Dunkl Algebra
&
Time-Dependent Quantum Systems

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RESEARCH

Dunkl-Schrödinger Equation with Time-Dependent Harmonic Oscillator Potential

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Time-dependent Dunkl–Schrödinger equation with an angular-dependent potential

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ORIGINAL PAPER

Time-dependent Dunkl–Pauli oscillator

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PAPER

One-dimensional Dunkl quantum mechanics: a path integral approach

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Abstract
In the present manuscript, we employ the Feynman path integral method to derive the propagator in one-dimensional Wigner-Dunkl quantum mechanics. To verify our findings we calculate the propagator associated with the free particle and the harmonic oscillator in the presence of the Dunkl derivative. We also deduce the energy spectra and the corresponding bound-state wave functions from the spectral decomposition of the propagator.

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A path integral treatment of time-dependent Dunkl quantum mechanics

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Wigner's Contribution (1950)

- In 1950, Wigner published his work *"Do the equations of motion determine the commutation relations?"* and posed the following fundamental question:

  Are the commutation relations in quantum mechanics (e.g. $[x, p] = i\hbar$) truly a fundamental postulate, or can they also be derived from the classical equations of motion?

- Wigner examined this question specifically for the **one-dimensional harmonic oscillator**.
- In his study, he showed that the standard Heisenberg algebra is **not the only possible structure**, and it can be modified to remain consistent with the classical equations of motion.

Yang's Contribution (1951)

- Addressed **the quantum harmonic oscillator** as a test case.
- Showed that consistency requires adding a **reflection operator** R with a free parameter.
- Defined a **generalized derivative**:

$$D_x^{(Yang)} = \frac{d}{dx} - \frac{\mu}{x} R$$

- Established the **parity-dependent deformation** of the oscillator algebra.

Historical Applications

- **1950s:** Wigner–Yang → Green → **Parastatistics** (parabosons, parafermions).
- **1960s–70s:** Connection to **Calogero–Moser–Sutherland models** (inverse-square interactions, reflection symmetry).
- **1980s–90s:** Revival in **deformed oscillator algebras** (Plyushchay and others).

Dunkl derivative

$$\hat{D}_j = \frac{\partial}{\partial x_j} + \frac{\mu_j}{x_j}\left(1 - \hat{R}_j\right),$$

μ_j Deformation (Wigner) parameter(s)

- A **differential–difference operator** combined with a **reflection group symmetry**.
- Generalizes the ordinary derivative by including reflection operators.
- Provides a natural algebraic tool for studying **parity- and symmetry-sensitive quantum systems**.

$$\hat{R}_j f(x_j) = f(-x_j),$$

$$\hat{R}_i \hat{R}_j = \hat{R}_j \hat{R}_i, \qquad \hat{R}_i x_j = -\delta_{ij} x_j \hat{R}_i, \qquad \hat{R}_i \frac{\partial}{\partial x_j} = -\delta_{ij} \frac{\partial}{\partial x_j} \hat{R}_i.$$

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RESEARCH

Dunkl-Schrödinger Equation with Time-Dependent Harmonic Oscillator Potential

A. Benchikha[1,2] · B. Hamil[2] · B. C. Lütfüoğlu[3] · B. Khantoul[4]

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Abstract

This paper presents an investigation into one- and three-dimensional harmonic oscillators with time-dependent mass and frequency, within the framework of the Dunkl formalism, which is constituted by replacing the ordinary derivative with the Dunkl derivative. To ascertain a general form of the wave functions the Lewis-Riesenfeld method was employed.

Time-Dependent Harmonic Oscillator in 1-D

$$H(t)\,\psi(x,t) = i\hbar\frac{\partial}{\partial t}\psi(x,t),$$

Lewis-Riesenfeld Method: A technique for solving the time-dependent system under certain condition

The existence of an invariant Hermitian operator, I(t), which satisfies

$$\frac{dI(t)}{dt} = \frac{\partial I(t)}{\partial t} + \frac{1}{i\hbar}[I(t), H(t)] = 0,$$

Such that a particular solution appears in the form of

$$\psi(x,t) = e^{i\eta(t)}\Phi(x,t),$$

where $\Phi_n(x,t)$ are the eigenfunctions of the Hermitian operator determined from

$$I(t)\,\Phi_n(x,t) = \lambda_n\Phi_n(x,t).$$

Here, λ_n represents the time-independent eigenvalues of $I(t)$, and $\eta(t)$ are phase functions, derived from the equation

$$\hbar\frac{d}{dt}\eta(t) = \langle\Phi(x,t)|\left(i\hbar\frac{\partial}{\partial t} - H(t)\right)|\Phi(x,t)\rangle. \tag{7}$$

We then replace the ordinary derivatives with the Dunkl ones, and for a time-dependent one-dimensional harmonic oscillator we get the Dunkl Hamiltonian:

$$H = -\frac{\hbar^2}{2M(t)}\left(\frac{\partial^2}{\partial x^2} + \frac{2\mu}{x} - \frac{\mu\,(1-R)}{x^2}\right) + \frac{1}{2}M(t)\omega^2\,(t)\,x^2. \tag{8}$$

We note that the Dunkl-Hamiltonian and the reflection operator commute with each other, so they should have common eigenfunctions and thus can be diagonalised simultaneously. This fact allows us to choose the eigenfunction $\psi\,(x,t)$ with a definite parity: $R\psi\,(x,t) = s\psi\,(x,t)\,1$, where $s = \pm 1$. Consequently, the Dunkl-Schrödinger equation takes the following form:

$$\left[\frac{\mathcal{P}^2}{2M(t)} + \frac{\hbar^2\left(\nu^2 - 1/4\right)}{2M\,(t)\,x^2} + \frac{1}{2}M(t)\omega^2\,(t)\,x^2\right]\psi^s\,(x,t) = i\hbar\frac{\partial\psi^s\,(x,t)}{\partial t}, \tag{9}$$

where $\mathcal{P} = \frac{\hbar}{i}\left(\frac{\partial}{\partial x} + \frac{\mu}{x}\right)$, and $\nu = \mu - \frac{s}{2}$.

It is worth emphasizing that this transformation simplifies the equation, allowing us to concentrate on solving for $\psi^s\,(x,t)$ and ultimately to understand the behavior of the system under the time-dependent harmonic potential.

Derivation of Dunkl-Invariant Operator $I^s(t)$

We assume $$I^s = \frac{1}{2}\Big(\alpha(t)\,T_1 + \beta(t)\,T_2 + \gamma(t)\,T_3\Big),$$

with real functions α, β, and γ and generators of the Lie algebra of the group $SL(2, R)$

$$T_1 = \mathcal{P}^2 + \frac{\hbar^2\left(\nu^2 - 1/4\right)}{x^2}, \qquad T_2 = x^2, \qquad T_3 = x\mathcal{P} + \mathcal{P}x,$$

which obey the following commutations

$$[T_1, T_2] = -2i\hbar T_3, \quad [T_2, T_3] = 4i\hbar T_2, \quad [T_1, T_3] = -4i\hbar T_1.$$

After solving the (4), we find that the real functions must satisfy the following system of coupled first-order linear differential equations:

$$\alpha = \rho^2, \tag{13}$$

$$\beta = \frac{1}{\rho^2} + M^2\dot{\rho}^2, \tag{14}$$

$$\gamma = -M\rho\dot{\rho}. \tag{15}$$

Using (13), (14) and (15) in (10), we obtain the Dunkl exact invariant as

$$I^s = \frac{1}{2}\left[\rho^2\left(\mathcal{P}^2 + \frac{\hbar^2\left(\nu^2 - 1/4\right)}{x^2}\right) + \left(\frac{1}{\rho^2} + M^2\dot{\rho}^2\right)x^2 - \rho\dot{\rho}M\left(x\mathcal{P} + \mathcal{P}x\right)\right], \tag{16}$$

where ρ is a real parameter satisfying the Ermakov-Pinney equation [75, 76]:

$$\ddot{\rho} + \frac{\dot{M}}{M}\dot{\rho} + \omega^2\rho = \frac{1}{M^2\rho^3}. \tag{17}$$

Depending to the form of the mass function the solution of the Ermakov-Pinney equation can be obtained either analytically or numerically.

To derive the explicit form of the wavefunctions, we have to identify a preferred basis of eigenstates for the Dunkl invariant by solving the following eigenvalue equation.

$$I^s \Phi_n^s = \overset{s}{\lambda}_n \Phi_n^s. \tag{18}$$

Now, we introduce a unitary transformation

$$\varphi_n^s(x) = U \Phi_n^s(x) = \exp\left(\frac{iM\dot{\rho}}{2\hbar\rho}x^2\right)\Phi_n^s(x), \tag{19}$$

which transforms the operator I^s to $\mathcal{I}^s$:

$$\mathcal{I}^s = U I^s U^\dagger = \frac{1}{2}\left[\rho^2\left(\mathcal{P}^2 + \frac{\hbar^2\left(\nu^2 - 1/4\right)}{x^2}\right) + \frac{1}{\rho^2}x^2\right]. \tag{20}$$

Then, using the substitution $\mathcal{I}^s \varphi_n^s(x) = \overset{s}{\lambda}_n \varphi_n^s(x)$ with $\varphi_n^s(x) = x^{-\mu}\phi_n^s(x)$ and the new variable $y = \frac{x}{\rho}$, we get the eigenvalue equation for the operator $\mathcal{I}^s$

$$\frac{\partial^2}{\partial y^2}\phi_n^s(x) + \frac{1}{\hbar^2}\left[2\lambda - \frac{\hbar^2\left(\nu^2 - 1/4\right)}{y^2} - y^2\right]\phi_n^s(x) = 0. \tag{21}$$

By putting the parameter $\lambda_n^s = \hbar\,(2n + 1 + \nu)$, with $n = 0, 1, 2...$, we express the solution of the above equation in terms of generalized Laguerre polynomials [80],

$$\phi_n^s\,(x) = \mathbf{N}_{s,n}\; y^{\mu+\frac{1-s}{2}}\, e^{-\frac{y^2}{2\hbar}}\, L_n^{\mu-s/2}\left(\frac{y^2}{\hbar}\right), \tag{22}$$

with the normalization constant, $\mathbf{N}_{s,n}$. Finally, combining $\varphi^s\,(x)$ with the wave function $\phi^s\,(x,t)$ and using the variable transformation $y = \frac{x}{\rho}$, we get the wave function as

$$\Phi_n^s\,(x,t) = \mathbf{N}_{s,n}\,\frac{x^{\frac{1-s}{2}}}{\rho^{\mu+\frac{1-s}{2}}}\,L_n^{\mu-\frac{s}{2}}\left(\frac{x^2}{\hbar\rho^2}\right)\exp\left[\left(iM\rho\dot{\rho} - 1\right)\frac{x^2}{2\hbar\rho^2}\right]. \tag{23}$$

Quantum Phase and Total Wave Function

We now focus on deriving the quantum phase using (7). To determine the phase function, $\eta_n^s(t)$, we use (9) and (19) in (7). This gives us

$$\dot{\eta}_n^s(t) = -\frac{\lambda_n^s}{\hbar M \rho^2}. \tag{24}$$

Then, we integrate it over time

$$\eta_n^s(t) = -\left(2n + 1 + \mu - \frac{s}{2}\right) \int_0^t \frac{dt'}{M(t')\,\rho(t')^2}. \tag{25}$$

We observe that the quantum phase varies due to the parity. Utilizing (5), (23) and (25) we find the normalized wavefunctions with the corresponding eigenvalues for both cases as follows:

