Eigenvalue bounds in the linear stability of the Ekman spiral

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- **B. Gerhat, O.O. Ibrogimov and P. Siegl** (Comm. Math. Phys. 2022)
  On the point spectrum in the Ekman boundary layer problem
- **B. Gerhat** (in preparation)
  The number of eigenvalues in the Ekman boundary layer problem

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Ekman boundary layer [Ekman (1905)]

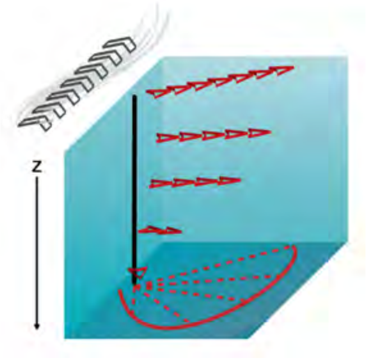
- **Navier–Stokes** in **rotating frame**

$$\left.\begin{aligned} \partial_t u - \nu\Delta u + \Omega e_3 \times u + \boxed{(u\cdot\nabla)u} + \nabla p &= 0 \\ \operatorname{div} u &= 0 \end{aligned}\right\}$$

- unknown **velocity** u and **pressure** p on **half space**

- surface and bottom **boundary conditions**

$$\left.\begin{aligned} u(t,x,y,0) &= (0,0,0) \\ \lim_{z\to\infty} u(t,x,y,z) &= (u_\infty,0,0) \end{aligned}\right\}$$



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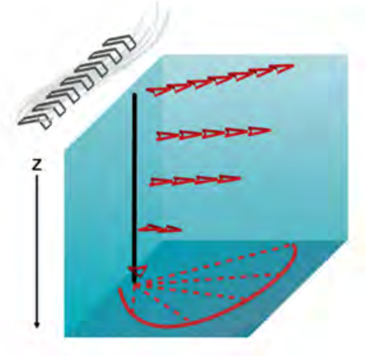

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Ekman spiral

$\delta = (2\nu/\Omega)^{\frac{1}{2}}$

$$u_E(z) = u_\infty\left(1 - e^{-\frac{z}{\delta}}\cos\left(\frac{z}{\delta}\right), e^{-\frac{z}{\delta}}\sin\left(\frac{z}{\delta}\right), 0\right), \qquad p_E(y) = -\Omega u_\infty y$$

Spectral problem

Non-selfadjoint operator matrix family

$R = u_\infty \delta / \nu \geq 0, \quad \alpha > 0$

$$\mathscr{T}(\lambda) = \begin{pmatrix} (-\partial_x^2 + \alpha^2)(-\partial_x^2 + \alpha^2 - \lambda) & 2\partial_x \\ 2\partial_x & -\partial_x^2 + \alpha^2 - \lambda \end{pmatrix} + \mathrm{i}\alpha R \begin{pmatrix} V(-\partial_x^2 + \alpha^2) + V'' & 0 \\ U' & V \end{pmatrix}$$

- constant **domain** in ambient **Hilbert space**

$$\operatorname{Dom} \mathscr{T} = \left\{ (f, g) \in H^4(\mathbb{R}_+) \times H^2(\mathbb{R}_+) \mid f(0) = f'(0) = g(0) = 0 \right\} \subseteq L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_+)$$

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- generalised **coefficients**

$u_E \to (U, V, 0)$

$$U', V, V', V'' \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$$

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- **structure** / **location** of **spectrum**

$$\sigma(\mathscr{T}) = \left\{ \lambda \in \mathbb{C} \mid 0 \in \sigma(\mathscr{T}(\lambda)) \right\},$$

$\forall\ \alpha > 0 : \operatorname{Re} \sigma(\mathscr{T}) \geq 0$ **?**

dependence on R **?**

Previous results

- **experiments**, non-rigorous **numerical stability** analysis [Faller (1963), Lilly (1966), Spooner (1982)]
- local **existence** and **uniqueness** [Giga–Inui–Mahalov–Matsui (2007), Giga–Saal (2013)]
- non-linear **stability analysis** [Giga–Saal (2015), Hess–Hieber–Mahalov–Saal (2010), Fischer–Saal (2013)]

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Spectral theory

[Greenberg–Marletta (2004), Marletta–Tretter (2007)]

