generalised norm resolvent convergence and quasi-unitary equivalence

How to define $R_n^\pm = (D_n \mp \mathrm{i})^{-1} \to R_\infty^\pm = (D_\infty \mp \mathrm{i})^{-1}$?

- Need identification operators $J_n \colon \mathscr{H}_n \longrightarrow \mathscr{H}_\infty$ and $\delta_n \to 0$ such that

$$\|J_n R_n - R_\infty J_n\| \le \delta_n \qquad \text{(a)}$$

$$\|(\mathrm{id}_{\mathscr{H}_n} - J_n^* J_n) R_n\| \le \delta_n \quad \text{(b)} \qquad \|(\mathrm{id}_{\mathscr{H}_\infty} - J_n J_n^*) R_\infty\| \le \delta_n \quad \text{(b')}$$

  We then say that $D_n \to D_\infty$ in generalised norm resolvent sense

- if $J_n = \tilde{J}_n \oplus \vec{J}_n$ respects functions and vector fields,

$$\begin{aligned}
\text{(b)} \iff \|u - J_n^* J_n u\|^2 &= \|\tilde{u} - \tilde{J}_n^* \tilde{J}_n \tilde{u}\|^2 + \|\vec{u} - \vec{J}_n^* \vec{J}_n \vec{u}\|^2 \\
&\le \delta_n^2 \|(D_n \mp \mathrm{i}) u\|^2 \\
&= \delta_n^2 \|d_n \tilde{u}\|^2 + \|d_n^* \vec{u}\|^2 + \|\tilde{u}\|^2 + \|\vec{u}\|^2
\end{aligned}$$

  for all $u = (\tilde{u}, \vec{u}) \in \operatorname{dom} D_n$

The identification operators on functions

- Identification operators on functions

$$(\tilde{J}_n^* \tilde{f})_e := \tilde{f}_e \otimes \mathbb{1}_n \, \mathrm{d}s, \qquad (\tilde{J}_n^* \tilde{f})_0 := 0.$$

- ajoint is

$$\langle \tilde{J}_n^* \tilde{f}, \tilde{u} \rangle_{\mathsf{L}_2(X_n)} = \sum_{e=1}^{r} \int_{X_{\infty,e}} \tilde{f}_e(s) \langle \mathbb{1}_n, \tilde{u}_e(s,\cdot) \rangle_{\mathsf{L}_2(Y_n)} \, \mathrm{d}s = \langle \tilde{f}, \tilde{J}_n \tilde{u} \rangle_{\mathsf{L}_2(X_\infty)},$$

so that

$$(\tilde{J}_n \tilde{u})_e(s) = \langle \tilde{u}_e(s,\cdot), \mathbb{1}_n \rangle_{\mathsf{L}_2(Y_n)} = \varepsilon_n^{-1/2} \int_{Y_n} \tilde{u}_e(s,y) \mathrm{d}y$$

is a scaled cross-sectional average of $\tilde{u}_e$ at s

- we have $\tilde{J}_n \tilde{J}_n^* \tilde{f} = \tilde{f}$

The identification operators on functions

- We need that $\mathrm{id}_{\mathscr{H}_n} - \tilde{J}_n^* \tilde{J}_n$ is small in some (weighted) operator norm

$$\|\tilde{u} - \tilde{J}_n^* \tilde{J}_n^0 \tilde{u}\|^2_{\mathsf{L}_2(X_n)} = \sum_{e=1}^{r} \int_{X_{\infty,e}} \big\|\tilde{u}_e(s,\cdot) - \langle \tilde{u}_e(s,\cdot), \mathbb{1}_n\rangle \mathbb{1}_n\big\|^2_{\mathsf{L}_2(Y_n)} \,\mathrm{d}s + \|\tilde{u}_0\|^2_{\mathsf{L}_2(X_{n,0})}$$

  where u_0 is the contribution of u on the central branching region
- first terms can be estimated as follows (Poincaré type estimate)

$$\sum_{e=1}^{r} \frac{1}{\lambda_2(Y_n)} \int_{X_{\infty,e}} \|\partial_2 u_e(s,\cdot)\|^2_{\mathsf{L}_2(Y_n)} \,\mathrm{d}s \le \frac{\varepsilon_n^2}{\pi^2} \sum_{e=1}^{r} \|\nabla u_e\|^2_{\mathsf{L}_2(X_{n,e},\mathbb{C}^2)},$$

  $\lambda_2(Y_n) = \pi^2/\varepsilon_n^2$ second (first non-zero) Neumann ev of $Y_n = [0, \varepsilon_n]$
- For last term (as for Laplacians in [Pos12])
  $\|\tilde{u}_0\|^2_{\mathsf{L}_2(X_{n,0})} \le \mathscr{O}(\varepsilon^2)\|\nabla \tilde{u}\|^2_{\mathsf{L}_2(X_n,\mathbb{C}^2)} + \mathscr{O}(\varepsilon)\|\tilde{u}\|^2_{\mathsf{L}_2(X_n)}$

