Magnetic Dirac operator in strips

submitted to strong magnetic fields

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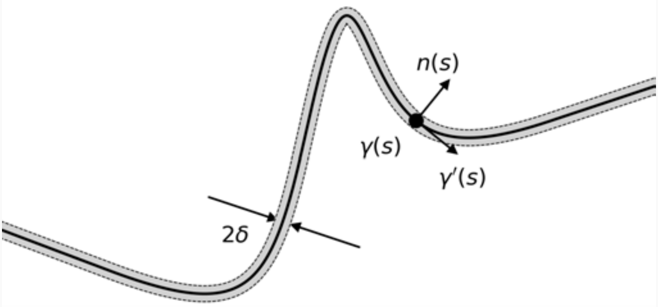
Context

The strip

Geometric objects : $\Omega = \Theta(\Omega_0)$,

- $\gamma : \mathbb{R} \to \mathbb{R}^2$ smooth, simple, straight at infinity.
- $\Theta : \Omega_0 = \mathbb{R} \times (-\delta, \delta) \ni (s, t) \mapsto \gamma(s) + t\mathbf{n}(s)$,



What about δ?

$\delta \in (0, \|\kappa\|_{L^\infty}^{-1})$. But, **no thin strip limit : $\delta \nrightarrow 0$.**

The magnetic field

Magnetic Objects

- Magnetic field : $B \equiv 1$.
- Potential field : There exists a unique $\phi \in W^{\infty,\infty}(\Omega, \mathbb{R}_-^*)$ such that

$$\begin{cases} \Delta\phi = B, & \text{on } \Omega, \\ \phi = 0, & \text{on } \partial\Omega. \end{cases}$$

Gauge Fixing

- Magnetic vector potential: $\mathbf{A} = \nabla\phi^{\perp}$.

Remarks

- The problem is gauge invariant, as Ω is simply connected.
- This choice of gauge simplifies the presentation.
- The magnetic vector potential $\mathbf{A}$ is bounded !

The magnetic Dirac operator

$$\mathscr{D}_h = \sigma \cdot (\mathbf{p} - A) = \begin{pmatrix} 0 & d_h \\ d_h^\times & 0 \end{pmatrix},$$

The Pauli Matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Cauchy-Riemann op.:

$$\partial_{\overline{z}} = \frac{\partial_1 + i\partial_2}{2}, \quad \partial_z = \frac{\partial_1 - i\partial_2}{2}.$$

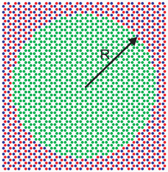
- $\mathbf{p} = -ih\nabla$, $(h > 0)$, $d_h = -2ih\partial_z - A_1 + iA_2$, $d_h^\times = -2ih\partial_{\overline{z}} - A_1 - iA_2$,

The infinite mass model :

$$\mathrm{Dom}(\mathscr{D}_h) = \{u \in H^1(\Omega; \mathbb{C}^2),\ \mathcal{B}u = u \text{ on } \partial\Omega\},$$

with

- $\mathcal{B}(s) = -i\sigma_3 \sigma \cdot \mathfrak{n}(s)$,
- $\mathfrak{n}(s)$ is the outward pointing normal at $s \in \partial\Omega$.



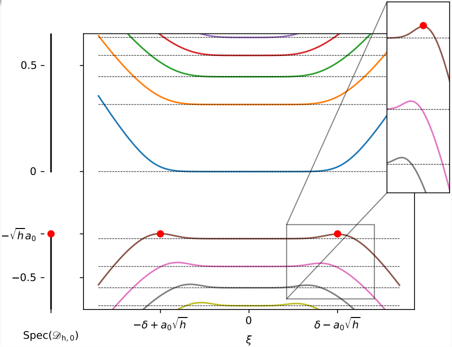
GQD in an infinite mass media. [Grujic et Al., Physical review B 84, 205441 (2011)]