- Even parity case: $s = +1$,

the wavefunction

$$\psi_n^+(x,t) = \sqrt{\frac{n!}{\hbar^{\mu+\frac{1}{2}}\Gamma\left(\mu+n+\frac{1}{2}\right)}}\rho^{-\left(\mu+\frac{1}{2}\right)} L_n^{\mu-\frac{1}{2}}\left(\frac{x^2}{\hbar\rho^2}\right) \tag{26}$$

$$\times \exp\left[\left(iM(t)\rho\dot{\rho}-1\right)\frac{x^2}{2\hbar\rho^2} - i\left(2n+\mu+\frac{1}{2}\right)\int_0^t \frac{dt'}{M(t')\rho(t')^2}\right], \tag{27}$$

and the eigenvalue

$$\lambda_n^+ = 2n + \mu + \frac{1}{2}. \tag{28}$$

- Odd parity: $s = -1$,
the wavefunction

$$\psi_n^-(x,t) = \sqrt{\frac{n!}{\hbar^{\mu+\frac{3}{2}}\Gamma\left(\mu+n+\frac{3}{2}\right)}}\rho^{-\left(\mu+\frac{3}{2}\right)} x L_n^{\mu+\frac{1}{2}}\left(\frac{x^2}{\hbar\rho^2}\right)$$

$$\times \exp\left[\left(iM(t)\rho\dot{\rho}-1\right)\frac{x^2}{2\hbar\rho^2} - i\left(2n+\mu+\frac{3}{2}\right)\int_0^t \frac{dt'}{M(t')\rho(t')^2}\right], \tag{29}$$

and the eigenvalue,

$$\lambda_n^- = 2n + \mu + \frac{3}{2}. \tag{30}$$

We note that in both cases, the eigenvalues depend on the Wigner constant. From the orthogonality relation of the Laguerre polynomials, it is straightforward to observe that for $\mu > -1/2$, the eigenfunction $\psi_n^{\pm}(x)$ satisfies

$$\int_{-\infty}^{+\infty} \psi_n^s(x)\left[\psi_n^s(x)\right]^* |x|^\mu \, dx = 1. \tag{31}$$

Based on the above result, it is clear that the eigenstates $\psi_n^+(x,t)$ and $\psi_n^-(x,t)$ of Hamiltonian H can be labeled by a single integer n, with the parity corresponding to that of the associated wave function. To simplify this, we introduce the generalized Hermite polynomials [36],

$$H_{2n+p}^{\mu}(x) = (-1)^n \sqrt{\frac{n!}{\Gamma(n+p+\mu+1/2)}}\, x^{\pi} L_n^{\mu-1/2+p}\left(x^2\right), \tag{32}$$

where $p = 0, 1$. Using this definition, the eigenfunctions of the Hamiltonian can be written as

$$\psi_n(x,t) = \frac{C_n}{\rho^{(\mu+1/2)}} H_{2n+p}^{\mu}\left(\frac{x}{\sqrt{\hbar}\rho}\right) \exp\left[\left(iM(t)\rho\dot{\rho} - 1\right)\frac{x^2}{2\hbar\rho^2} - i\left(2n+p+\mu+\frac{1}{2}\right)\int_0^t \frac{dt'}{M(t')\rho(t')^2}\right], \tag{33}$$

where C_n is the normalization constant. On the other hand, when $\mu = 0$, we obtain the ordinary time-dependent harmonic oscillator in one dimension wavefunction [81].

$$\psi_N(x,t) = \frac{C_N}{\rho^{1/2}} H_N\left(\frac{x}{\sqrt{\hbar}\rho}\right) \exp\left[\left(iM(t)\rho\dot{\rho} - 1\right)\frac{x^2}{2\hbar\rho^2} - i\left(N+\frac{1}{2}\right)\int_0^t \frac{dt'}{M(t')\rho(t')^2}\right]. \tag{34}$$

Application

$$M(t) = M_0 e^{kt}. \tag{35}$$

Thus, the harmonic oscillator with an exponentially increasing mass has the Dunkl-Caldirola-Kanai Hamiltonian

$$H = -\frac{\hbar^2}{2M_0}\left(\frac{\partial^2}{\partial x^2} + \frac{2\mu}{x}\frac{\partial}{\partial x} - \frac{\mu(1-s)}{x^2}\right)e^{-kt} + \frac{1}{2}M_0\omega_0^2 e^{kt}x^2, \tag{36}$$

where M_0, k and ω_0 are real positive constants. In this case the solution of the Ermakov-Pinney (17) takes the form

$$\rho = \frac{e^{-\frac{k}{2}t}}{\sqrt{M_0\Omega}}, \tag{37}$$

where

$$\Omega = \sqrt{\omega_0^2 - \frac{k^2}{4}}, \tag{38}$$

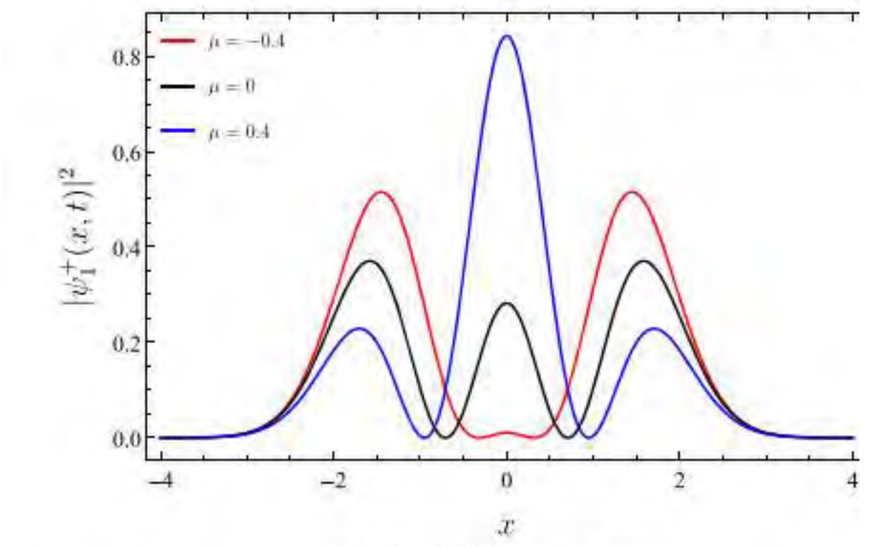
and the phase expressed is given by

$$\eta_n^s(t) = -\left(2n + 1 + \mu - \frac{s}{2}\right)\Omega t. \tag{39}$$

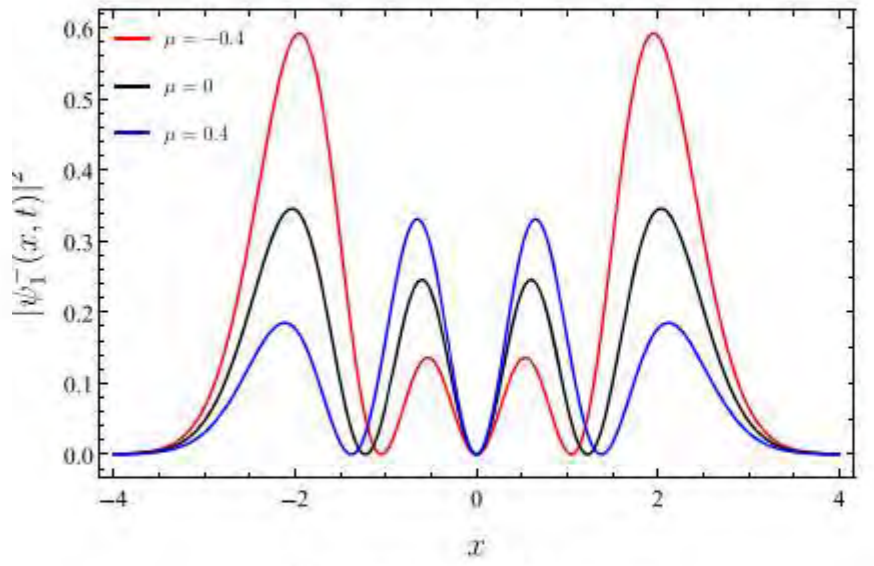
Utilizing the above result, we obtain both the even and odd wave functions of the Dunkl-Caldirola-Kanai oscillator:

$$\psi_n^+(x,t) = \sqrt{\frac{n!(M_0\Omega)^{\mu+\frac{1}{2}}}{\hbar^{\mu+\frac{1}{2}}\Gamma\left(\mu+n+\frac{1}{2}\right)}} L_n^{\mu-\frac{1}{2}}\left(\frac{M_0\Omega x^2 e^{kt}}{\hbar}\right)$$
$$\times\exp\left\{\left(\frac{-ik}{\Omega}-1\right)\frac{M_0\Omega x^2}{2\hbar}e^{kt}+\left[\frac{k}{2}\left(\mu+\frac{1}{2}\right)-i\left(2n+\mu+\frac{1}{2}\right)\Omega\right]t\right\}. \tag{40}$$

$$\psi_n^-(x,t) = \sqrt{\frac{n!(M_0\Omega)^{\mu+\frac{3}{2}}}{\hbar^{\mu+\frac{3}{2}}\Gamma\left(\mu+n+\frac{3}{2}\right)}} x L_n^{\mu+\frac{1}{2}}\left(\frac{M_0\Omega x^2}{\hbar} e^{kt}\right)$$
$$\times\exp\left\{\left(\frac{-ik}{\Omega}-1\right)\frac{M_0\Omega x^2}{2\hbar}e^{kt}+\left[\frac{k}{2}\left(\mu+\frac{3}{2}\right)-i\left(2n+\mu+\frac{3}{2}\right)\Omega\right]t\right\}. \tag{41}$$



(a) Positive parity case.



(b) Negative parity case.

Fig. 1 The square module of the function $\psi_1^s(x,t)$

Time-Dependent Isotropic Harmonic Oscillator in Three Dimensions

$$\mathcal{H} = -\frac{\hbar^2}{2M(t)}\left(\hat{D}_1^2+\hat{D}_2^2+\hat{D}_3^3\right)+\frac{M(t)}{2}\omega^2(t)\left(x_1^2+x_2^2+x_3^2\right),$$

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Time-dependent Dunkl–Schrödinger equation with an angular-dependent potential

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Abstract:

In this paper, we investigate the analytical solution of the time-dependent Schrödinger equation for a harmonic oscillator with time-dependent mass and frequency, coupled with angular-dependent potential energy by utilizing the Dunkl derivatives. To obtain the solution, we employ the Lewis–Riesenfeld invariant methodology. Our approach broadens the scope of quantum mechanical analyses, offering exact solutions and new insights into dynamic quantum systems under varying conditions.

2. Time-Dependent Dunkl–Schrödinger Equation with Angular-Dependent Potential

In this section, we delve into the exact solutions of the time-dependent Dunkl-Schrödinger equation with an angular-dependent potential:

$$\mathcal{H}(t)\Psi(\vec{r},t) = i\frac{\partial}{\partial t}\Psi(\vec{r},t). \tag{3}$$

The Hamiltonian for this system is given by

$$\mathcal{H} = -\frac{1}{2M(t)}(\hat{D}_1^2 + \hat{D}_2^2 + \hat{D}_3^2) + \frac{1}{2}M(t)\omega^2(t)(x_1^2 + x_2^2 + x_3^2) + \frac{U(\theta,\varphi)}{M(t)(x_1^2 + x_2^2 + x_3^2)}. \tag{4}$$

Here, $M(t)$ and $\omega(t)$ denote the time-dependent mass and frequency, respectively, while $U(\theta,\varphi)$ represents the angular-dependent potential[123–126]

$$U(\theta,\varphi) = \frac{b}{2}\cot^2\theta + \frac{a}{\sin^2\theta\sin^2\varphi}, \tag{5}$$

with real coupling constants a and b.