The identification operators on vector fields

- Identification operator on vector fields

$$(\vec{J}_n^* \vec{f})_e := \vec{f}_e \otimes \mathbb{1}_n \, \mathrm{d}s, \qquad (\vec{J}_n^* \vec{f})_0 := 0.$$

- As for functions, its ajoint is (scaled cross-sect. average)

$$(\vec{J}_n \vec{u})_e(s) = \langle \vec{u}_{e,1}(s,\cdot), \mathbb{1}_n \rangle_{\mathsf{L}_2(Y_n)} = \varepsilon_n^{-1/2} \int_{Y_n} \vec{u}_{e,1}(s,y) \mathrm{d}y$$

- Again, $\vec{J}_n \vec{J}_n^* \vec{f} = \vec{f}$. Moreover

$$\|\vec{u} - \vec{J}_n^* \vec{J}_n \vec{u}\|^2_{\mathsf{L}_2(X_n)} = \sum_{e=1}^{r} \int_{X_{\infty,e}} \Big(\big\|\vec{u}_{e,1}(s,\cdot) - \langle \vec{u}_{e,1}(s,\cdot), \mathbb{1}_n \rangle \mathbb{1}_n \big\|^2_{\mathsf{L}_2(Y_n)} + \big\|\vec{u}_{e,2}(s,\cdot)\big\|^2_{\mathsf{L}_2(Y_n)} \Big) \mathrm{d}s + \|\vec{u}_0\|^2_{\mathsf{L}_2(X_{n,0},\mathbb{C}^2)},$$

where the first two terms can be estimated as before by $(\vec{u}_{e,2}(s,\cdot)\restriction_{Y_n} = 0)$

$$\frac{1}{\lambda_2(Y_n)} \sum_{e=1}^{r} \int_{X_{\infty,e}} \|\partial_2 \vec{u}_{e,1}(s,\cdot)\|^2_{\mathsf{L}_2(Y_n)} \mathrm{d}s + \frac{1}{\lambda_1^{\mathrm{D}}(Y_n)} \sum_{e=1}^{r} \int_{X_{\infty,e}} \|\partial_2 \vec{u}_{e,2}(s,\cdot)\|^2_{\mathsf{L}_2(Y_n)}$$

Gaffney estimates

- For $\|\vec{u}_0\|^2_{\mathsf{L}_2(X_{n,0},\mathbb{C}^2)}$, need more ideas: recycle scalar estimates! Kato's inequality stating roughly that if a scalar estimate

$$\|\tilde{u}_0\|^2_{\mathsf{L}_2(X_{n,0})} \le \mathcal{O}(\varepsilon^2)\|\nabla \tilde{u}\|^2_{\mathsf{L}_2(X_n,\mathbb{C}^2)} + \mathcal{O}(\varepsilon)\|\tilde{u}\|^2_{\mathsf{L}_2(X_n)}$$

holds then it holds (with same constants) also for vector fields

$$\|\vec{u}_0\|^2_{\mathsf{L}_2(X_{n,0},\mathbb{C}^2)} \le \mathcal{O}(\varepsilon^2)\|\nabla \vec{u}\|^2_{\mathsf{L}_2(X_n,\mathbb{C}^{2\times 2})} + \mathcal{O}(\varepsilon)\|\vec{u}\|^2_{\mathsf{L}_2(X_n,\mathbb{C}^2)}.$$

- estimate $\|\nabla \vec{u}\|^2_{\mathsf{L}_2(X_n,\mathbb{C}^{2\times 2})}$ in terms of $\|\operatorname{div} \vec{u}\|^2_{\mathsf{L}_2(X_n)} + \|\vec{u}\|^2_{\mathsf{L}_2(X_n,\mathbb{C}^2)}$
- Key ingredient: $X \subset \mathbb{R}^d$, $\partial X \in C^2$, tan. vector fields ($\vec{u} \cdot \vec{n}\restriction_{\partial X} = 0$)

$$\|\operatorname{div} \vec{u}\|^2_{\mathsf{L}_2(X_n)} + \|\operatorname{rot} \vec{u}\|^2_{\mathsf{L}_2(X_n,\mathbb{C}^{\binom{d}{2}})} = \|\nabla \vec{u}\|^2_{\mathsf{L}_2(X_n,\mathbb{C}^{d\times d})} + \int_{\partial X} \mathrm{II}_{\partial X}(\vec{u},\vec{u})\,\mathrm{d}\sigma$$