The straight strip

Fibered operators

Proposition $\mathrm{sp}_{\mathrm{ess}}(\mathscr{D}_h) = \mathrm{sp}_{\mathrm{ess}}(\mathscr{D}_{h,0})$ ($\mathscr{D}_{h,0}$ the op. on Ω_0).

$$\mathscr{D}_{h,0} = \int^{\oplus} \mathscr{D}_{h,0,\xi} \mathrm{d}\xi \,, \qquad \text{Fourier.}$$



Fibered op. $(I = (-\delta, \delta))$

$$\mathscr{D}_{h,0,\xi} = (\xi + t)\sigma_1 + \sigma_2 D_t$$

$$\mathrm{Dom}(\mathscr{D}_{h,0,\xi}) = \{\psi = (\psi_1, \psi_2) \in H^1(I), \\ \psi_1(\pm\delta) = \mp\psi_2(\pm\delta)\} \,.$$

Theorem [letreust:hal-04691264]

$$\mathrm{sp}(\mathscr{D}_{h,0}) = \mathbb{R} \setminus (-\lambda^-_{\mathrm{ess}}(h), \lambda^+_{\mathrm{ess}}(h)) \,,$$

$$\lambda^+_{\mathrm{ess}}(h) = 2\sqrt{\frac{h}{\pi}} e^{-\delta^2/h}(1 + o(1)) \quad \text{and} \quad \lambda^-_{\mathrm{ess}}(h) = a_0\sqrt{h} + \mathscr{O}(h^\infty) \,,$$

for some $a_0 \in (0, \sqrt{2})$ (size of the gap on the half-space).

Positive Energy Bound States?

Main results

Assumption : Unique and non-degenerate minimum for ϕ ($x_{\min} \in \Omega$).

$$\phi_{\min} = \phi(x_{\min}) = \min_{\overline{\Omega}} \phi < \min_{\overline{\Omega_0}} \phi_0 = -\delta^2/2 \,,$$
$$\liminf_{\substack{|x|\to\infty \\ x\in\Omega}} \phi(x) = \min_{\overline{\Omega_0}} \phi_0 > \phi_{\min} \,.$$

$$\lambda_k^{\mathrm{eff}}(h) = \inf_{\substack{W\subset\mathscr{H}^2(\Omega) \\ \dim W = k}} \sup_{u\in W\setminus\{0\}} \frac{h\|u\|^2_{\partial\Omega}}{\|e^{-\phi/h}u\|^2} \,, \quad k \geq 1 \,.$$

$\mathscr{H}^2(\Omega)$: "Holomorphic functions on Ω with $L^2(\partial\Omega)$-trace."

Theorem [L.T., Raymond, Royer]

Consider $N \in \mathbb{N}^*$. There exists $h_0 > 0$ such that for all $h \in (0, h_0)$,

1. The operator $\mathscr{D}_h$ has at least N positive discrete eigenvalues (counted with multiplicities).
2. Denoting the first N eigenvalues by $(\lambda_k^+(h))_{k\in[\![1,N]\!]}$, we have for all $k \in [\![1, N]\!]$

$$\lambda_k^+(h) \underset{h\to 0}{\sim} \lambda_k^{\mathrm{eff}}(h) \,.$$

Asymptotic of $\lambda_k^{\mathrm{eff}}(h)$

Theorem [L.T., Raymond, Royer]

$$\lambda_k^{\mathrm{eff}}(h) = h^{1-k} e^{2\phi_{\min}/h} \left(\frac{d^k_{\mathscr{H}}}{d^k_{\mathscr{B}}}\right)^2 (1 + o_{h\to 0}(1))\,, \quad k \geq 1\,.$$

$\mathscr{B}^2(\mathbb{C}) = \{u \in \mathscr{O}(\mathbb{C}) : N_{\mathscr{B}}(u) < +\infty\}$: Segal-Bargmann space.

$$N_{\mathscr{B}}(u) = \left(\int_{\mathbb{R}^2} |u(y_1 + iy_2)|^2 e^{-\mathrm{Hess}_{x_{\min}}\phi(y,y)} \mathrm{d}y\right)^{1/2}\,.$$

