Discrete complex scale invariance in few-body problems

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Introduction

Introduction

- Today I will talk about breakdown of continuous scale invariance.
- Here continuous scale invariance refers to the invariance under a transformation like

$$x \mapsto \mathrm{e}^{t} x \tag{1}$$

where t is an arbitrary constant.

- In quantum mechanics, this invariance can be broken in several ways.
- A well-known breaking pattern is the breakdown of continuous scale invariance to discrete scale invariance:

$$x \mapsto \mathrm{e}^{t} x, \quad \text{where} \quad t \in t_{*}\mathbb{Z} = \{0, \pm t_{*}, \pm 2t_{*}, \cdots\} \tag{2}$$

Here t_{*} is a model-dependent parameter. In this case, one can show that S-matrix possesses a geometric sequence of bound-state poles at

$$E_n = E_0\,\mathrm{e}^{2nt_{*}} \tag{3}$$

- A typical realization of this infinite bound-state tower is the Efimov effect of three particles in three dimensions [Efimov '70].

Introduction

- Today I will discuss yet another breaking pattern of continuous scale invariance: discrete complex scale invariance (discrete phase invariance):

$$x \mapsto \mathrm{e}^{t} x, \quad \text{where} \quad t \in it_*\mathbb{Z} = \{0, \pm it_*, \pm i2t_*, \cdots\} \tag{4}$$

  In this case, one can show that S-matrix possesses circularly distributed simple poles at

$$E_n = E_0 \, \mathrm{e}^{i2nt_*} \tag{5}$$

- In the rest of the talk, I will introduce several few-body models that enjoy this invariance.

Plan of the talk

Introduction

Discrete complex scale invariance: A toy example

Few-body examples

Summary

Discrete complex scale invariance: A toy example

Discrete complex scale invariance: A toy example

- A typical example of scale-invariant models in quantum mechanics is the inverse-square-potential problem on the half line.
- Consider the Hamiltonian
$$H = -\frac{d^2}{dr^2} + \frac{\lambda}{r^2}, \quad \lambda \in \mathbb{R} \tag{6}$$
- Under the scale transformation $r \mapsto \mathrm{e}^t r$, this Hamiltonian transforms as
$$H \mapsto \mathrm{e}^{-2t} H \tag{7}$$
- Let ψ_E be an energy eigenfunction satisfying
$$H\psi_E(r) = E\psi_E(r) \tag{8}$$
Under the scale transformation $r \mapsto \mathrm{e}^t r$, this equation transforms as
$$\mathrm{e}^{-2t} H\psi_E(\mathrm{e}^t r) = E\psi_E(\mathrm{e}^t r) \tag{9}$$
or, equivalently,
$$H\psi_E(\mathrm{e}^t r) = \mathrm{e}^{2t} E\psi_E(\mathrm{e}^t r) \tag{10}$$
- This implies the following scaling law of eigenfunction:
$$\psi_{\mathrm{e}^{2t} E}(r) \propto \psi_E(\mathrm{e}^t r) \tag{11}$$

Discrete complex scale invariance: A toy example

- Next consider the asymptotic behavior of $\psi_E(r)$:

$$\psi_E(r) \to \mathrm{e}^{-i\sqrt{E}r} + S(E)\,\mathrm{e}^{+i\sqrt{E}r} \quad \text{as} \quad r \to \infty \tag{12}$$

  where $S(E)$ is the S-matrix (reflection amplitude off the boundary $r = 0$).

- Substituting this into the scaling law $\psi_{\mathrm{e}^{2t}\,E}(r) \propto \psi_E(\mathrm{e}^t\,r)$, we get

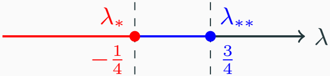
$$S(\mathrm{e}^{2t}\,E) = S(E) \tag{13}$$

  This is the scaling law of S-matrix.

- In the following, I will discuss the breakdown of continuous scale invariance and its consequences from the viewpoint of the scaling law.

Discrete complex scale invariance: A toy example

$$H = -\frac{d^2}{dr^2} + \frac{\lambda}{r^2}$$



- Let us first consider the case $\lambda < \lambda_*$. In this case, the S-matrix turns out to satisfy the following discrete scaling law:

$$S(\mathrm{e}^{2nt_*} E) = S(E), \quad \text{where} \quad t_* = \frac{\pi}{\sqrt{\lambda_* - \lambda}} \tag{14}$$

- Suppose that the S-matrix possesses a simple pole at $E = E_0$:

$$S(E) = \frac{\text{const}}{E - E_0} + O(1) \quad \text{as} \quad E \to E_0 \tag{15}$$