Gaffney estimates

- For $\|\vec{u}_0\|^2_{\mathsf{L}_2(X_{n,0},\mathbb{C}^2)}$, need more ideas: recycle scalar estimates! Kato's inequality stating roughly that if a scalar estimate

$$\|\tilde{u}_0\|^2_{\mathsf{L}_2(X_{n,0})} \le \mathscr{O}(\varepsilon^2)\|\nabla\tilde{u}\|^2_{\mathsf{L}_2(X_n,\mathbb{C}^2)} + \mathscr{O}(\varepsilon)\|\tilde{u}\|^2_{\mathsf{L}_2(X_n)}$$

holds then it holds (with same constants) also for vector fields

$$\|\vec{u}_0\|^2_{\mathsf{L}_2(X_{n,0},\mathbb{C}^2)} \le \mathscr{O}(\varepsilon^2)\|\nabla\vec{u}\|^2_{\mathsf{L}_2(X_n,\mathbb{C}^{2\times 2})} + \mathscr{O}(\varepsilon)\|\vec{u}\|^2_{\mathsf{L}_2(X_n,\mathbb{C}^2)}.$$

- estimate $\|\nabla\vec{u}\|^2_{\mathsf{L}_2(X_n,\mathbb{C}^{2\times 2})}$ in terms of $\|\operatorname{div}\vec{u}\|^2_{\mathsf{L}_2(X_n)} + \|\vec{u}\|^2_{\mathsf{L}_2(X_n,\mathbb{C}^2)}$
- $d = 2$, $\operatorname{rot}\vec{u} = 0$, $\vec{u}\cdot\vec{n}\restriction_{\partial X} = 0$, $X_n = \varepsilon_n X_1 \subset \mathbb{R}^2$ $(\varepsilon_1 = 1)$, scale invariant! $\mathrm{II}_{\partial X_n} < 0$ only on $Z_n = \partial X_n \cap X_{n,0}$

$$\|\operatorname{div}\vec{u}\|^2_{\mathsf{L}_2(X_1)} = \|\nabla\vec{u}\|^2_{\mathsf{L}_2(X_1,\mathbb{C}^{2\times 2})} + \int_{\partial X_{1,0}} \mathrm{II}_{\partial X_1,0}(\vec{u},\vec{u})\,\mathrm{d}\sigma$$

$$\ge \|\nabla\vec{u}\|^2_{\mathsf{L}_2(X_1,\mathbb{C}^{2\times 2})} - \frac{1}{2}\Big(\|\nabla\vec{u}\|^2_{\mathsf{L}_2(X_1,\mathbb{C}^{2\times 2})}\Big) - 4\kappa_-^2\|\vec{u}\|^2_{\mathsf{L}_2(X_n)}$$

The sandwiched resolvent difference

For $\|J_n R_n^{\pm} - R_\infty^{\pm} J_n\| \le \delta_n$ use equivalent statement

$$\bigl|\langle J_n^* f, D_n u\rangle_{\mathscr{H}_n} - \langle J_n^* f, D_n u\rangle_{\mathscr{H}_n}\bigr| \le \delta_n \|(D_\infty \mp \mathrm{i}) f\|_{\mathscr{H}_\infty} \|(D_n \mp \mathrm{i}) u\|_{\mathscr{H}_n}$$

$$\begin{aligned}\langle J_n^* f, D_n u\rangle_{\mathscr{H}_n} - \langle J_n^* f, D_n u\rangle_{\mathscr{H}_n} &= \bigl(\langle \tilde{J}_n^* \tilde{f}, d_n^* \vec{u}\rangle_{\mathscr{H}_n} - \langle \vec{J}_n^* d_\infty \tilde{f}, \vec{u}\rangle_{\mathscr{H}_n}\bigr)\\ &\quad + \bigl(\langle \vec{J}_n^* \vec{f}, d_n \tilde{u}\rangle_{\mathscr{H}_n} - \langle \tilde{J}_n^* d_\infty^* \vec{f}, \tilde{u}\rangle_{\mathscr{H}_n}\bigr).\end{aligned}$$

First difference: (second analogue, second is similar to scalar case)

$$\begin{aligned}&\langle \tilde{J}_n^* \tilde{f}, d_n \vec{u}\rangle_{\mathscr{H}_n} - \langle \vec{J}_n^* d_\infty \tilde{f}, \vec{u}\rangle_{\mathscr{H}_n}\\ &= \sum_{e=1}^r \int_{X_{\infty,r}} \Bigl(\tilde{f}_e(s)\langle \mathbb{1}_n, -(\partial_1 \vec{u}_{e,1}(s,\cdot) + \partial_2 \vec{u}_{e,2}(s,\cdot))\rangle_{\mathsf{L}_2(Y_n)} - \tilde{f}_e'(s)\langle \mathbb{1}_n, \vec{u}_{e,1}(s,\cdot)\rangle_{\mathsf{L}_2(Y_n)}\Bigr)\,\mathrm{d}s\\ &= -\sum_{e=1}^r \Bigl(\int_{X_{\infty,r}} \Bigl(\tilde{f}_e(s)\langle \mathbb{1}_n, \partial_1 \vec{u}_{e,1}(s,\cdot)\rangle_{\mathsf{L}_2(Y_n)} - \tilde{f}_e'(s)\langle \mathbb{1}_n, \vec{u}_{e,1}(s,\cdot)\rangle_{\mathsf{L}_2(Y_n)}\Bigr)\,\mathrm{d}s\\ &\qquad - \int_{X_{\infty,e}} \tilde{f}_e(s)\langle \mathbb{1}_n, \partial_2 \vec{u}_{2,e}(s,\cdot))\rangle_{\mathsf{L}_2(Y_n)}\,\mathrm{d}s\Bigr)\end{aligned}$$