Distances to closed convex spaces

$d^k_{\mathscr{H}} = \mathrm{dist}_{\mathscr{H}^2(\Omega)}(0, \mathbb{X}_k)$ and $d^k_{\mathscr{B}} = \mathrm{dist}_{\mathscr{B}(\mathbb{C})}(0, \mathbb{Y}_k)$.

$$\mathbb{X}_k = \{u \in \mathscr{H}^2(\Omega)\,, \forall j \in [\![0, k-2]\!]\,,\ u^{(j)}(z_{\min}) = 0\,,\ u^{(k-1)}(z_{\min}) = 1\}\,,$$
$$\mathbb{Y}_k = \{u \in \mathbb{C}[X]\,, \deg u = k-1\,, u^{(k-1)}(0) = 1\}\,.$$

Selected Elements of the Proofs

Proof Outline

Following [**barbaroux:hal-02889558**] :

1. **Gauge Change [Thaller]:** $e^{\sigma_3\phi/h}\sigma\cdot\mathbf{p}e^{\sigma_3\phi/h} = \sigma\cdot(\mathbf{p}-A)\,.$
2. **Min-Max Principle for Positive Eigenvalues:**

$$\lambda_k^+(h) = \inf_{\substack{W\subset H^1(\Omega,\mathbb{C})\\ \dim W=k}} \sup_{v\in W\setminus\{0\}} \rho_+(v)\,,\quad k\geq 1\,,$$

where $\rho_+(v) = \frac{h\|v\|^2_{\partial\Omega}+\sqrt{h^2\|v\|^4_{\partial\Omega}+4\|ve^{-\phi/h}\|^2\|he^{-\phi/h}2\partial_{\overline{z}}v\|^2}}{2\|e^{-\phi/h}v\|^2}.$

3. **Projection onto $H^1(\Omega)\cap\mathscr{O}(\Omega)$:** cancels the term $\|he^{-\phi/h}2\partial_{\overline{z}}v\|^2$.
4. **Analysis of $\lambda_k^{\mathrm{eff}}(h)$:**

$$\lambda_k^{\mathrm{eff}}(h) = \inf_{\substack{W\subset H^1(\Omega)\cap\mathscr{O}(\Omega)\\ \dim\, W=k}} \sup_{u\in W\setminus\{0\}} \frac{h\|u\|^2_{\partial\Omega}}{\|e^{-\phi/h}u\|^2}\,,\quad k\geq 1\,.$$

Key Differences: Bounded vs. Strip

1. Existence of essential spectrum. Consequences on the characterization of the eigenvalues.
2. Existence and bounds on the potential ϕ [**bonlavigne:hal-03678608**]: $\phi \notin H^1(\Omega)$.
3. Closedness of the form

$$v \in H^1(\Omega) \longmapsto \|v\|_{\partial\Omega}^2 + \|ve^{-\phi/h}\|^2 + \|e^{-\phi/h} 2\partial_{\bar{z}} v\|^2 .$$

   The main challenge is to study the completion of $H^1(\Omega) \cap \mathscr{O}(\Omega)$ with respect to the norm $v \longmapsto \|v\|_{\partial\Omega}$:

   How is the Hardy space $\mathscr{H}^2(\Omega)$ defined?

4. Asymptotic study of $\lambda_k^{\mathrm{eff}}(h)$.
   In [**barbaroux:hal-02889558**], strong use of the fact that polynomials belong to $\mathscr{H}^2(\Omega)$ when Ω is bounded.

Questions?

Steps for Constructing the Hardy Space on Ω

1. Construct the Hardy space on the straight strip $\Omega_0 = \mathbb{R} \times (-\delta, \delta)$.
   A Paley-Wiener Theorem: $\mathscr{H}^2(\Omega_0) = \mathscr{O}(\Omega_0) \cap L^\infty((-\delta, \delta)_y, L^2(\mathbb{R}_x))$.
   A nice exercice adapted from
   [Rudin, Real and Complex Analysis, Chap. 19].
2. Construct a bilholomorphism between Ω_0 and Ω.
   With a bounded derivative $\rightsquigarrow$ preserving both infinite parts.
3. Pushing $\mathscr{H}^2(\Omega_0)$ onto Ω.