To facilitate the solution process, we use the spherical coordinates

$$x_1 = r\cos\varphi\sin\theta, \quad x_2 = r\sin\varphi\sin\theta, \quad x_3 = r\cos\theta, \tag{6}$$

and transform the Hamiltonian into the following form:

$$\mathcal{H} = \frac{1}{2M(t)}\mathbf{p}^2 + \frac{1}{2}M(t)\omega^2(t)r^2 + \frac{\hbar^2\delta(\delta-1)+2U(\theta,\varphi)}{2M(t)r^2}, \tag{7}$$

with the momentum operator in three dimensions[61]

$$\mathbf{p}^2 = \mathcal{P}_{r,\delta}^2 + \frac{L_D^2}{r^2}, \tag{8}$$

and the operator, $\mathcal{P}_{r,\delta}$,

$$\mathcal{P}_{r,\delta} = \frac{\hbar}{i}\left(\frac{\partial}{\partial r} + \frac{\delta}{r}\right). \tag{9}$$

Here, the Dunkl-angular momentum operator L_D^2 reads as follows:

$$\begin{aligned} L_D^2 = & -\hbar^2\Bigg\{\frac{\partial^2}{\partial\theta^2} + 2\left[\left(\frac{1}{2}+\mu_1+\mu_2\right)\cot\theta - \mu_3\tan\theta\right]\frac{\partial}{\partial\theta} - \frac{\mu_3}{\cos^2\theta}(1-\hat{R}_3) \\ & + \frac{1}{\sin^2\theta}\Bigg(\frac{\partial^2}{\partial\varphi^2} + 2(\mu_2\cot\varphi - \mu_1\tan\varphi)\frac{\partial}{\partial\varphi} \\ & - \frac{\mu_1}{\cos^2\varphi}(1-\hat{R}_1) - \frac{\mu_2}{\sin^2\varphi}(1-\hat{R}_2)\Bigg)\Bigg\}, \end{aligned} \tag{10}$$

where

$$\delta = \mu_1 + \mu_2 + \mu_3 + 1, \quad \text{and} \quad [P_{r,\delta}, r] = i\hbar. \tag{11}$$

Now, we express the time-dependent Schrödinger equation which we aim to solve

$$i\hbar\frac{\partial}{\partial t}\psi(r,\theta,\varphi,t)=\left[\frac{1}{2M(t)}\mathbf{p}^2+\frac{1}{2}M(t)\omega^2(t)r^2+\frac{\hbar^2\delta(\delta-1)}{2M(t)r^2}+\frac{U(\theta,\varphi)}{M(t)r^2}\right]\psi(r,\theta,\varphi,t), \tag{12}$$

with the angular-dependent potential given in Eq. (5).

2.1. *Exact solution*

To solve Eq. (12), we employ the Lewis–Riesenfeld invariant approach.

$$\frac{dI(t)}{dt}=\frac{\partial I(t)}{\partial t}+\frac{1}{i\hbar}[I(t),\mathcal{H}(t)]=0. \tag{13}$$

This allows us to relate the solution of the time-dependent Schrödinger equation, $\Psi(\vec{r},t)$, to the solution of the invariant's eigenvalue problem

$$I(t)\Upsilon(\vec{r},t)=E_{n,l,m}\Upsilon(\vec{r},t), \tag{14}$$

with a phase factor

$$\Psi(\vec{r},t)=e^{i\eta(t)}\Upsilon(\vec{r},t), \tag{15}$$

where $\eta(t)$ can be determined from the following equation:

$$\hbar\frac{d}{dt}\eta_n(t)=\langle\Upsilon_n(\vec{r},t)|i\hbar\frac{\partial}{\partial t}-H|\Upsilon_n(\vec{r},t)\rangle. \tag{16}$$

To construct the exact invariant for the quantum system, we then introduce the generators $\{T_1, T_2, T_3\}$ in the following explicit forms:

$$T_1 = \mathbf{p}^2 + \left(\frac{\hbar^2\delta(\delta-1) + 2U(\theta,\varphi)}{r^2}\right), \tag{17}$$

$$T_2 = r^2, \tag{18}$$

$$T_3 = \frac{1}{2}(r\mathcal{P}^2_{r,\delta} + \mathcal{P}^2_{r,\delta}r), \tag{19}$$

which verifies the following commutation relations:

$$[T_1, T_2] = -2i\hbar T_3, \quad [T_2, T_3] = 4i\hbar T_2, \quad [T_1, T_3] = -4i\hbar T_1. \tag{20}$$

with the assumption that the invariant follows

$$I = \frac{1}{2}(\alpha T_1 + \beta T_2 + \gamma T_3).$$

Subsequently, utilizing Eq. (13), we obtain a system of equations whose solution is contingent upon the real-time variation of the functions α, β and γ

$$\alpha = \rho^2, \tag{22}$$

$$\beta = \frac{1}{\rho^2} + M^2\dot{\rho}^2, \tag{23}$$

$$\gamma = -M\rho\dot{\rho}. \tag{24}$$

Thus, the invariant takes the following explicit form:

$$I = \frac{1}{2}\Bigg[\left(\frac{1}{\rho^2} + M^2\dot{\rho}^2\right)r^2 + \rho^2\left(\mathbf{p}^2 + \frac{\hbar^2\delta(\delta-1) + 2U(\theta,\varphi)}{r^2}\right) - \rho\dot{\rho}M(r\mathcal{P}_{r,\delta} + \mathcal{P}_{r,\delta}r)\Bigg], \tag{25}$$

where ρ satisfies the Ermakov–Pinney equation given by

$$\ddot{\rho} + \frac{\dot{M}}{M}\dot{\rho} + \omega^2(t)\rho = \frac{1}{M^2\rho^3}. \tag{26}$$

To solve the eigenvalue equation given in Eq. (14), we introduce the following unitary transformation:

$$\Upsilon(\vec{r}) = S\mathcal{F}(\vec{r}) = e^{\frac{iM\dot{\rho}}{2\hbar\rho}r^2}\mathcal{F}(\vec{r}), \tag{27}$$

so the invariant transforms as

$$I' = \frac{1}{2}\left[\rho^2\left(\mathbf{p}^2 + \frac{\hbar^2\delta(\delta-1) + 2U(\theta,\varphi)}{r^2}\right) + \frac{1}{\rho^2}r^2\right]. \tag{28}$$

We then express the multi-variable function $\mathcal{F}(\vec{r})$ as the multiplication of radial, polar and azimuthal functions

$$\mathcal{F}(r,\theta,\varphi) = r^{-\delta}R(r)\Theta(\theta)\Phi(\varphi). \tag{29}$$

$$
-\left[\frac{\partial^2}{\partial\varphi^2}+2(\mu_2\cot\varphi-\mu_1\tan\varphi)\frac{\partial}{\partial\varphi}-\frac{\mu_1}{\cos^2\varphi}(1-\hat{R}_1)-\frac{\mu_2}{\sin^2\varphi}(1-\hat{R}_2)\right.
$$

$$
\left.-\frac{2a}{\hbar^2\sin^2\varphi}\right]\Phi(\varphi)=m^2\Phi(\varphi), \tag{30}
$$

$$
-\left[\frac{\partial^2}{\partial\theta^2}-2\left(\mu_3\tan\theta-\left(\frac{1}{2}+\mu_1+\mu_2\right)\cot\theta\right)\frac{\partial}{\partial\theta}-\frac{m^2}{\sin^2\theta}\right.
$$

$$
\left.-\frac{\mu_3}{\cos^2\theta}(1-\hat{R}_3)-b\cot^2\theta\right]\Theta(\theta)=\lambda\Theta(\theta), \tag{31}
$$

$$
-\rho^2\left[\frac{\partial^2}{\partial r^2}-\frac{\delta(\delta-1)+\lambda}{r^2}-\frac{1}{\hbar^2}\frac{r^2}{\rho^4}\right]R(r)=\frac{2E_{n,l,m}}{\hbar^2}R(r). \tag{32}
$$

Here, m^2 and $\lambda=l(l+1)$ represent the separation constants, where m, λ, and l are the standard magnetic quantum numbers. The general form of the azimuthal eigenfunction reads as follows:

$$
\Phi(\varphi)=\mathcal{C}_\varphi\cos(\varphi)^{\frac{1-e_1}{2}}\sin(\varphi)^{\frac{1}{2}-\mu_2+\alpha}P_{n_\varphi}^{(\alpha,\mu_1-\frac{e_1}{2})}(\cos 2\varphi), \tag{33}
$$

where

$$
\alpha=\sqrt{\left(\mu_2-\frac{e_2}{2}\right)^2+\frac{2a}{\hbar^2}}, \tag{34}
$$

and $P_n^{(p,q)}$ is the generalized Jacobi polynomial.[127] Here, $R_1\Phi_m(\varphi) = e_1\Phi_m(\varphi)$ and $R_1\Phi_m(\varphi) = e_2\Phi_m(\varphi)$, where e_1 and e_1 attain the values ± 1. Besides, n_φ is a positive integer related to the separation constant m^2 as follows:

$$m^2 = \left[2n_\varphi + 1 + \mu_1 - \frac{e_1}{2} + \alpha\right]^2 - (\mu_1 + \mu_2)^2. \tag{35}$$

For the action of the reflection operator, R_3, on the polar function $\Theta(\theta)$, we employ $R_3\Theta(\theta) = e_3\Theta(\theta)$, where e_3 attains the value ± 1. Thus, the polar equation $\Theta(\theta)$ takes the following form:

$$\Theta(\theta) = \mathcal{C}_\theta \cos(\theta)^{\frac{1-e_3}{2}} \sin(\theta)^{\beta-\mu_1-\mu_2} P_{n_\theta}^{(\beta,\mu_3-\frac{e_3}{2})}(\cos 2\theta), \tag{36}$$

where

$$\beta = \sqrt{m^2 + \frac{b}{\hbar^2} + (\mu_1 + \mu_2)^2}, \tag{37}$$

for a nonnegative integer n_θ. Here, we see that the separation constant, λ, is interrelated in terms of n_θ with the following relation:

$$\lambda = \left[2n_\theta + 1 + \mu_3 - \frac{e_3}{2} + \beta\right]^2 - \frac{b}{\hbar^2} - \left(\frac{1}{2} + \mu_1 + \mu_2 + \mu_3\right)^2. \tag{38}$$

In order to eliminate the time-dependent function ρ from the radial equation, we use the transformation $\frac{r}{\rho} = \varkappa$. We then rewrite Eq. (32) as

$$\left\{\frac{\partial^2}{\partial \varkappa^2} + \frac{1}{\hbar^2}\left[2E_{n,l,m} - \varkappa^2 - \frac{\hbar^2(\delta(\delta-1)+\lambda)}{\varkappa^2} -\right]\right\} R(\varkappa) = 0. \tag{39}$$

The solution to this equation is a straightforward matter of calculation, and the result is as follows:

$$R(\varkappa) = \mathcal{C}_r \varkappa^{\sigma+\frac{1}{2}} e^{-\frac{\varkappa^2}{2\hbar}} L_n^{\sigma}\left(\frac{\varkappa^2}{\hbar}\right). \tag{40}$$

Here, C_r is the normalization constant and the eigenvalue of the invariant reads as follows:

$$E_{n,l,m} = \hbar(2n + \sigma + 1), \quad \text{where} \quad n = 1, 2, 3, \ldots, \tag{41}$$

and

$$\sigma = \sqrt{\frac{1}{4} + \delta(\delta - 1) + \lambda}. \tag{42}$$

To calculate the quantum phase $\eta(t)$, we substitute the Hamiltonian

$$\mathcal{H} = \frac{I}{M\rho^2} - \left(\frac{1}{2M\rho^4} + \frac{M\dot{\rho}^2}{2\rho^2}\right)x^2 - \frac{\dot{\rho}}{2\rho}(r\mathcal{P}_{r,\delta} + \mathcal{P}_{r,\delta}r) + \frac{M\omega^2 x^2}{2}, \tag{43}$$

in Eq. (16). We find

$$\hbar\dot{\eta}(t) = -\frac{E}{M\rho^2} + \langle\mathcal{F}|i\hbar\frac{\partial}{\partial t} - \frac{\dot{\rho}}{2\rho}(r\mathcal{P}_{r,\delta} + \mathcal{P}_{r,\delta}r)|\mathcal{F}\rangle. \tag{44}$$

By employing the unitary transformation, as defined in Eq. (27), and taking into account the Ermakov–Pinney equation, as presented in Eq. (26), one can perform straightforward calculations to obtain

$$\langle\mathcal{F}|i\hbar\frac{\partial}{\partial t} - \frac{\dot{\rho}}{2\rho}(r\mathcal{P}_{r,\delta} + \mathcal{P}_{r,\delta}r)|\mathcal{F}\rangle = 0. \tag{45}$$