The sandwiched resolvent difference II

Treat $\langle \tilde{J}_n^* \tilde{f}, d_n^* \vec{u} \rangle_{\mathscr{H}_n} - \langle \vec{J}_n^* d_\infty \tilde{f}, \vec{u} \rangle_{\mathscr{H}_n}$:

$$\begin{aligned}
&\langle \tilde{J}_n^* \tilde{f}, d_n \vec{u} \rangle_{\mathscr{H}_n} - \langle \vec{J}_n^* d_\infty \tilde{f}, \vec{u} \rangle_{\mathscr{H}_n} \\
&= \sum_{e=1}^{r} \int_{X_{\infty,r}} \Big(\tilde{f}_e(s) \langle \mathbb{1}_n, -(\partial_1 \vec{u}_{e,1}(s,\cdot) + \partial_2 \vec{u}_{e,2}(s,\cdot)) \rangle_{\mathsf{L}_2(Y_n)} - \tilde{f}_e'(s) \langle \mathbb{1}_n, \vec{u}_{e,1}(s,\cdot) \rangle_{\mathsf{L}_2(Y_n)} \Big) \, \mathrm{d}s \\
&= -\sum_{e=1}^{r} \Big(\int_{X_{\infty,r}} \Big(\tilde{f}_e(s) \langle \mathbb{1}_n, \partial_1 \vec{u}_{e,1}(s,\cdot) \rangle_{\mathsf{L}_2(Y_n)} - \tilde{f}_e'(s) \langle \mathbb{1}_n, \vec{u}_{e,1}(s,\cdot) \rangle_{\mathsf{L}_2(Y_n)} \Big) \, \mathrm{d}s \\
&\qquad - \int_{X_{\infty,e}} \tilde{f}_e(s) \underbrace{\langle \mathbb{1}_n, \partial_2 \vec{u}_{2,e}(s,\cdot)) \rangle_{\mathsf{L}_2(Y_n)}}_{=0} \, \mathrm{d}s \Big) \quad \text{(int. by parts)} \\
&= -\sum_{e=1}^{r} \tilde{f}_e(0) \int_{\partial X_{n,e}} \mathbb{1}_n \vec{u}_e^* \cdot \vec{n} \, \mathrm{d}\sigma \quad \text{(f cont, boundary of vertex region!)} \\
&= -\tilde{f}(0) \sum_{e=1}^{r} \int_{\partial X_{n,e}} \mathbb{1}_n \vec{u}_e^* \cdot \vec{n} \, \mathrm{d}\sigma \\
&= \tilde{f}(0) \int_{\partial X_{n,\mathbf{0}}} \mathbb{1}_n \vec{u}_0^* \cdot \vec{n} \, \mathrm{d}\sigma = \tilde{f}(0) \int_{X_{n,\mathbf{0}}} \mathbb{1}_n \operatorname{div} \vec{u}_0^* \, \mathrm{d}x \qquad \text{(divergence theorem)}
\end{aligned}$$

Main results

Theorem ([EP25], arXiv:2502.19904)

$$D_n \to D_\infty \quad \textit{(gen. norm res. conv) of order } \mathscr{O}(\varepsilon_n^{1/2}).$$

The first order operator D_n on functions and irrotational tangential vector fields on a (smoothened) ε_n-neighbourhood converges in strong norm resolvent sense to the first order operator D_∞ on the underlying metric graph of order $\mathscr{O}(\varepsilon_n^{1/2})$

Corollary ([EP25])

We also have $D_n^2 \to D_\infty^2$ (first component is Neumann Laplacian/standard (Kirchhoff) Laplacian)

Corollary ([EP25],[PZ24])

*We have $\mathrm{d}_{\mathrm{Hausd}}\big((\sigma(D_n)-\mathrm{i})^{-1},(\sigma(D_\infty)-\mathrm{i})^{-1}\big) \le \mathscr{O}(\varepsilon_n^{1/2})$;
Suitable (sandwiched) operator functions (heat operator, spectral projections) converge as well in norm.*

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Thank you for your attenion!