Negative Energy Bound States?

Assumption : γ is analytic and κ tends to zero at infinity.

The essential spectrum stays the same.

Theorem [barbaroux:hal-02889558]

There exists $h_0 > 0$ such that for all $h \in (0, h_0)$,

1. The operator $\mathscr{D}_h$ has at least one negative discrete eigenvalue.
2. Denoting the first eigenvalue by $-\lambda_1^-(h)$, we have,

$$\lambda_1^-(h) = a_0\sqrt{h} + h^{\frac{3}{2}}\mathfrak{c}_0\lambda + o(h^{\frac{3}{2}}) \,,$$

where $\mathfrak{c}_0 > 0$ and

$$\lambda = \min\left(\lambda_1\left(D_s^2 - \frac{\kappa(s)^2}{12(1-\delta\kappa)^2}\right), \lambda_1\left(D_s^2 - \frac{\kappa(s)^2}{12(1+\delta\kappa)^2}\right)\right) < 0 \,.$$

On the 'Homogeneous' Magnetic Dirac Operators

The operators $\mathscr{D}_{\mathbb{R}^2}$ and $\mathscr{D}_{\mathbb{R}^2_+}$ act as $\sigma \cdot (-i\nabla - \mathbf{A}_0)$ on

$$\mathrm{Dom}(\mathscr{D}_{\mathbb{R}^2}) = \left\{\varphi \in L^2(\mathbb{R}^2, \mathbb{C}^2) \,\middle|\, (-i\nabla - \mathbf{A}_0)\varphi \in L^2(\mathbb{R}^2)\right\},$$

$$\mathrm{Dom}(\mathscr{D}_{\mathbb{R}^2_+}) = \left\{\varphi \in L^2(\mathbb{R}^2_+, \mathbb{C}^2) \,\middle|\, (-i\nabla - \mathbf{A}_0)\varphi \in L^2(\mathbb{R}^2),\ \sigma_1\varphi = \varphi\ \partial\mathbb{R}^2_+\right\},$$

$\mathbb{R}^2_+ = \mathbb{R} \times \mathbb{R}_+$, $\mathbf{A}_0 = (-x_2, 0)^T$.

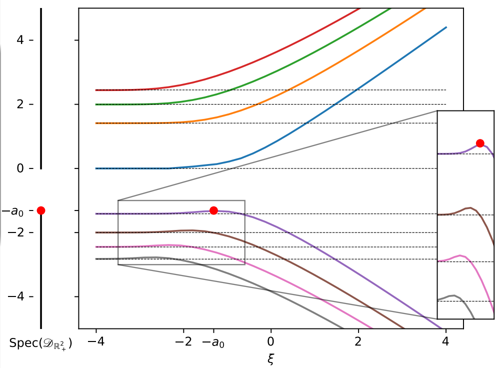
Theorem [barbaroux:hal-02889558]

$\mathscr{D}_{\mathbb{R}^2}$ and $\mathscr{D}_{\mathbb{R}^2_+}$ are s.a.,

$$\mathrm{sp}(\mathscr{D}_{\mathbb{R}^2}) = \{\pm\sqrt{2k} \mid k \in \mathbb{N}\},$$

$$\mathrm{sp}(\mathscr{D}_{\mathbb{R}^2_+}) = (-\infty, -a_0] \cup [0, +\infty).$$

$0 < a_0 < \sqrt{2}$



Positive Eigenvalues: Lower Bound

The magnetic Cauchy-Riemann operators :

$$d_h = -2ih\partial_z - A_1 + iA_2, \qquad d_h^\times = -2ih\partial_{\overline{z}} - A_1 - iA_2.$$