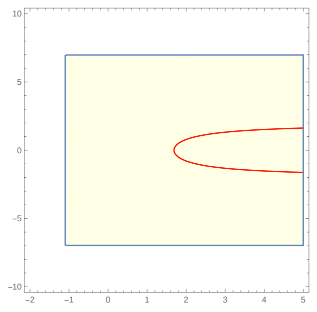
$$\sigma_{\mathrm{ess}}(\mathscr{T}) = \left\{\lambda \in \mathbb{C} \mid \exists\, \xi \in \mathbb{R} : (\xi^2+\alpha^2)(\xi^2+\alpha^2-\lambda)^2 + 4\xi^2 = 0\right\}$$

- **numerical range** enclosure

$$\sigma(\mathscr{T}) \subseteq \left\{\lambda \in \mathbb{C} \mid \mathrm{Re}\,\lambda \geq \gamma,\ |\mathrm{Im}\,\lambda| \leq \eta\right\}$$

- only **discrete eigenvalues** in **exterior** of $\sigma_{\mathrm{ess}}(\mathscr{T})$

$$\#\sigma(\mathscr{T}) \setminus \sigma_{\mathrm{ess}}(\mathscr{T}) = \,?$$



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Theorem [G–Ibrogimov–Siegl (2022)]

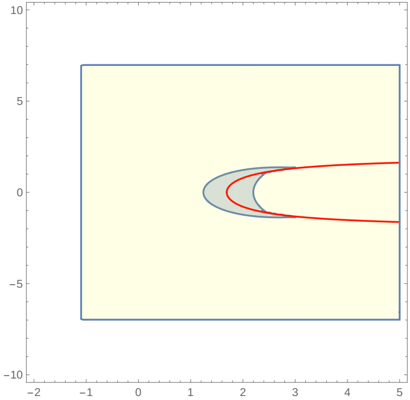
$$\Sigma(\mathscr{T}) \subseteq \left\{ \lambda \in \mathbb{C} \setminus \sigma_{\mathrm{ess}}(\mathscr{T}) \mid r(\mathscr{Q}(\lambda)) \geq 1 \right\}$$

is **bounded** and **discrete** with

$$r(\mathscr{Q}(\lambda)) = \mathscr{O}\left(|\lambda|^{-\frac{1}{2}}\right), \quad \lambda \to \infty$$

Theorem [G (prep)]

U', V, V'' **decay exponentially** $\implies$ $\#\Sigma(\mathscr{T}) < \infty$



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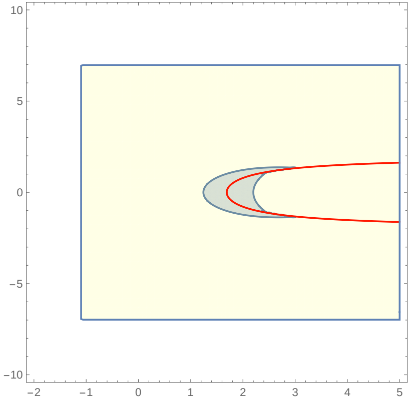

Conjecture / work in progress

$$\forall\ \varepsilon > 0 : \#\Sigma(\mathscr{T}) \leq \boxed{R^2 C_\varepsilon(\alpha)} \left(\int_0^\infty \mathrm{e}^{\varepsilon x} W(x) \mathrm{d}x \right)^2, \quad W := \max\{|V|, |V''|, |U'|\}$$

Strategy

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- **methods** from analysis of **Schrödinger** operators

⚠ operator **family** with **matrix structure**

⚠ free operator is **non-selfadjoint** and **4th order**

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  - ⚠ operator **family** with **matrix structure**
  - ⚠ free operator is **non-selfadjoint** and **4th order**
  - ⚠ **perturbation** is **differential** operator
- **holomorphic** family of **HS integral operators**

$$\mathbb{C}\setminus\sigma_{\mathrm{ess}}(\mathscr{T}) \ni \lambda \quad\longmapsto\quad \boxed{\mathscr{Q}(\lambda) = \mathrm{i}\alpha R\,\mathscr{V}_1\,\mathscr{L}(\lambda)^{-1}\,\mathscr{V}_2} \in \mathscr{S}_2$$

Birman–Schwinger principle

$\Sigma(\mathscr{T}) = \sigma(\mathscr{T}) \cap \Omega$

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[Birman, Schwinger, Simon, Abramov–Aslanyan–Davies (2001), ...]