- Then, combining these we get

$$S(E) = S(\mathrm{e}^{2nt_*} E) = \frac{\text{const}}{\mathrm{e}^{2nt_*} E - E_0} + O(1) \quad \text{as} \quad \mathrm{e}^{2nt_*} E \to E_0 \tag{16}$$

or, equivalently,

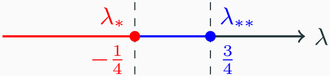
$$S(E) = \frac{\text{const} \times \mathrm{e}^{-2nt_*}}{E - E_0\, \mathrm{e}^{-2nt_*}} + O(1) \quad \text{as} \quad E \to E_0\, \mathrm{e}^{-2nt_*} \tag{17}$$

from which we find the following bound-state energies **[Case '50]**:

$$E_n = E_0\, \mathrm{e}^{-2nt_*} = E_0 \exp\left(-\frac{2n\pi}{\sqrt{\lambda_* - \lambda}}\right) \tag{18}$$

Discrete complex scale invariance: A toy example

$$H = -\frac{d^2}{dr^2} + \frac{\lambda}{r^2}$$



- Let us next consider the case $\lambda \in (\lambda_*, \lambda_{**})$. In this case, the S-matrix turns out to satisfy the following scaling law:

$$S(\mathrm{e}^{i2nt_*}\,E) = S(E), \quad \text{where} \quad t_* = \frac{\pi}{\sqrt{\lambda - \lambda_*}} \tag{19}$$

- Again suppose that the S-matrix has a simple pole at $E = E_0$:

$$S(E) = \frac{\text{const}}{E - E_0} + O(1) \quad \text{as} \quad E \to E_0 \tag{20}$$

- Then we have

$$S(E) = S(\mathrm{e}^{i2nt_*}\,E) = \frac{\text{const} \times \mathrm{e}^{-i2nt_*}}{E - E_0\,\mathrm{e}^{-i2nt_*}} + O(1) \tag{21}$$

  from which we find the following circularly distributed simple poles:

$$E_n = E_0\,\mathrm{e}^{-i2nt_*} = E_0 \cos\left(\frac{2n\pi}{\sqrt{\lambda - \lambda_*}}\right) - iE_0 \sin\left(\frac{2n\pi}{\sqrt{\lambda - \lambda_*}}\right) \tag{22}$$

- Note that, since $\sqrt{\lambda - \lambda_*} < 1$, there exists only one bound state ($n = 0$) on the physical sheet ($0 \leq \arg E_n < 2\pi$).

Discrete complex scale invariance: A toy example

- The above results can be derived by solving the Schrödinger equation under the following boundary condition **[cf. Meetz '64]**:

$$\lim_{r\to 0}\left[\frac{\psi(r)}{r^{\frac{1}{2}-\nu}} + g r^{1-2\nu}\frac{d}{dr}\left(\frac{\psi(r)}{r^{\frac{1}{2}-\nu}}\right)\right] = 0, \quad \nu = \sqrt{\lambda-\lambda_*} \in (0,1) \qquad (23)$$

  where g is a real dimensionful parameter.

- Under this boundary condition, the S-matrix takes the following form:

$$S(k) = -\frac{(+ik/\kappa_0)^{\frac{1}{2}-\nu} - \mathrm{sgn}(g)(+ik/\kappa_0)^{\frac{1}{2}+\nu}}{(-ik/\kappa_0)^{\frac{1}{2}-\nu} - \mathrm{sgn}(g)(-ik/\kappa_0)^{\frac{1}{2}+\nu}}, \quad k = \sqrt{E} \qquad (24)$$

  where $\kappa_0(>0)$ is a dimensionful constant determined by g and ν.

- The above S-matrix satisfies the following properties:

$$\overline{S(k)}S(k) = 1 \qquad (25a)$$

$$\overline{S(k)} = S(-k) \qquad (25b)$$

$$S(\mathrm{e}^{i\frac{n\pi}{\nu}}k) = S(k) \qquad (25c)$$

Few-body examples

Few-body examples

- Discrete complex scale invariance is easily realized in the presence of a codimension-2 flux. A typical example is the one-body Aharonov-Bohm problem in 2d:

$$H_{\mathrm{AB}} = -\frac{\hbar^2}{2m}\left(\boldsymbol{\nabla} + \frac{iq}{\hbar}\boldsymbol{A}\right)^2, \tag{26}$$

where $\boldsymbol{\nabla} = (\partial/\partial x, \partial/\partial y)$. Here $\boldsymbol{A} = (A_x, A_y)$ is a background gauge field satisfying $\partial_x A_y - \partial_y A_x = \Phi\delta(x)\delta(y)$. In the polar coordinate system (r, θ) given by $(x, y) = (r\cos\theta, r\sin\theta)$, such a gauge field $\boldsymbol{A}$ can be written as

$$\boldsymbol{A} = \frac{\Phi}{2\pi}\boldsymbol{\nabla}\theta. \tag{27}$$

In this gauge, the Hamiltonian can be written as

$$H_{\mathrm{AB}} = \frac{\hbar^2}{2m} r^{-1/2}\left(-\frac{\partial^2}{\partial r^2} + \frac{(-i\partial_\theta + \alpha)^2 - 1/4}{r^2}\right) r^{1/2}, \quad \alpha = \frac{q\Phi}{2\pi\hbar}. \tag{28}$$

- Under the assumption of rotational invariance, the radial Hamiltonian is

$$H = -\frac{d^2}{dr^2} + \frac{(n+\alpha)^2 - 1/4}{r^2}, \quad n \in \mathbb{Z} \tag{29}$$

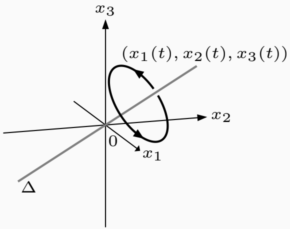
Hence, for $|n + \alpha| \in (0, 1)$, continous scale invariance can be broken to discrete complex scale invariance.