Proposition

1. The operator $(d_h, \mathrm{Dom}(d_h) = H_0^1(\Omega))$ is closed with a closed range.
2. The adjoint $(d_h^*, \mathrm{Dom}(d_h^*))$ acts as $d_h^\times$ on

$$\mathrm{Dom}(d_h^*) = \{u \in L^2(\Omega) : \partial_{\overline{z}} u \in L^2(\Omega)\} = \ker(d_h^*) + H^1(\Omega),$$

   with $\ker(d_h^*) = \{e^{-\phi/h} v \mid v \in \mathscr{O}(\Omega) \cap L^2(\Omega)\}$.
3. We have $\ker(d_h^*)^\perp \cap \mathrm{Dom}(d_h^*) = \{d_h w \mid w \in H_0^1(\Omega) \cap H^2(\Omega)\}$.
4. There exist $h_0, c > 0$ s.t., for all $h \in (0, h_0)$, $u \in \mathrm{Dom}(d_h^*) \cap \ker(d_h^*)^\perp$,

$$\|d_h^* u\|_{L^2(\Omega)} \geq \sqrt{2h}\|u\|_{L^2(\Omega)},$$

$$\|d_h^* u\|_{L^2(\Omega)} \geq ch^2 \left(\|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^2(\partial\Omega)}\right).$$

Minimax: Heuristic

1. Let $u = (u_1, u_2) \in \mathrm{Dom}(\mathscr{D}_h)$ and $\lambda > 0$ be s.t $\mathscr{D}_h u = \lambda u$. Then,

$$\begin{aligned} d_h^\times u_1 &= \lambda u_2\,, \qquad & i(\mathfrak{n}_1 + i\mathfrak{n}_2)u_1 = u_2 \text{ on } \partial\Omega\,, \\ d_h u_2 &= \lambda u_1\,, \end{aligned}$$

2. Then, u_1 satisfies

$$d_h d_h^\times u_1 = \lambda^2 u_1\,, \qquad i(\mathfrak{n}_1 + i\mathfrak{n}_2)u_1 = \lambda^{-1} d_h^\times u_1 \text{ on } \partial\Omega\,.$$

3. Therefore, u_1 is a critical point of the quadratic form

$$\mathcal{Q}_\lambda \colon u \mapsto \|d_h^\times u\|^2 + h\lambda\|u\|^2_{L^2(\partial\Omega)} - \lambda^2\|u\|^2,$$

   and $\mathcal{Q}_\lambda(u_1) = 0$.

Question: is the inverse true ?

Minimax: Step 1 :Completing the Space

Notation

- Let
$$\mathscr{H}^2_{h,A} = \{u \in L^2(\Omega) \mid d_h^\times u = 0,\ u_{|\partial\Omega} \in L^2(\partial\Omega)\}$$
be the magnetic Hardy space, endowed with $\|\cdot\|_{L^2(\partial\Omega)}$.
- Define the space
$$\mathfrak{H}_{h,A} = H^1(\Omega) + \mathscr{H}^2_{h,A},$$
with norm $\|\cdot\|^2_{\mathfrak{H}_{h,A}} = \|\cdot\|^2_{L^2(\partial\Omega)} + \|\cdot\|^2_{L^2(\Omega)} + \|d_h^\times \cdot\|^2_{L^2(\Omega)}$.

- For $A = 0$, $\mathscr{H}^2_{h,A}$ is the Hardy space on Ω, which is a Hilbert space.

Proposition

(1) $(\mathscr{H}^2_{h,A}, \|\cdot\|_{L^2(\partial\Omega)})$ and $(\mathfrak{H}_{h,A}, \|\cdot\|_{\mathfrak{H}_{h,A}})$ are Hilbert spaces, embedded in $L^2(\Omega)$.

(2) $H^1(\Omega)$ is dense in $\mathfrak{H}_{h,A}$.