Thus,

$$\dot{\eta}(t) = -\frac{E}{\hbar M\rho^2}, \tag{46}$$

and hence, the phase factor becomes

$$\eta(t) = -(2n+1+\sigma)\int_0^t \frac{dt'}{M(t')\rho(t')^2}. \tag{47}$$

Finally, the total wave function reads as follows:

$$\begin{aligned}\Psi(r,\theta,\varphi,t) =& \mathcal{C}_{r,\theta,\varphi} \exp\left[\frac{(iM\rho\dot{\rho}-1)r^2}{2\hbar\rho^2} - i\left((2n+1+\sigma)\int_0^t \frac{dt'}{M(t')\rho(t')^2}\right)\right] \\ &\times \frac{r^{\sigma-\delta+\frac{1}{2}}}{\rho^{\sigma-\delta+1}} L_n^{\sigma}\left(\frac{r^2}{\hbar\rho^2}\right) \cos(\varphi)^{\frac{1-\epsilon_1}{2}} \sin(\varphi)^{\frac{1}{2}-\mu_2+\alpha} P_{n_\varphi}^{(\alpha,\mu_1-\frac{\epsilon_1}{2})}(\cos 2\varphi) \\ &\times \cos(\theta)^{\frac{1-\epsilon_3}{2}} \sin(\theta)^{\beta-\mu_1-\mu_2} P_{n_\theta}^{(\beta,\mu_3-\frac{\epsilon_3}{2})}(\cos 2\theta). \end{aligned} \tag{48}$$

By making specific selections, the wave function can be normalized through

$$\begin{aligned}&\int_0^{2\pi}\int_0^{\pi}\int_0^{+\infty} r^{2+2\mu_1+2\mu_2+2\mu_3} |\sin(\theta)|^{2\mu_1+2\mu_2} |\cos(\theta)|^{2\mu_3} |\sin(\varphi)|^{2\mu_2} \\ &\times |\cos(\varphi)|^{2\mu_1} \psi_{n'}\psi_n dr d\theta d\varphi = \delta_{n,n'}. \end{aligned} \tag{49}$$

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ORIGINAL PAPER

Time-dependent Dunkl–Pauli oscillator

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Abstract: This study explores the time-dependent Dunkl–Pauli oscillator in two dimensions. We constructed the Dunkl–Pauli Hamiltonian, which incorporates a time-varying magnetic field and a harmonic oscillator characterized by time-dependent mass and frequency, initially in Cartesian coordinates. Subsequently, we reformulated the Hamiltonian in polar coordinates and analyzed the eigenvalues and eigenfunctions of the Dunkl angular operator, deriving exact solutions using the Lewis–Riesenfeld invariant method. Additionally, we examine the Dunkl–Caldirola–Kanai oscillator as a specific example of a time-dependent system with deformed symmetries, illustrating the effects of time-dependent mass and damping on wave function evolution. Our findings regarding the total quantum phase factor and wave functions reveal the significant impact of Dunkl operators on quantum systems, yielding precise expressions for both wave functions and energy spectra. This work enhances the understanding of quantum systems with deformed symmetries and suggests avenues for future research in quantum mechanics and mathematical physics.

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ORIGINAL PAPER

Dunkl–Pauli equation in the presence of a magnetic field

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2. Time-dependent Dunkl–Pauli oscillator in two dimensions

The usual Pauli equation for a spin-1/2 particle of mass m in a potential $V(r)$ is given by [1]:

$$\left[\frac{1}{2m}\left(\vec{\sigma_j}\cdot\vec{\pi_j}\right)+V(r)\right]\psi\left(\vec{r},t\right)=i\hbar\frac{\partial}{\partial t}\psi\left(\vec{r},t\right). \tag{1}$$

Here, σ_j represents the Pauli matrices and $\vec{\pi}$ denotes the kinetic momentum term, expressed as:

$$\vec{\pi}=\vec{p}-\frac{e}{c}\vec{A}=\left(p_1-\frac{e}{c}A_1,p_2-\frac{e}{c}A_2\right), \tag{2}$$

where e is the charge and c is the speed of light. In this case, the spinor wave function consists of two components:

$$\psi=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}. \tag{3}$$

Now, we consider the time-dependent Pauli equation with a harmonic oscillator potential in two dimensions, which represents the non-relativistic limit of the Dirac equation in standard quantum mechanics:

$$\left[\frac{1}{2m(t)}\left(\vec{\sigma_j}\cdot\vec{\pi_j}\right)+\frac{1}{2}m(t)\omega^2(t)r^2\right]\psi\left(\vec{r},t\right)=i\hbar\frac{\partial}{\partial t}\psi\left(\vec{r},t\right), \tag{4}$$

where $m(t)$ denotes the time-dependent mass, and $\omega(t)$ is the time-dependent frequency of the harmonic oscillator.

To incorporate the Dunkl formalism, we replace the ordinary momentum operator with the Dunkl momentum operator ($\hbar=1$):

$$p_j=\frac{1}{i}D_j, \tag{5}$$

where the Dunkl derivatives are defined as:

$$D_j=\frac{\partial}{\partial x_j}+\frac{\nu_j}{x_j}\left(1-R_j\right),\quad \text{for } j=1,2. \tag{6}$$

Employing the Dunkl second derivative

$$D_j^2=\frac{\partial^2}{\partial x_j^2}+\frac{2\nu_j}{x_j}\frac{\partial}{\partial x_j}-\frac{\nu_j}{x_j^2}\left(1-R_j\right), \tag{8}$$

the time-dependent Dunkl–Pauli Hamiltonian in Cartesian coordinates is expressed as:

$$H=\frac{1}{2m(t)}\left[\pi_1^2+\pi_2^2+\sigma_3\left[\pi_1,\pi_2\right]\right]+\frac{1}{2}m(t)\omega^2(t)r^2. \tag{9}$$

We then adopt the symmetric gauge for the vector potential:

$$e\overrightarrow{A} = \frac{eB(t)}{2c}\left(-y\hat{i} + x\hat{j}\right), \tag{10}$$

where $\left(\hat{i}, \hat{j}\right)$ represents the Cartesian orthonormal basis. Applying the Dunkl-deformed algebra and the definitions provided above, we derive the time-dependent Dunkl–Pauli Hamiltonian in the following form [90]:

$$H = -\frac{1}{2m(t)}\Delta_D + \frac{m(t)\Omega^2(t)}{2}\left(x^2 + y^2\right) + i\frac{\omega_c(t)}{2}$$
$$(xD_2 - yD_1) - \frac{e}{2m(t)c}(1 + \nu_1 R_1 + \nu_2 R_2)\sigma_z B(t), \tag{11}$$

where the Dunkl–Laplacian is given by:

$$\Delta_D = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{2\nu_1}{x}\frac{\partial}{\partial x} + \frac{2\nu_2}{y}\frac{\partial}{\partial y}$$
$$-\frac{\nu_1}{x^2}(1 - R_1) - \frac{\nu_2}{y^2}(1 - R_2). \tag{12}$$

Here,

$$\omega_c(t) = \frac{eB(t)}{m(t)c}, \tag{13}$$

is the time-dependent cyclotron frequency, and

$$\Omega^2(t) = \omega(t)^2 + \frac{\omega_c(t)^2}{4}, \tag{14}$$

is the modified harmonic oscillator frequency.

Solutions in polar coordinates

$$x = r\cos\theta \text{ and } y = r\sin\theta. \tag{15}$$

$$\begin{aligned} H(t) = &-\frac{1}{2m(t)}\frac{\partial^2}{\partial r^2} - \frac{1+2\nu_1+2\nu_2}{2m(t)r}\frac{\partial}{\partial r} \\ &+ \frac{m(t)}{2}\Omega^2(t)r^2 + \frac{\mathcal{J}_\theta^2 - 2\nu_1\nu_2(1-R_1R_2)}{2m(t)r^2} \\ &+ \frac{\omega_c(t)}{2}\left[\mathcal{J}_\theta - g_s(1+\nu_1R_1+\nu_2R_2)\cdot\mathbf{S}_z\right]. \end{aligned} \tag{16}$$

Here, $\mathbf{S}_z = \frac{\sigma_z}{2}$ represents the spin-$\frac{1}{2}$ operator, and g_s is the free electron g-factor, with $g_s = 2$. The angular operator $\mathcal{J}_\theta$ is defined as [49, 50, 52, 53]:

$$\mathcal{J}_\theta = i\left(\frac{\partial}{\partial\theta} + \nu_2\cot\theta(1-R_2) - \nu_1\tan\theta(1-R_1)\right). \tag{17}$$

$$\psi_{m_s}(r,\theta,t) = \phi(r,\theta,t)\chi_{m_s}, \tag{20}$$

where χ_{m_s} represents the spin function:

$$\mathbf{S}_z\chi_{ms} = \frac{m_s}{2}\chi_{m_s} \quad \text{for} \quad m_s = \pm 1, \tag{21}$$

and

$$\chi_{+1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \chi_{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{22}$$

Subsequently, the time-dependent Dunkl–Pauli oscillator equation is expressed as:

$$
\left[-\frac{1}{2m(t)}\frac{\partial^2}{\partial r^2}-\frac{1+2\nu_1+2\nu_2}{2m(t)r}\frac{\partial}{\partial r}+\frac{m(t)}{2}\Omega^2(t)r^2 \right.
$$

$$
+\frac{\mathcal{J}_\theta^2-2\nu_1\nu_2(1-R_1R_2)}{2m(t)r^2}+\frac{\omega_c(t)}{2}\Big(\mathcal{J}_\theta-m_s \tag{23}
$$

$$
\left.(1+\nu_1R_1+\nu_2R_2)\Big)\right]\phi(r,\theta,t)=i\frac{\partial}{\partial t}\phi(r,\theta,t).
$$

To eliminate the last time-dependent term $\frac{\omega_c(t)}{2}\Big(\mathcal{J}_\theta-m_s(1+\nu_1R_1+\nu_2R_2)\Big)$, we apply the transformation:

$$
\phi(r,\theta,t)=e^{-\frac{i}{2}(\mathcal{J}_\theta-m_s(1+\nu_1R_1+\nu_2R_2))\int^t\omega_c(t')dt'}F(r,\theta,t), \tag{24}
$$

which simplifies Eq. (23) to:

$$
\left\{\frac{\mathbf{P}^2}{2m(t)}+\frac{\delta(\delta-1)-2\nu_1\nu_2(1-R_1R_2)}{2m(t)r^2}+\frac{m(t)}{2}\Omega^2(t)r^2\right\}
$$

$$
F(r,\theta,t)=i\frac{\partial}{\partial t}F(r,\theta,t), \tag{25}
$$

where

$$
\mathbf{P}^2=\mathcal{P}^2+\frac{\mathcal{J}_\theta^2}{r^2}, \tag{26}
$$

$$
\mathcal{P}=\frac{\hbar}{i}\left[\frac{\partial}{\partial r}+\frac{\delta}{r}\right], \tag{27}
$$

$$
\delta=\nu_1+\nu_2+\frac{1}{2}. \tag{28}
$$

We observed that the time-dependent Dunkl–Pauli equation mimics the time-dependent Dunkl– Shrödinger equation.