- **Birman–Schwinger** principle

$$\boxed{\lambda \in \Sigma(\mathscr{T}) \iff \lambda \in \sigma(I + \mathscr{Q})}$$

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- **Birman–Schwinger** principle

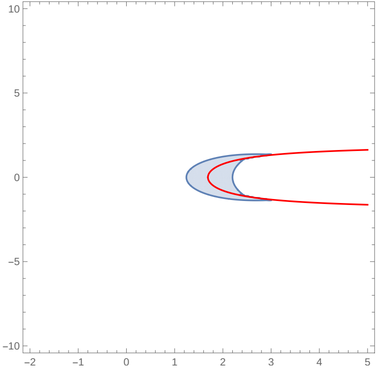
$$\lambda \in \Sigma(\mathscr{T}) \iff \lambda \in \sigma(I + \mathscr{Q})$$

- bounded **spectral enclosure** $(\mathscr{Q} \to \mathscr{U}^{-1}\mathscr{Q}\mathscr{U})$

$$\|\mathscr{Q}(\lambda)\| \le \|\mathscr{Q}(\lambda)\|_{\mathrm{HS}} = \mathscr{O}\left(|\lambda|^{-\frac{1}{2}}\right), \quad \lambda \to \infty$$

- both $\mathscr{T}$ and $I + \mathscr{Q}$ have **spectrum** of **finite type** in Ω

$$\mathrm{m}_{\mathrm{alg}}(\lambda, \mathscr{T}) = \mathrm{m}_{\mathrm{alg}}(\lambda, I + \mathscr{Q})$$



No EV accumulation

Relate to holomorphic function [Dunford–Schwarz, Gohberg–Krein]

$$\lambda \in \Sigma(\mathscr{T}) \iff \lambda \in \sigma(I + \mathscr{Q}) \iff \det_2\left(1 + \mathscr{Q}(\lambda)\right) = 0$$

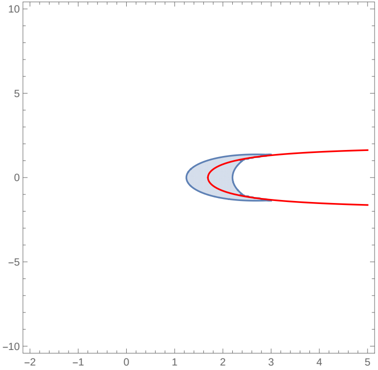
- preserves **algebraic multiplicities** [Frank (2018)]
- consider **two** connected **components**

$$\Omega = \Omega_+ \cup \Omega_-, \qquad (\alpha^2, \infty) \subseteq \Omega_+$$

- **extend** holomorphically to small **neighbourhood**

$$f_\pm : (\Omega_\pm)_\varepsilon \to \mathbb{C}, \qquad f_\pm(\lambda) = \det_2\left(1 + \mathscr{Q}(\lambda)\right), \qquad \lambda \in \Omega_\pm$$

- roots **do not accumulate** in **interior** of domain !



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$$\lambda \in \Sigma(\mathscr{T}) \iff \det\nolimits_2\big(1 + \mathscr{Q}(\lambda)\big) = 0$$ (including multiplicities)

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- bound in **Schrödinger** case [Frank–Laptev–Safronov (2016)]

$$\forall\ \varepsilon>0\ :\ \#\sigma_{\mathrm{disc}}(-\Delta_{\mathrm{D}}^{\mathbb{R}_+}+q) \le \frac{1}{\varepsilon^2}\left(\int_0^\infty \mathrm{e}^{\varepsilon x}|q(x)|\mathrm{d}x\right)^2$$

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- more **complicated geometry**

$$\xi^2 - \lambda \quad \leftrightarrow \quad (\xi^2 + \alpha^2)(\xi^2 + \alpha^2 - \lambda)^2 + 4\xi^2$$

- suitable **biholomorphic** transformation with **controlled derivative**

$$\text{"}\ \sqrt{\Omega_\pm} \to \{\operatorname{Im} z > 0\}\ \text{"}$$

Thank you !

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[2] Frank, R. L. (2018). Eigenvalue bounds for Schrödinger operators with complex potentials. III. *Trans. Amer. Math. Soc.*, 370(1):219–240.

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