Minimax: Step 2 : Study of the minmax levels of $\mathcal{Q}_\lambda$

Notation

For $k \geq 1$,

$$\ell_k(\lambda) = \inf_{\substack{W \subset \mathfrak{H}_{h,A} \\ \dim W = k}} \sup_{u \in W \setminus \{0\}} \mathcal{Q}_\lambda(u)\,,$$

and

$$\mu_+^k = \inf_{\substack{W \subset \mathfrak{H}_{h,A} \\ \dim W = k}} \sup_{u \in W \setminus \{0\}} \rho_+(u)\,.$$

Remark:

- For $\lambda > 0$, $\mathcal{Q}_\lambda$ is bounded below and closed.
- For $u \in \mathfrak{H}_{h,A} \setminus \{0\}$, $\rho_+(u)$ is the only positive root of $\lambda \mapsto \mathcal{Q}_\lambda(u)$.

For $k \geq 1$, if $\mu_+^k > 0$, μ_+^k is the only positive root of $\lambda \mapsto \ell_k(\lambda)$.

Minimax: Step 3 :The Link Between These Objects

Main Proposition

Let $\lambda > 0$. Then, the map

$$\mathscr{J}_\lambda : \begin{cases} \ker \mathscr{L}_\lambda & \longrightarrow & \ker(\mathscr{D}_h - \lambda) \\ u & \longmapsto & \begin{pmatrix} u \\ \frac{d_h^\times u}{\lambda} \end{pmatrix} \end{cases}$$

is well-defined and is an isomorphism.

Corollary

For $k \geq 1$, we have

$$\lambda_+^k(h) = \mu_+^k(h).$$

Heuristic for the Asymptotics of $\lambda_k^{\mathrm{eff}}(h)$: Laplace's Method

Let $v \in \mathscr{H}^2(\Omega) \setminus \{0\}$ with $v(x_{\min} + y) \sim_{y\to 0} \frac{v^{(m)}(x_{\min})y^m}{m!}$.

$$
\begin{aligned}
&\int_\Omega e^{-2\phi/h}|v|^2 \mathrm{d}x \\
&= h e^{-2\phi_{\min}/h} \int_{\Omega_h} e^{-2(\phi-\phi_{\min})(x_{\min}+y\sqrt{h})/h} |v(x_{\min} + y\sqrt{h})|^2 \mathrm{d}y \\
&\sim_{h\to 0} h e^{-2\phi_{\min}/h} \int_{\mathbb{R}^2} e^{-\mathrm{Hess}_{\min}\phi(y,y)} \left| \frac{v^{(m)}(x_{\min})y^m \sqrt{h}^m}{m!} \right|^2 \mathrm{d}y \\
&\sim_{h\to 0} h^{1+m} e^{-2\phi_{\min}/h} \frac{|v^{(m)}(x_{\min})|^2}{(m!)^2} \int_{\mathbb{R}^2} e^{-\mathrm{Hess}_{\min}\phi(y,y)} |y|^{2m} \mathrm{d}y \,.
\end{aligned}
$$

Consequence:

$$
\frac{h\|v\|^2_{\partial\Omega}}{\|e^{-\phi/h}v\|^2} \sim_{h\to 0} h^{-m} e^{2\phi_{\min}/h} c_v \,,
$$

with $c_v = \frac{\|v\|^2_{\partial\Omega}(m!)^2}{|v^{(m)}(x_{\min})|^2 \int_{\mathbb{R}^2} e^{-\mathrm{Hess}_{\min}\phi(y,y)}|y|^{2m}\mathrm{d}y}$.

Some ideas

1. For $k = 1$, set $m = 0$ and select v as the minimizer of $d^k_{\mathscr{H}} = \mathrm{dist}_{\mathscr{H}^2(\Omega)}(0, \mathbb{X}_k)$ with

$$\mathbb{X}_k = \{u \in \mathscr{H}^2(\Omega)\,, \forall j \in [\![0, k-2]\!]\,,\ u^{(j)}(z_{\min}) = 0\,,\ u^{(k-1)}(z_{\min}) = 1\}\,.$$

2. For $k \geq 1$, too rough to take v s.t. $v(x_{\min} + y) \sim_{y \to 0} \frac{v^{(m)}(x_{\min}) y^m}{m!}$,

Better take (v_h) s.t. $v_h(x_{\min} + \sqrt{h} y) \sim_{h \to 0} P(y) \in \mathbb{C}_{k-1}[Y]$.