3. Lewis–Riesenfeld invariant approach

To obtain an exact solution, we utilize the Lewis-Riesenfeld invariant approach [103], which requires that the Hamiltonian of the system and the invariant $I(t)$ satisfy the Lewis–Riesenfeld invariant equation:

$$\frac{dI(t)}{dt} = \frac{\partial I(t)}{\partial t} + \frac{1}{i}[I(t), \tilde{H}(t)] = 0, \tag{29}$$

where the Hamiltonian is given by on the left-side of Eq. (25) as:

$$\tilde{H}(t) = \frac{\mathbf{P}^2}{2m(t)} + \frac{\delta(\delta-1) - 2v_1v_2(1-R_1R_2)}{2m(t)r^2} + \frac{m(t)}{2}\Omega^2(t)r^2. \tag{30}$$

According to this approach the solution of Eq. (25) can be related to the solution of the eigenvalue problem for the invariant:

$$I(t)\mathcal{F}(r,\theta,t) = \varepsilon\mathcal{F}(r,\theta,t), \tag{31}$$

with the associated phase factor given by

$$F(r,\theta,t) = e^{i\eta(t)}\mathcal{F}(r,\theta,t), \tag{32}$$

where the phase $\eta(t)$ is determined by:

$$\frac{d}{dt}\eta(t) = \left\langle \mathcal{F}(r,\theta,t)\middle| i\frac{\partial}{\partial t} - \tilde{H}\middle|\mathcal{F}(r,\theta,t)\right\rangle. \tag{33}$$

To find the exact invariants for the system, we define three generators T_1, T_2 and T_3 from the Lie algebra of the group SL(2,R), which satisfy the following commutation relations:

$$[T_1, T_2] = -2i\hbar T_3, \quad [T_2, T_3] = 4i\hbar T_2, \quad [T_1, T_3] = -4i\hbar T_1. \tag{34}$$

The invariant is then constructed using the ansatz:

$$I(t) = \frac{1}{2}\left(\alpha T_1 + \beta T_2 + \gamma T_3\right). \tag{35}$$

By applying Eq. (29), we obtain a set of coupled equations whose solutions are expressed in terms of real time-dependent functions α, β, and γ:

$$\begin{cases} \alpha = \rho^2, \\ \beta = \dfrac{1}{\rho^2} + m^2\dot{\rho}^2, \\ \gamma = -m\rho\dot{\rho}. \end{cases} \tag{36}$$

Thus, the invariant can be explicitly written as:

$$I = \frac{1}{2}\left[\left(\frac{1}{\rho^2} + m^2\dot{\rho}^2\right)r^2 + \rho^2 \left(\mathbf{P}^2 + \frac{\delta(\delta-1) - 2v_1v_2(1-R_1R_2)}{r^2}\right) - \rho\dot{\rho}m(r\mathcal{P} + \mathcal{P}r)\right], \tag{37}$$

where ρ obeys the Ermakov–Pinney equation [104]:

$$\ddot{\rho} + \frac{\dot{m}}{m}\dot{\rho} + \Omega^2\rho = \frac{1}{m^2\rho^3}. \tag{38}$$

following unitary transformation:

$$\mathcal{F}(r,\theta,t) = U(r)\mathcal{G}(r,\theta), \tag{39}$$

with the unitary operator

$$U(r) = \exp\left(\frac{im\dot{\rho}}{2\rho}r^2\right). \tag{40}$$

This transformation modifies the the operator I into I', according to the relation:

$$I' = U^{+}IU = \frac{1}{2}\left[\rho^2\left(\mathbf{P}^2 + \frac{\delta(\delta-1) - 2\nu_1\nu_2(1 - R_1R_2)}{r^2}\right) + \frac{1}{\rho^2}r^2\right]. \tag{41}$$

Consequently, Eq. (31) maps to:

$$I'(t)\mathcal{G}(r,\theta) = \varepsilon\mathcal{G}(r,\theta). \tag{42}$$

Assuming

$$\mathcal{G}(r,\theta) = r^{-\delta}\mathcal{Q}(r)\Theta_\epsilon(\theta), \tag{43}$$

the wave equation given in Eq. (42) separates into two differential equations:

$$\mathcal{J}_\theta\Theta_\epsilon(\theta) = \lambda_\epsilon\Theta_\epsilon(\theta), \tag{44}$$

$$\left[\frac{\partial^2}{\partial\varkappa^2} - \varkappa^2 - \frac{(\sigma_l^\epsilon)^2 - \frac{1}{4}}{\varkappa^2} + 2\varepsilon_{n,l}^\epsilon\right]\mathcal{Q}(\varkappa) = 0, \tag{45}$$

in terms of the new variable $\varkappa = \frac{r}{\rho}$, and

$$\sigma_l^\epsilon = \sqrt{\lambda_\epsilon^2 + (\nu_1 + \epsilon\nu_2)^2}. \tag{46}$$

Here, $\epsilon = \epsilon_1\epsilon_2 = \pm 1$, with each $\epsilon_i = \pm 1$, representing the eigenvalues associated with the reflection operators R_i. The stationary Schrödinger equation (45) corresponds to the de Alfaro-Fubini-Furlan model (AFF), which includes a confining harmonic potential term, sometimes referred to as an isotonic oscillator [105–109]. It is described by the Hamiltonian operator:

$$H_{\upsilon_l^\epsilon} = -\frac{d^2}{d\varkappa^2} + \varkappa^2 + \frac{\upsilon_l^\epsilon}{\varkappa^2}, \tag{47}$$

where $\upsilon_l^\epsilon = (\sigma_l^\epsilon)^2 - \frac{1}{4}$. The Hamiltonian (47) serves as the compact generator of the dynamical conformal symmetry of the AFF model. Together with the second-order differential operators:

$$\mathcal{C}_{\upsilon_l^\epsilon}^- = -\left(\frac{d}{d\varkappa} + \varkappa\right)^2 + \frac{\upsilon_l^\epsilon}{\varkappa^2}; \tag{48}$$

$$\mathcal{C}_{\upsilon_l^\epsilon}^+ = -\left(-\frac{d}{d\varkappa} + \varkappa\right)^2 + \frac{\upsilon_l^\epsilon}{\varkappa^2}, \tag{49}$$

the system satisfies the commutation relations of the $sl(2, R)$ algebra

$$\left[H_{\upsilon_l^\epsilon}, \mathcal{C}_{\upsilon_l^\epsilon}^\pm\right] = \pm 4\mathcal{C}_{\upsilon_l^\epsilon}^\pm, \qquad \left[\mathcal{C}_{\upsilon_l^\epsilon}^-, \mathcal{C}_{\upsilon_l^\epsilon}^+\right] = 8H_{\upsilon_l^\epsilon}. \tag{50}$$

The operators $\mathcal{C}_{\upsilon_l^\epsilon}^\pm$ act as the ladder operators of the quantum AFF system.

- **First case**, $\epsilon = 1$**:**
  This case corresponds to $\epsilon_1 = \epsilon_2 = 1$ or $\epsilon_1 = \epsilon_2 = -1$ subcases, and the solution for $\Theta_{+1}(\theta)$ are given in [90] as follows:

$$\begin{aligned}\Theta_{+1}(\theta) = a_l\mathbf{P}_l^{(\nu_1+1/2,\nu_2+1/2)}(-2\cos\theta) \\ \pm \acute{a}_l \sin\theta\cos\theta\mathbf{P}_{l-1}^{(\nu_1+1/2,\nu_2+1/2)}(-2\cos\theta),\end{aligned} \tag{51}$$

where $\mathbf{P}_l^{(\alpha,\beta)}$ are Jacobi polynomials, the parameters a_l and $\acute{a}_l$ are normalization factors, and the eigenvalues are:

$$\lambda_+ = \pm 2\sqrt{l(l+\nu_1+\nu_2)}, \qquad l \in \{1,2,3,\cdots\}. \tag{52}$$

- **Second case,** $\epsilon = -1$**:**
  This case involves two sub-cases: $(\epsilon_1, \epsilon_2) = (+1, -1)$ or $(\epsilon_1, \epsilon_2) = (-1, +1)$. Here, the angular eigenfunction $\Theta_{-1}(\theta)$ is given by:

$$\begin{aligned}\Theta_{-1}(\theta) = b_l\mathbf{P}_{l-1/2}^{(\nu_1+1/2,\nu_2-1/2)}(-2\cos\theta) \\ \pm \acute{b}_l \sin\theta\mathbf{P}_{l-1/2}^{(\nu_1-1/2,\nu_2+1/2)}(-2\cos\theta).\end{aligned} \tag{53}$$

The parameters B_l and $\acute{B}_l$ are normalization factors, and the corresponding eigenvalues are:

$$\lambda_- = \pm 2\sqrt{(l+\nu_1)(l+\nu_2)}, \; l \in \{1/2, 3/2, 5/2, \cdots\}. \tag{54}$$

Subsequently, the solution of Eq. (45) is given by:

$$\mathcal{Q}(\varkappa) = \mathcal{C}_r \varkappa^{\sigma_l^\epsilon+\frac{1}{2}} e^{-\frac{\varkappa^2}{2}} L_n^{\sigma_l^\epsilon}\left(\varkappa^2\right), \tag{55}$$

where the eigenvalues of the invariant are

$$\varepsilon_{n,l}^{\epsilon} = 2n + \sigma_l^\epsilon + 1. \qquad n = 0, 1, \cdots. \tag{56}$$

Here, $\mathcal{C}_r$ represents the normalization constant.

4. Total quantum phase and wave function

By applying a unitary transformation and utilizing the Ermakov-Pinney equation, we simplify the expression in Eq. (33), yielding:

$$\dot{\eta}(t) = -\frac{\varepsilon_{n,l}}{m\rho^2} + \left\langle \mathcal{F}(r,\theta,t) \middle| i\frac{\partial}{\partial t} - \frac{\dot{\rho}}{2\rho}(r\mathcal{P} + \mathcal{P}r) \middle| \mathcal{F}(r,\theta,t)) \right\rangle. \tag{57}$$

It is straightforward to show that:

$$\left\langle \mathcal{F}(r,\theta,t) \middle| i\frac{\partial}{\partial t} - \frac{\dot{\rho}}{2\rho}(r\mathcal{P} + \mathcal{P}r) \middle| \mathcal{F}(r,\theta,t) \right\rangle = 0, \tag{58}$$

so the phase can be written as:

$$\eta_{l,n}^{\epsilon}(t) = -\left(2n + \sigma_l^{\epsilon} + 1\right) \int^{t} \frac{dt'}{m(t')\rho(t')^2}, \tag{59}$$

We note that the quantum phase changes varies due to the parity. Combining these results, we find that the total time-dependent wave function for Eq. (4) takes the form:

$$\begin{aligned} \psi_{n,l,m_s}^{\epsilon_1,\epsilon_2}(r,\theta,t) = \mathcal{C}_r \frac{r^{\sigma_l^{\epsilon}-\delta+\frac{1}{2}}}{\rho^{\sigma_l^{\epsilon}+\frac{1}{2}}} \exp\Bigg\{ (im\rho\dot{\rho} - 1)\frac{r^2}{2\rho^2} \\ + i\Bigg[\frac{1}{2}(\lambda_\epsilon - m_s(1 + \nu_1\epsilon_1 + \nu_2\epsilon_2)) \int^{t} \omega_c(t')dt' \\ - \left(2n + \sigma_l^{\epsilon} + 1\right) \int^{t} \frac{dt'}{m(t')\rho(t')^2}\Bigg]\Bigg\} \\ \times L_n^{\sigma_l^{\epsilon}}\left(\frac{r^2}{\rho^2}\right)\Theta_\epsilon(\theta)\chi_{m_s}. \end{aligned} \tag{60}$$

Using the total wave function and Eq. (24), we obtain the total phase factor $\mu_{l,n}^{\epsilon}(t)$ in the following form:

$$\begin{aligned} \mu_{l,n}^{\epsilon}(t) = \frac{1}{2}(\lambda_\epsilon - m_s(1 + \nu_1\epsilon_1 + \nu_2\epsilon_2)) \\ \int^{t} \omega_c(t')dt' - \left(2n + \sigma_l^{\epsilon} + 1\right) \int^{t} \frac{dt'}{m(t')\rho(t')^2}. \end{aligned} \tag{61}$$

The wave function in Eq. (60) depends on the values of ϵ, so the solutions have to be distinguished into two cases:

- **First case**, $\epsilon = 1$:

  In this case, we have $\sigma_l^+ = \sqrt{\lambda_+^2 + (v_1 + v_2)^2}$ and $\lambda_+ = \pm 2\sqrt{l(l + v_1 + v_2)}$.