- The one-body Aharonov-Bohm problem is easily generalized to few-body problems by considering a codimension-2 flux in configuration space (domain of many-body wavefunction).
- A well known example is the two-body problem in two dimensions, where the flux is penetrating through the set of two-body coincidence points:

$$\Delta = \{(x_1, x_2) \in \mathbb{R}^2 \times \mathbb{R}^2 : x_1 = x_2\} \tag{30}$$

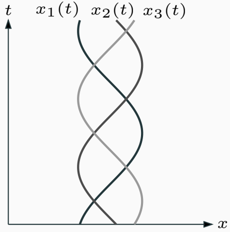
- A less known example is the three-body problem in one dimension, where the flux is penetrating through the set of three-body coincidence points:

$$\Delta = \{(x_1, x_2, x_3) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} : x_1 = x_2 = x_3\} \tag{31}$$



(a) Configuration-space picture

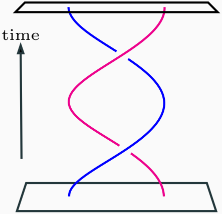
$\Leftrightarrow$



(b) Spacetime picture

Few-body examples

- Two-body problem in 2d **[cf. Jo-Lee '96]**



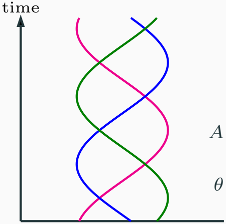
$$H = -\sum_{j=1}^{2} \frac{1}{2m_j} \left(\boldsymbol{\nabla}_j + i\boldsymbol{A}_j\right)^2 \tag{32}$$

$$A = \frac{\alpha}{2\pi} d\theta$$

$$\theta = \arctan\left(\frac{y_1 - y_2}{x_1 - x_2}\right)$$

$$F = dA = \frac{\alpha}{2\pi} \delta^2(\boldsymbol{r}_1 - \boldsymbol{r}_2)(dx_1 - dx_2) \wedge (dy_1 - dy_2)$$

- Three-body problem in 1d **[cf. Ohya '23]**



$$H = -\sum_{j=1}^{3} \frac{1}{2m_j} \left(\frac{\partial}{\partial x_j} + iA_j\right)^2 \tag{33}$$

$$A = \frac{\alpha}{2\pi} d\theta$$

$$\theta = \arctan\left(\sqrt{\frac{m_1 m_2 m_3}{m_1 + m_2 + m_3}} \frac{m_1(x_1 - x_3) - m_2(x_2 - x_3)}{m_1 m_2 (x_1 - x_2)}\right)$$

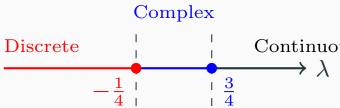
$$F = dA = \frac{\alpha}{2\pi} \delta(x_1 - x_2)\delta(x_2 - x_3)(dx_1 \wedge dx_2 + dx_2 \wedge dx_3 + dx_3 \wedge dx_1)$$

Summary

Summary

- The $1/r^2$-potential problem has three distinct "phases".

$$H = -\frac{d^2}{dr^2} + \frac{\lambda}{r^2}$$



- Continuous scale invariant phase: $\lambda > 3/4$.
  The S-matrix satisfies the scaling law $S(\mathrm{e}^{2t}\,E) = S(E)$, whose solution is just a constant.
- Discrete scale invariant phase: $\lambda < -1/4$.
  The S-matrix satisfies the scaling law $S(\mathrm{e}^{2nt_*}\,E) = S(E)$ and possesses a geometric sequence of bound-state poles at $E_n = E_0\,\mathrm{e}^{-2nt_*}$.
- Discrete complex scale invariant phase: $\lambda \in (-1/4, 3/4)$.
  The S-matrix satisfies the scaling law $S(\mathrm{e}^{i2nt_*}\,E) = S(E)$ and possesses circularly distributed simple poles at $E_n = E_0\,\mathrm{e}^{i2nt_*}$.

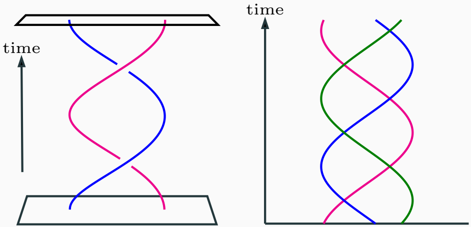
Summary

- Discrete complex scale invariance is easily realized in few-body problems by introducing a codimension-2 flux in configuration space. Examples are:

(1) One-body Aharonov-Bohm problem in 2d.

(2) Two-body problem in 2d.

(3) Three-body problem in 1d.



- (2) is essentially equiavalent to the two-anyon problem. (3) is a generalization of 2d anyons.