3. The Taylor expansion of $v \in \mathscr{H}^2(\Omega)$ is a natural object, but polynomials do not belong to $\mathscr{H}^2(\Omega)$.

Define $\mathrm{Tayl}^k_{\mathscr{H}^2(\Omega)}$: $\|\cdot\|_{\partial\Omega}$-orthogonal projection onto the orthogonal of

$$\widetilde{\mathbb{X}}_{k+1} = \{u \in \mathscr{H}^2(\Omega)\,, \forall j \in [\![0, k-1]\!]\,,\ u^{(j)}(z_{\min}) = 0\}\,.$$

4. ...(Explicit expression for $\mathrm{Tayl}^k_{\mathscr{H}^2(\Omega)}$ and Cauchy formula.)

Formulas for the Terms $(d_{\mathscr{B}}^k)_k$

Lemma

$$(d_{\mathscr{B}}^k)^2 = \frac{\pi(B(x_{\min}))^{k-1}}{2^{k-1}(k-1)!(\det \mathrm{Hess}_{x_{\min}}\phi)^{k-\frac{1}{2}}}\,, k \geq 1\,.$$

For $m \geq 0$, we have $(d_{\mathscr{B}}^{m+1}) = N_{\mathscr{B}}(P_m)/m!$, where

$$P_m \colon z \mapsto \begin{cases} z^m & \text{if } a = b\,, \\ \left|\frac{b-a}{ab}\right|^{m/2} He_m\left(z\sqrt{\left|\frac{ab}{b-a}\right|}\right) & \text{if } a \neq b\,, \end{cases}$$

and

$$N_{\mathscr{B}}(P_m)^2 = \frac{2\pi m!(a+b)^m}{(ab)^{m+\frac{1}{2}}}\,,$$

$a/2, b/2$ are the eigenvalues of $\mathrm{Hess}_{x_{\min}}\phi$,

He_m are the probabilist's Hermite polynomials.

Anisotropic case ($a \neq b$)

[1990, van Eijndhoven and Meyers]

New orthogonality relations for the Hermite polynomials and related Hilbert spaces. J. Math. Anal. Appl.

Formulas for the Terms $(d^k_{\mathscr{H}})_k$

Lemma

$$d^k_{\mathscr{H}} = \frac{\sqrt{2\pi}}{(k-1)!}|\varphi'(0)|^{k-\frac{1}{2}}\,, k \geq 1\,.$$

$$d^k_{\mathscr{H}} = \inf\left\{ \frac{\|u\|_{\mathscr{H}^2(\Omega)}}{|u^{(k-1)}(z_{\min})|}\,, u \in \mathscr{H}^2(\Omega)\,, \begin{array}{l} u^{(j)}(z_{\min}) = 0\,, \forall j \in [\![0, k-2]\!]\,, \\ u^{(k-1)}(z_{\min}) \neq 0 \end{array} \right\}.$$

Λz^{k-1} realizes the minima where Λ is the isometric isomorphism defined by

$$\begin{array}{rccc} \Lambda\colon & \mathscr{H}^2(\mathbb{D}) & \longrightarrow & \mathscr{H}^2(\Omega) \\ & u & \longmapsto & \left[z \mapsto \sqrt{\varphi'(z)}\,u \circ \varphi(z)\right] \end{array},$$

φ being a biholomorphism from the unit disk $\mathbb{D}$ to Ω such that $\varphi(0) = z_{\min}$.

Proposition [letreust:hal-04691264]

$$\left(\frac{d^k_{\mathscr{H}}}{d^k_{\mathscr{B}}}\right)^2 = \frac{2^k|\varphi'(0)|^{2k-1}(\det \mathrm{Hess}_{x_{\min}}\phi)^{k-\frac{1}{2}}}{(k-1)!B(x_{\min})^{k-1}}\,, k \geq 1.$$