  1. For $\epsilon_1 = 1$, $\epsilon_2 = 1$, the wave function is:

$$\begin{aligned}\psi^{+,+}_{n,l,m_s}(r,\theta,t) =& \mathcal{C}_r \frac{r^{\sigma_l^+ - \delta + \frac{1}{2}}}{\rho^{\sigma_l^+ + \frac{1}{2}}} \exp\Bigg\{ (im\rho\dot{\rho} - 1)\frac{r^2}{2\rho^2} \\ & + i\left[\frac{1}{2}(\lambda_+ - m_s(1 + v_1 + v_2)) \int^t \omega_c(t')dt' \right. \\ & \left. - (2n + \sigma_l^+ + 1)\int^t \frac{dt'}{m(t')\rho(t')^2}\right]\Bigg\} \\ & \times L_n^{\sigma_l^+}\left(\frac{r^2}{\rho^2}\right)\Theta_{+1}(\theta)\chi_{m_s}.\end{aligned} \tag{62}$$

2. For $\epsilon_1 = -1, \epsilon_2 = -1$, the wave function is:

$$\begin{aligned}\psi^{-,-}_{n,l,m_s}(r,\theta,t) =& \mathcal{C}_r \frac{r^{\sigma_l^+ - \delta + \frac{1}{2}}}{\rho^{\sigma_l^+ + \frac{1}{2}}} \exp\Bigg\{ (iM\rho\dot{\rho} - 1)\frac{r^2}{2\rho^2} \\ & + i\left[\frac{1}{2}(\lambda_+ - m_s(1 - v_1 - v_2)) \int^t \omega_c(t')dt' \right. \\ & \left. - (2n + \sigma_l^+ + 1)\int_0^t \frac{dt'}{M(t')\rho(t')^2}\right]\Bigg\} \\ & \times L_n^{\sigma_l^+}\left(\frac{r^2}{\rho^2}\right)\Theta_{+1}(\theta)\chi_{m_s}.\end{aligned} \tag{63}$$

- **Second case**, $\epsilon = -1$:

  In this case, we have $\sigma_l^- = \sqrt{\lambda_-^2 + (v_1 - v_2)^2}$ and $\lambda_- = \pm 2\sqrt{(l + v_1)(l + v_2)}$.

  1. For $\epsilon_1 = -1$, $\epsilon_2 = 1$, the wave function is:

$$\begin{aligned}\psi^{-,+}_{n,l,m_s}(r,\theta,t) =& \mathcal{C}_r \frac{r^{\sigma_l^- - \delta + \frac{1}{2}}}{\rho^{\sigma_l^- + \frac{1}{2}}} \exp\Bigg\{ (im\rho\dot{\rho} - 1)\frac{r^2}{2\rho^2} \\ & + i\left[\frac{1}{2}(\lambda_- - m_s(1 - v_1 + v_2)) \int^t \omega_c(t')dt' \right. \\ & \left. -- (2n + \sigma_l^- + 1)\int^t \frac{dt'}{m(t')\rho(t')^2}\right]\Bigg\} \\ & \times L_n^{\sigma_l^-}\left(\frac{r^2}{\rho^2}\right)\Theta_{-1}(\theta)\chi_{m_s}.\end{aligned} \tag{64}$$

2. For $\epsilon_1 = 1$, $\epsilon_2 = -1$, the wave function is:

$$\begin{aligned}\psi^{+,-}_{n,l,m_s}(r,\theta,t) =& \mathcal{C}_r \frac{r^{\sigma_l^- - \delta + \frac{1}{2}}}{\rho^{\sigma_l^- + \frac{1}{2}}} \exp\Bigg\{ (iM\rho\dot{\rho} - 1)\frac{r^2}{2\rho^2} \\ & + i\left[\frac{1}{2}(\lambda_- - m_s(1 + v_1 - v_2)) \int^t \omega_c(t')dt' \right. \\ & \left. -- (2n + \sigma_l^- + 1)\int^t \frac{dt'}{M(t')\rho(t')^2}\right]\Bigg\} \\ & \times L_n^{\sigma_l^-}\left(\frac{r^2}{\rho^2}\right)\Theta_{-1}(\theta)\chi_{m_s}.\end{aligned} \tag{65}$$

Two limiting cases:

For the time-independent case, where

$$\omega_c = C^{st}, \text{ and } m = m_0, \tag{66}$$

we take $\omega(t) \to 0$. Thus, we obtain:

$$\Omega = \frac{\omega_c}{2}, \text{ and } \rho = \frac{1}{\sqrt{\frac{1}{2} m_0 \omega_c}}. \tag{67}$$

Then, the total phase $\mu_{l,n}^{\epsilon}(t)$ becomes the same with the energy spectrum $E_{n,l}^{\epsilon}$ of the stationary Dunkl–Pauli oscillator equation

$$\mu_{l,n}^{\epsilon}(t) = -E_{n,l}^{\epsilon} t, \tag{68}$$

with

$$E_{n,l}^{\epsilon} = \frac{\omega_c}{2}\left[2n + \sigma_l^{\epsilon} + 1 - \lambda_\epsilon + m_s(1 + \nu_1\epsilon_1 + \nu_2\epsilon_2)\right]. \tag{69}$$

This energy spectrum result depends on the eigenvalues ϵ_1 and ϵ_2 of the reflection operators R_1 and R_2, which aligns with the findings presented in reference [90].

Now, let us consider the isotropic two-dimensional Dunkl operators, with the anisotropic oscillator potential

$$V(x, y) = \frac{1}{2} m(t)\left[\omega_x^2(t) x^2 + \omega_y^2(t) y^2\right]. \tag{70}$$

In this case, the time-dependent Dunkl–Pauli oscillator equation can be expressed as:

$$\Big[-\frac{1}{2m(t)}\frac{\partial^2}{\partial r^2} - \frac{1+4\nu}{2m(t)r}\frac{\partial}{\partial r} + \frac{m(t)}{2}\Omega_\theta^2(t) r^2 + \frac{\mathcal{J}_\theta^2 - 2\nu^2(1 - R_1R_2)}{2m(t)r^2} + \frac{\omega_c(t)}{2}\left(\mathcal{J}_\theta - m_s(1 + \nu(R_1 + R_2))\right)\Big]\phi(r,\theta,t) = i\frac{\partial}{\partial t}\phi(r,\theta,t), \tag{71}$$

where

$$\Omega_\theta^2(t) = \omega_x^2(t)\cos^2\theta + \omega_y^2(t)\sin^2\theta + \frac{\omega_c^2(t)}{4}. \tag{72}$$

The resolution of this problem requires further investigation and will be addressed in future research.

5. Dunkl–Caldirola–Kanai oscillator

the time-dependent mass is given by:

$$m(t) = M_0 e^{kt}. \tag{73}$$

Here, M_0 is the constant mass, k is the constant damping coefficient and $B(t) = B_0 e^{kt}$ is the chosen function. Under these conditions, the Dunkl–Caldirola–Kanai Hamiltonian for the system takes the form:

$$\begin{aligned} H = & -\frac{e^{-kt}}{2M_0}\frac{\partial^2}{\partial r^2} - e^{-kt}\frac{1+2\nu_1+2\nu_2}{2M_0 r}\frac{\partial}{\partial r} + \frac{M_0 e^{kt}}{2}\Omega_0^2 r^2 \\ & + e^{-kt}\frac{\mathcal{J}_\theta^2 - 2\nu_1\nu_2(1-R_1R_2)}{2M_0 r^2} \\ & + \frac{\omega_{0c}}{2}\Big(\mathcal{J}_\theta - m_s(1+\nu_1R_1+\nu_2R_2)\Big), \end{aligned} \tag{74}$$

where

$$\omega_{0c} = \frac{eB_0}{cm_0}, \qquad \Omega_0^2 = \frac{\omega_{0c}^2 - k^2}{4} + \omega_0^2, \qquad \omega(t) = \omega_0. \tag{75}$$

(75)

In this case, the solution of the Ermakov-Pinney equation is given by:

$$\rho(t) = \frac{e^{-\frac{k}{2}t}}{\sqrt{M_0\Omega_0}}, \tag{76}$$

and the corresponding total phase is expressed as:

$$\mu_{l,n}^{\epsilon} = \left[\frac{1}{2}\omega_{0c}(\lambda_\epsilon - m_s(1+\nu_1\epsilon_1+\nu_2\epsilon_2)) - \Omega_0\left(2n+\sigma_l^\epsilon+1\right)\right]t. \tag{77}$$

Using the above result, the wave functions for the Dunkl–Caldirola–Kanai oscillator is derived as :

$$\begin{aligned} \psi_{n,l,m_s}^{\epsilon}(r,\theta,t) = & \mathcal{C}_r (M_0\Omega_0)^{\frac{1}{2}\left(\sigma_l^\epsilon+\frac{1}{2}\right)} r^{\sigma_l^\epsilon-\delta+\frac{1}{2}} \\ & \exp\left\{-\left(1+i\frac{k}{2\Omega_0}\right)\frac{M_0\Omega_0 e^{kt} r^2}{2}\right. \\ & +\left[i\left[\frac{\omega_{0c}}{2}(\lambda_\epsilon - m_s(1+\nu_1\epsilon_1+\nu_2\epsilon_2))\right.\right. \\ & \left.\left.\left.-\Omega_0\left(2n+\sigma_l^\epsilon+1\right)\right] + \frac{k}{2}\left(\sigma_l^\epsilon+\frac{1}{2}\right)\right]t\right\} \\ & \times L_n^{\sigma_l^\epsilon}\left(M_0\Omega_0 e^{kt} r^2\right)\Theta_\epsilon(\theta)\chi_{m_s}. \end{aligned} \tag{78}$$

It is worth noting that the general solution for the $\rho(t)$ parameter, given by Eq. (76), is independent of both parity and Wigner deformation parameters. Figure 1 illustrates the time evolution of $\rho(t)$ for different values of the damping coefficient k.

The figure exhibits an exponential decay with the decay rate increasing as k increases, as predicted by Eq. (76). The black solid curve ($k = 1/2$) shows the slowest decay, while the orange dashed curve ($k = 2$) decays the fastest. This behavior reflects the impact of damping: larger values of k correspond to stronger dissipation, causing $\rho(t)$ to decrease more rapidly.

In comparison to the standard Caldirola-Kanai oscillator, the Dunkl–Caldirola–Kanai system introduces additional parameters such as reflection operators and Wigner parameters, which modify the overall dynamics. However,

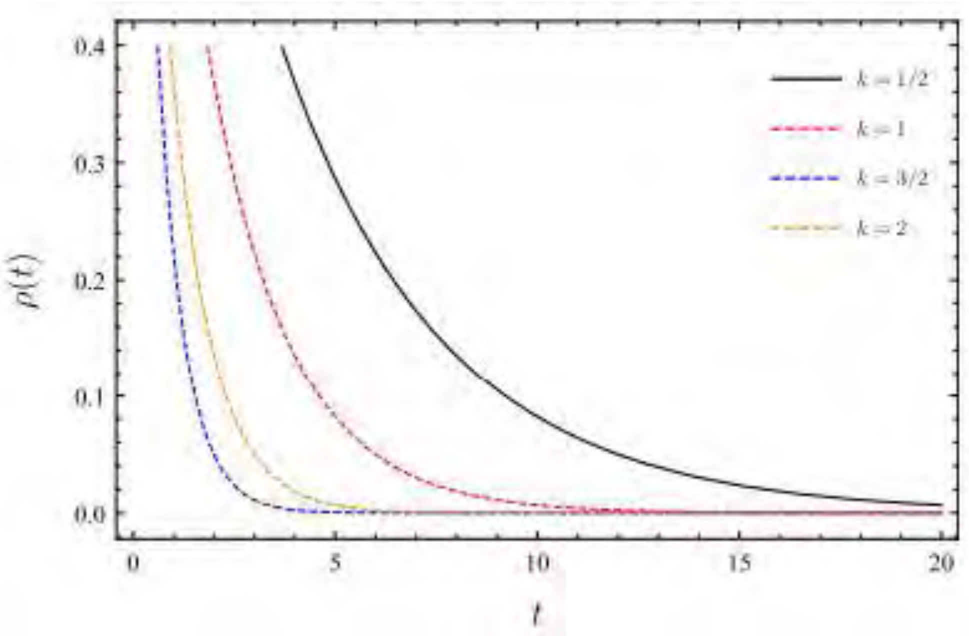

Fig. 1 Time evaluation of $\rho(t)$ for different values of the damping coefficient k where $\Omega_0^2 = M_0 = 1$

the general dissipative behavior remains similar, with $\rho(t)$ approaching zero as $t \to \infty$, indicating the suppression of oscillations over time. The primary distinction lies in the presence of parity-dependent modifications, which can lead to subtle deviations in the decay profile depending on the chosen parameters.

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One-dimensional Dunkl quantum mechanics: a path integral approach

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Abstract

In the present manuscript, we employ the Feynman path integral method to derive the propagator in one-dimensional Wigner-Dunkl quantum mechanics. To verify our findings we calculate the propagator associated with the free particle and the harmonic oscillator in the presence of the Dunkl derivative. We also deduce the energy spectra and the corresponding bound-state wave functions from the spectral decomposition of the propagator.

2. Quantum mechanics with Dunkl derivative

The WDQM, which we consider in this paper, is defined by the following commutation relation between the position and momentum operators,

$$[\hat{x}, \hat{p}] = i\hbar\,(1 + 2\nu R),$$

where $\hbar$ is Planck's reduced constant, ν is the deformation parameter and is sometimes called the Wigner parameter in reference to historical connection. Here, R denotes the reflection or parity operator defined as [34, 69],

$$R = (-1)^{x\frac{d}{dx}}, \qquad R^2 = 1.$$

The action of the reflection operator on such function $g(x)$ it rotates around the origin,

$$Rg(x) = g(-x).$$

$$\hat{p} = \frac{\hbar}{i} D_x, \quad \hat{x} = x,$$

where D_x is the Dunkl derivative, defined as,

$$D_x = \frac{d}{dx} + \frac{\nu}{x}(1 - R).$$

inner product in the Hilbert space associated with one-dimensional WDQM is defined by,

$$\langle \Phi | \varphi \rangle = \int_{-\infty}^{\infty} \Phi^*(x) \varphi(x) |x|^{2\nu} dx, \qquad \nu > -\frac{1}{2}, \tag{6}$$

where $|x|^{2\nu}$ is a weight function. Under these assumptions, the completeness and orthogonality relations can be extended as follows:

$$\int_{-\infty}^{\infty} dx |x|^{2\nu} |x\rangle \langle x| = 1, \tag{7}$$

$$\langle u | x \rangle = \frac{1}{|x|^{2\nu}} \delta(x - u). \tag{8}$$

Now, let us introduce a new momentum operator $\hat{\mathcal{P}}$ [70–73],

$$\hat{\mathcal{P}} = \frac{\hbar}{i}\left(\frac{\partial}{\partial x} + \frac{\nu}{x}\right), \tag{9}$$

whose action on ket $|\mathcal{P}\rangle$ is,

$$\hat{\mathcal{P}}|\mathcal{P}\rangle = \mathcal{P}|\mathcal{P}\rangle;\ [\hat{x}, \hat{\mathcal{P}}] = i\hbar. \tag{10}$$

Then, on the coordinate space the eigenfunction for the new momentum operator, $\hat{\mathcal{P}}$, takes the following form,

$$\langle x|\ \hat{\mathcal{P}}|\mathcal{P}\rangle = \mathcal{P}\langle x|\ \mathcal{P}\rangle = \frac{\hbar}{i}\left(\frac{\partial}{\partial x} + \frac{\nu}{x}\right)\langle x|\ \mathcal{P}\rangle, \tag{11}$$

and it can be solved to obtain formal momentum eigenvectors,

$$\langle |x||\ \mathcal{P}\rangle = \frac{1}{\sqrt{2\pi\hbar}}\ |x|^{-\nu} e^{\frac{i}{\hbar}\mathcal{P}\ \ |x|} = \frac{1}{\sqrt{2\pi\hbar}} y^{-\nu} e^{\frac{i}{\hbar}\mathcal{P}\ \ y}, \quad \text{with } y = |x|. \tag{12}$$

In light of this, the identity operator for the eigenstates of the latter momentum operator reads:

$$\int_{-\infty}^{\infty} d\mathcal{P}|\mathcal{P}\rangle\langle\mathcal{P}| = 1. \tag{13}$$

We would like to emphasize that the objective of introducing the momentum operator, $\hat{\mathcal{P}}$, is to express the Dunkl-Hamiltonian in a familiar form that would allow us to investigate non-relativistic WDQM problems using the Feynman's path integral method.

3. Propagator in one-dimensional Wigner-Dunkl quantum mechanics

In a non-relativistic quantum mechanical system, described by a Hamiltonian $\hat{H}$, the propagator or the transition amplitude from an initial state, $|x_a\rangle$, to a final state, $|x_b\rangle$, can be determined by the solution of the following equation,

$$\left(\hat{H} - i\hbar\frac{\partial}{\partial t_b}\right)K(x_b, t_b; x_a, t_a) = i\hbar\delta(x_b - x_a)\delta(t_b - t_a). \tag{14}$$

In the context of WDQM, it is necessary to replace the conventional Hamiltonian with the Dunkl-Hamiltonian, which takes the following form:

$$\hat{H} = -\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial x^2} + \frac{2\nu}{x}\frac{\partial}{\partial x} - \frac{\nu}{x^2}(1 - R)\right] + V(x), \tag{15}$$

where m denotes the rest mass. It is noteworthy that the Dunkl-Hamiltonian operator differs from the conventional one by incorporating two additional terms The first of these terms includes the first derivative with respect to the coordinate variable, while the second term depends on the reflection operator and distinguishes the odd parity solutions.

Before formulating the propagator using Feynman path integrals in one-dimensional WDQM, we want to express the Dunkl-Hamiltonian in a form similar to that used in traditional quantum mechanics. For this purpose, using the Dunkl-momentum operator, we rewrite the Dunkl-Hamiltonian operator as follows:

$$\hat{H} = \frac{\hat{\mathcal{P}}^2}{2m} + \frac{\hbar^2 \nu}{2mx^2}(\nu - R) + V(x). \tag{16}$$

Subsequently, to eliminate the reflection operator, we introduce a regulation function

$$f(x) = f_l(x) f_r(x), \tag{17}$$

that specifies the parity

$$R f_r(x) = s f_r(x), \qquad \text{with} \quad s = \pm 1. \tag{18}$$

Then, using the Duru-Kleinert method [74], we obtain a new form for the Hamiltonian,

$$\hat{H}_s^E = f_l(x)(\hat{H}_s - E)f_r(x), \tag{19}$$

with the fixed-energy amplitude,

$$G_s(x_b, x_a, E) = \left\langle x_b \,\middle|\, f_r(x)\frac{1}{f_l(x)(\hat{H}_s - E - i\epsilon)f_r(x)}f_l(x) \,\middle|\, x_a \right\rangle = \frac{i}{\hbar}\int_0^\infty dT f_r(x_b)f_l(x_a)K_s^E(x_b, x_a; T), \tag{20}$$

where the Feynman kernel $K_s^E(x_b, x_a; T)$ is defined by [75]:

$$K_s^E(x_b, x_a; T) = \langle x_b|\, e^{-\frac{i}{\hbar}\hat{H}_s^E T}|x_a\rangle. \tag{21}$$

To obtain a path integral representation for $K_s^E(x_b, x_a; T)$, we use the standard method of discrediting the time interval T into $N + 1$ infinitesimal equal parts,

$$\varepsilon = \frac{T}{N + 1} = \tau f_l(x_j)f_r(x_{j-1}). \tag{22}$$

We then apply the Trotter's formula, $e^{-\frac{i}{\hbar}\hat{H}_s^E T} = [e^{-\frac{i}{\hbar}\varepsilon\hat{H}_s^E}]^{N+1}$ and insert the closure relations (7) and (13) between each pair of exponential in order to eliminate the operators $\hat{x}$ and $\hat{\mathcal{P}}$. With the help of the scalar product,

$$\langle |x| | \mathcal{P}\rangle = \frac{1}{\sqrt{2\pi\hbar}} |x|^{-\nu} e^{\frac{i}{\hbar}\mathcal{P} \ |x|}, \tag{23}$$

which allows it to pass from one base to another one, we obtain the kernel, K_q^E, in the following discrete form,

$$\begin{aligned} K_s^E(y_b, y_a; T) = \lim_{N\to\infty} \int \prod_{j=1}^{N} y_j^{2\nu} dy_j \prod_{n=1}^{N+1} \frac{d\mathcal{P}_j}{2\pi\hbar\,(y_j y_{j-1})^{\nu}} \exp \Bigg\{ \frac{i}{\hbar} \sum_{j=1}^{N} [\mathcal{P}_j(y_j - y_{j-1}) \\ - \tau f_l(y_j) f_r(y_{j-1}) \left(\frac{\mathcal{P}_j^2}{2m} + \frac{\hbar^2\left(\lambda_s^2 - \frac{1}{4}\right)}{y_j^2} + V(y_j) - E \right) \Bigg] \Bigg\}, \end{aligned} \tag{24}$$

where $y_a = y_0, y_b = y_{N+1}, t_a = t_0, t_b = t_{N+1}$ and,

$$\lambda_s = \nu - \frac{s}{2}. \tag{25}$$

For the odd case we have $\lambda_{-1} = \nu + \frac{1}{2}$ and in the even case $\lambda_{+1} = \nu - \frac{1}{2}$. By establishing a symmetrical measure, For the odd case we have $\lambda_{-1} = \nu + \frac{1}{2}$ and in the even case $\lambda_{+1} = \nu - \frac{1}{2}$. By establishing a symmetrical measure,

$$\prod_{j=1}^{N} y_j^{2\nu} = \frac{1}{(y_a y_b)^\nu} \prod_{j=1}^{N+1} (y_j y_{j-1})^\nu, \tag{26}$$

the partial kernel then becomes,

$$K_s^E = \lim_{N\to\infty} \frac{1}{(y_a y_b)^\nu} \int \prod_{j=1}^{N} dy_j \prod_{n=1}^{N+1} \frac{d\mathcal{P}_j}{2\pi\hbar} \exp \left\{ \frac{i}{\hbar} \sum_{j=1}^{N} [\mathcal{P}_j(y_j - y_{j-1}) \right.$$
$$\left. - \tau f_l(y_j) f_r(y_{j-1}) \left(\frac{\mathcal{P}_j^2}{2m} + \frac{\hbar^2 \left(\lambda_s^2 - \frac{1}{4}\right)}{y_j^2} + V(y_j) - E \right) \right] \Bigg\}, \tag{27}$$

After evaluating the integrals with respect to $\mathcal{P}_j$, we arrive at,

$$K_s^E = \lim_{N\to\infty} \frac{1}{(y_a y_b)^\nu} \sqrt{\frac{m}{2\pi i\tau f_l(y_b) f_r(y_a)\hbar}} \int \prod_{j=1}^{N} \left[\sqrt{\frac{m}{2\pi i\tau f(y_j)\hbar}}\, dy_j \right]$$
$$\times \exp\left\{ \frac{i}{\hbar} \sum_{j=1}^{N} \left[\frac{m(\Delta y_j)^2}{2\tau f_l(y_j) f_r(y_{j-1})} - \tau f_l(y_j) f_r(y_{j-1}) \left(\frac{\hbar^2\left(\lambda_s^2 - \frac{1}{4}\right)}{y_j^2} + V(y_j) - E \right) \right] \right\}. \tag{28}$$

As long as the condition $f(y) = f_l(y) f_r(y)$ is satisfied, the functions $f_l(y)$ and $f_r(y)$ can be chosen arbitrarily [75]. Thus, we can adopt Kleinert's approach by assuming $f_l(y) \equiv f_r(y) \equiv 1$. Then, the Feynman kernel $K_s^E(x_b, x_a; T)$ reads:

$$K_s^E(y_b, y_a; T) = e^{\frac{i}{\hbar}ET} K_s(y_b, y_a; T), \tag{29}$$

with

$$K_s(y_b, y_a; T) = \frac{1}{(y_a y_b)^\nu} \int Dy \exp\left[\frac{i}{\hbar} \int_0^T dt \left[\frac{m}{2}\dot{y}^2 - V_{eff}(y) \right] \right], \tag{30}$$

$$V_{eff}(y) = \frac{\hbar^2\left(\lambda_s^2 - \frac{1}{4}\right)}{2my^2} + V(y). \tag{31}$$

Here, we note that the obtained kernel, given in equation (31), is the same as the conventional Feynman path integral of the standard quantum mechanics, except with two terms These terms are the prefactor $(y_a y_b)^{-\nu}$ and the additional repulsive singular potential term, $\frac{\hbar^2}{2my^2}\left(\lambda_s^2 - \frac{1}{4}\right)$, which exists in the effective potential. We have to underline the fact that the effective potential, given in (30), differs from Junker's findings [67]. In addition, due to the effective potential the time evolution amplitude possesses only a complicated time-sliced path integral involving Bessel functions. According to [75], the effective potential requires the regularization,

$$\varepsilon\hbar^2\frac{\lambda_s^2 - \frac{1}{4}}{2my_j^2} \to i\hbar \log I_{\lambda_s}\left(\frac{m}{i\varepsilon\hbar} y_j y_{j-1}\right). \tag{32}$$

To be more precise, in the current form, the kernel propagator is split into two parity variants, even and odd, designated by the eigenvalues s. Here, the even parity case propagator:

$$K_+(y_b, y_a; T) = \frac{1}{(y_a y_b)^\nu} \int Dy \exp\left\{ \frac{i}{\hbar} \int_0^T dt \left[\frac{m}{2}\dot{y}^2 - \frac{\hbar^2\left(\alpha^2 - \frac{1}{4}\right)}{2my^2} - V(y) \right] \right\}, \tag{33}$$

and the odd parity case propagator:

$$K_-(y_b, y_a; T) = \frac{1}{(y_a y_b)^\nu} \int Dy \exp\left\{ \frac{i}{\hbar} \int_0^T dt \left[\frac{m}{2}\dot{y}^2 - \frac{\hbar^2\left(\beta^2 - \frac{1}{4}\right)}{2my^2} - V(y) \right] \right\}, \tag{34}$$

stand with,

$$\alpha = \nu - \frac{1}{2}, \quad \beta = \nu + \frac{1}{2}. \tag{35}$$

Taking into account (12), (33) and (34), we can write the Dunkl path integral kernel as a sum of even and odd kernel propagators as follows [67]:

$$K(x_b, x_a; T) = K_+(x_b, x_a; T) + sgn(x_b x_a) K_-(x_b, x_a; T), \tag{36}$$

where $sgn(x)$ represents the sign function. Please note that if the Hamiltonian (15) has a complete set of eigenfunctions $\Psi_{n,s}$ associated with eigenvalues E_n^s, we then can apply the following spectral decomposition:

$$K(x_b, x_a; T) = \sum_n \{[\Psi_{n,+1}(x_b)\Psi^*_{n,+1}(x_a)]e^{-\frac{i}{\hbar}E_n^+T} + [\Psi_{n,-1}(x_b)\Psi^*_{n,-1}(x_a)]e^{-\frac{i}{\hbar}E_n^-T}\}, \tag{37}$$

where E_n^+ and E_n^- are the energy eigenvalues related to the even and odd wave functions, respectively.

4. Applications

To demonstrate the efficacy of our method, we apply it to two fundamental problems: the free particle case and the harmonic oscillator.

4.1. Free particle

This section aims to derive the Dunkl kernel propagator for the free particle scenario. In one dimension, with $V(x) = 0$, we express the Dunkl kernel propagator via (36) as follows:

$$K(x_b, x_a; T) = \frac{1}{|x_a x_b|^{\nu}} \int D|x| \exp\left\{\frac{i}{\hbar}\int_0^T dt\left[\frac{m}{2}\dot{x}^2 - \frac{\hbar^2\left(\alpha^2 - \frac{1}{4}\right)}{2mx^2}\right]\right\}$$
$$+ \frac{1}{|x_a x_b|^{\nu}} sng(x_a x_b) \int D|x| \exp\left\{\frac{i}{\hbar}\int_0^T dt\left[\frac{m}{2}\dot{x}^2 - \frac{\hbar^2\left(\beta^2 - \frac{1}{4}\right)}{2mx^2}\right]\right\}. \quad (38)$$

We observe that the latter propagator is identical to the propagator of a free particle moving in two dimensions, as referenced in [71]. Consequently, we express the final form of the propagator as follows:

$$K(x_b, x_a; T) = \frac{1}{|x_a x_b|^{\nu-\frac{1}{2}}} \frac{m}{2i\hbar T} \exp\left[\frac{im}{2\hbar T}(x_a^2 + x_b^2)\right]\left\{I_{\nu-\frac{1}{2}}\left(\frac{m|x_a x_b|}{i\hbar T}\right) + sng\,(x_a x_b) I_{\nu+\frac{1}{2}}\left(\frac{m|x_a x_b|}{i\hbar T}\right)\right\}, \tag{39}$$

where $I_\mu(x)$ is the modified Bessel functions of the first kind. Then, we use the relation between $I_\mu(x)$ and deformed exponential function $E_\mu(x)$,

$$E_\nu(x) = \Gamma\left(\nu + \frac{1}{2}\right)\left(\frac{2}{|x|}\right)^{\nu-\frac{1}{2}}\left(I_{\nu-\frac{1}{2}}(|x|) + sng\,(x) I_{\nu+\frac{1}{2}}(|x|)\right), \tag{40}$$

which results in,

$$K(x_b, x_a; T) = \frac{1}{\Gamma(\nu + 1/2)}\left(\frac{m}{2i\hbar T}\right)^{\nu+\frac{1}{2}} \exp\left[\frac{im}{2\hbar T}(x_a^2 + x_b^2)\right] E_\nu\left(\frac{m x_a x_b}{i\hbar T}\right). \tag{41}$$

It is important to underline that (41) is the same as equation (52) in [67].

4.2. Harmonic oscillator

We now examine the harmonic potential problem. In this case, with $V(x) = \frac{1}{2}m\omega^2x^2$, the kernel propagator reduces to:

$$K(x_b, x_a; T) = \frac{1}{|x_a x_b|^\nu}\int D|x|\exp\left\{\frac{i}{\hbar}\int_0^T dt\left[\frac{m}{2}\dot{x}^2 - \frac{1}{2}m\omega^2x^2 - \frac{\hbar^2\left(\alpha^2 - \frac{1}{4}\right)}{2mx^2}\right]\right\}$$
$$+ \frac{1}{|x_a x_b|^\nu} sng(x_a x_b)\int D|x|\exp\left\{\frac{i}{\hbar}\int_0^T dt\left[\frac{m}{2}\dot{x}^2 - \frac{1}{2}m\omega^2x^2 - \frac{\hbar^2\left(\beta^2 - \frac{1}{4}\right)}{2mx^2}\right]\right\}, \qquad (42)$$

where ω is the frequency of the oscillator. This corresponds in fact to the d-dimensional isotropic harmonic oscillator that is perturbed by an inverse square potential. Using the Hille-Hardy formula:

$$\sum_{n=0}^{\infty}\frac{n!L_n^\mu(x)L_n^\mu(y)z^n}{\Gamma(n+\mu+1)}e^{-\frac{x+y}{2}} = \frac{(xyz)^{-\mu/2}}{(1-z)}\exp\left[-\frac{1}{2}(x+y)\frac{1+z}{1-z}\right]I_\mu\left(\frac{2\sqrt{xyz}}{1-z}\right), \qquad (43)$$

we obtain a spectral decomposition of the kernel propagators,

$$
\begin{aligned}
K(x_b, x_a; T) = {} & \frac{1}{|x_a x_b|^{\nu-\frac{1}{2}}} \frac{m\omega}{\hbar} \sum_n \frac{n!}{\Gamma\left(n+\nu+\frac{1}{2}\right)} \left(\frac{m\omega |x_a x_b|}{\hbar}\right)^{\nu-\frac{1}{2}} \\
& \times L_n^{\nu-\frac{1}{2}}\left(\frac{m\omega}{\hbar} x_a^2\right) L_n^{\nu-\frac{1}{2}}\left(\frac{m\omega}{\hbar} x_b^2\right) \exp\left[-\frac{m\omega (x_a^2 + x_b^2)}{2\hbar} - i\omega T\left(2n+\nu+\frac{1}{2}\right)\right] \\
& + sng\,(x_a x_b) \frac{1}{|x_a x_b|^{\nu-\frac{1}{2}}} \frac{m\omega}{\hbar} \sum_n \frac{n!}{\Gamma\left(n+\nu+\frac{3}{2}\right)} \left(\frac{m\omega |x_a x_b|}{\hbar}\right)^{\nu+\frac{1}{2}} \\
& \times L_n^{\nu+\frac{1}{2}}\left(\frac{m\omega}{\hbar} x_a^2\right) L_n^{\nu+\frac{1}{2}}\left(\frac{m\omega}{\hbar} x_b^2\right) \exp\left[-\frac{m\omega (x_a^2 + x_b^2)}{2\hbar} - i\omega T\left(2n+\nu+\frac{3}{2}\right)\right].
\end{aligned}
\tag{44}
$$

After the algebra, we obtain the normalized wave functions and their energy eigenvalues, that describe the one-dimensional quantum harmonic oscillator parity-dependent dynamics as follows:

- Even parity solution:

$$\Psi_{n,\nu}^{+}(x) = \sqrt{\frac{n!}{\Gamma\left(n+\nu+\frac{1}{2}\right)}}\left(\frac{m\omega}{\hbar}\right)^{\frac{\nu}{2}+\frac{1}{4}} \exp\left[-\frac{m\omega}{2\hbar}x^2\right] L_n^{\nu-\frac{1}{2}}\left(\frac{m\omega}{\hbar}x^2\right), \tag{45}$$

$$E_n^{+} = \hbar\omega\left(2n+\nu+\frac{1}{2}\right), \qquad n = 0,\, 1,\, 2,\ldots \tag{46}$$

- Odd parity solution:

$$\Psi_{n,\nu}^{-}(x) = \sqrt{\frac{n!}{\Gamma\left(n+\nu+\frac{3}{2}\right)}}\left(\frac{m\omega}{\hbar}\right)^{\frac{\nu}{2}+\frac{3}{4}} x \exp\left[-\frac{m\omega}{2\hbar}x^2\right] L_n^{\nu+\frac{1}{2}}\left(\frac{m\omega}{\hbar}x^2\right), \tag{47}$$

$$E_n^{-} = \hbar\omega\left(2n+\nu+\frac{3}{2}\right), \qquad n = 0,\, 1,\, 2,\ldots \tag{48}$$

Before ending the manuscript, it is worth mentioning that the same topic was recently investigated by Junker in [67]. In that work, Junker did not construct the Dunkl path integral for the free-particle propagator. Instead, he directly used spectral decompositions to find the free-particle propagator and then inserted it into the path integral representation of the interacting system. He used the asymptotic relation of the deformed exponential function to derive the effective potential. As a result, he found that the expression for the propagator is identical to the usual Feynman path integral in standard quantum mechanics, with the only difference being the prefactor $\frac{1}{|x_a x_b|^{\nu}}$ and the additional repulsive singular potential $\frac{\nu^2}{mx^2}$. In our paper, we construct the path integral for WDQM, starting with the introduction of a new momentum operator $\mathcal{P}$ that satisfies the usual Heisenberg uncertainty relation. We then introduced a regulation function to eliminate the reflection operator. As a result, we found that the expression for the propagator is identical to Junker's propagator, with the only difference being the repulsive singular potential, which is dependent on the eigenvalue 's' of the reflection operator. In our paper, we follow the usual steps to find the wave functions using the path integral formalism: We begin by identifying initial and final states. We then divide the time interval into N small segments. For each time slice, we consider all possible paths. We integrate over all intermediate positions and take the limit as N approaches infinity. Next, we express the propagator in terms of energy eigenstates (spectral decomposition). In the final step, we extract wave functions and energies from this representation. We conclude that our methodology can be used efficiently to study other problems of physics, and may be extended within higher dimensions using the polar and the spherical coordinates.

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A path integral treatment of time-dependent Dunkl quantum mechanics

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Abstract

This study addresses a significant gap in the literature by extending the path integral formalism to time-dependent systems within the Wigner–Dunkl framework, deriving exact propagator solutions for analytically solvable cases. By employing generalized canonical transformations, we reformulated the path integral to develop an explicit expression for the propagator. This formalism is applied to specific cases, including a Dunkl-harmonic oscillator with time-dependent mass and frequency. Solutions for the Dunkl–Caldirola-Kanai oscillator and a model with a strongly pulsating mass are derived, providing exact propagator expressions and corresponding wave functions. These findings extend the utility of Dunkl operators in quantum mechanics, offering new insights into the dynamics of time-dependent quantum systems and possibly find application in quantum optics, plasma physics, and other fields.

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