A connection between quantum dot Dirac operators and $\overline{\partial}$-Robin Laplacians

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Prague, August 28, 2025

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References

[D., 2025] J. Duran, *The $\overline{\partial}$-Robin Laplacian*, arXiv:2507.16895, (2025).

[D., Mas, Sanz-Perela, 2025] J. Duran, A. Mas, T. Sanz-Perela, *A connection between quantum dot Dirac operators and $\overline{\partial}$-Robin Laplacians in the context of shape optimization problems*, arXiv:2507.18698, (2025).

References

[D., 2025] J. Duran, *The $\overline{\partial}$-Robin Laplacian*, arXiv:2507.16895, (2025).

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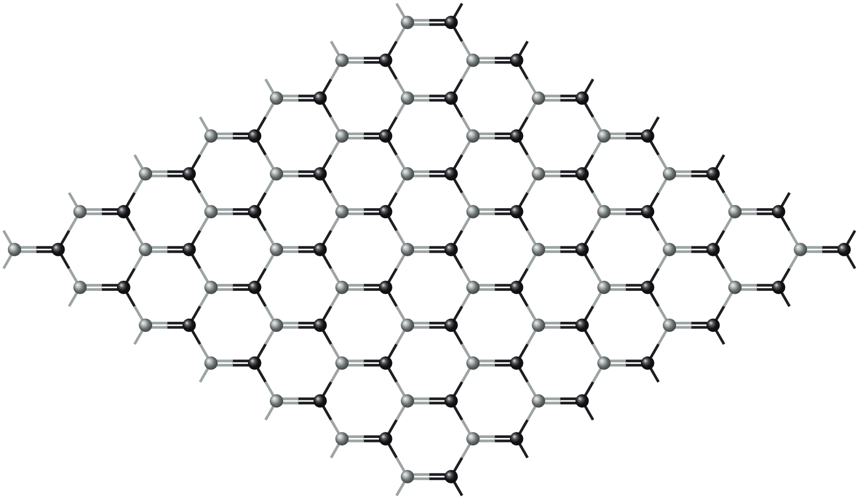
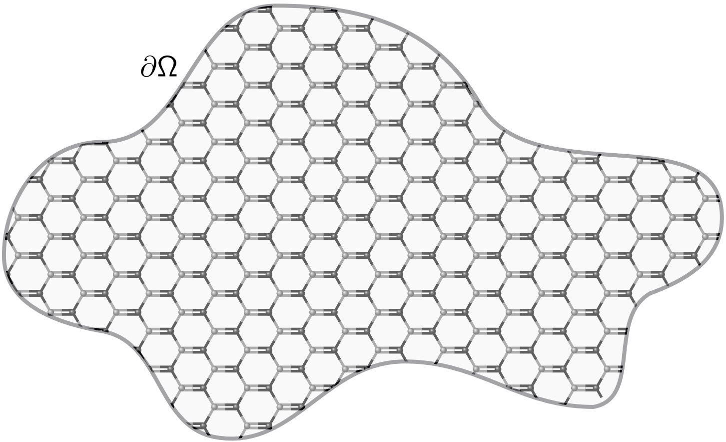
[Antunes, Benguria, Lotoreichik, Ourmières-Bonafos, 2021] P. R. S. Antunes, R. D. Benguria, V. Lotoreichik, T. Ourmières-Bonafos, *A Variational Formulation for Dirac Operators in Bounded Domains. Applications to Spectral Geometric Inequalities*, Communications in Mathematical Physics, 386 (2021), 781–818.

Graphene



Graphene quantum dot

$\Omega \subset \mathbb{R}^2$



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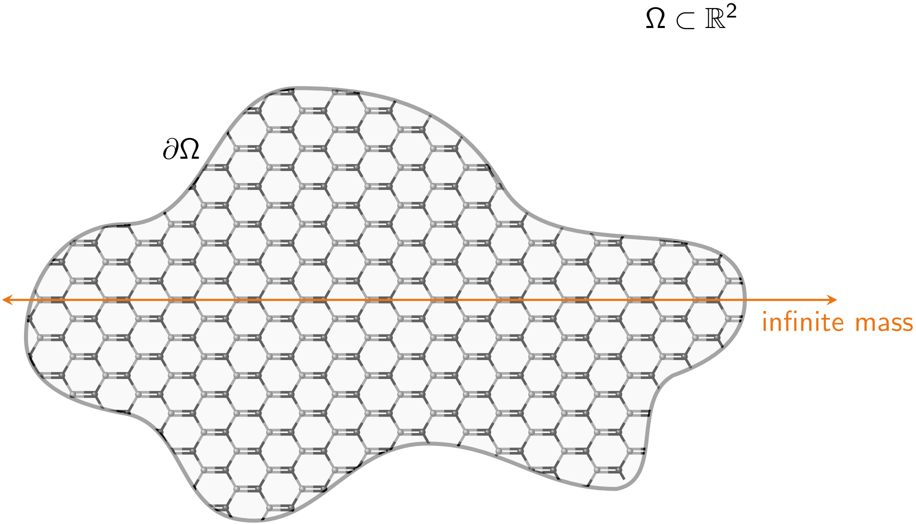
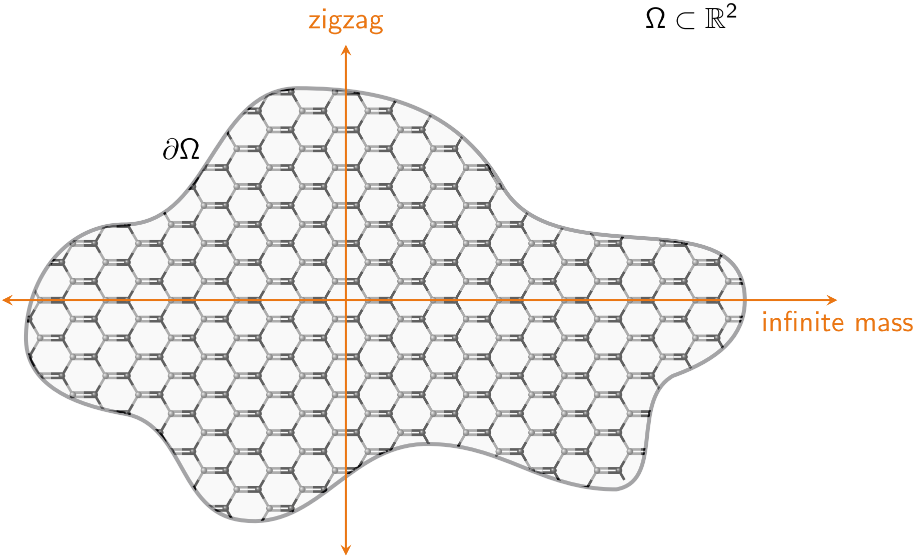
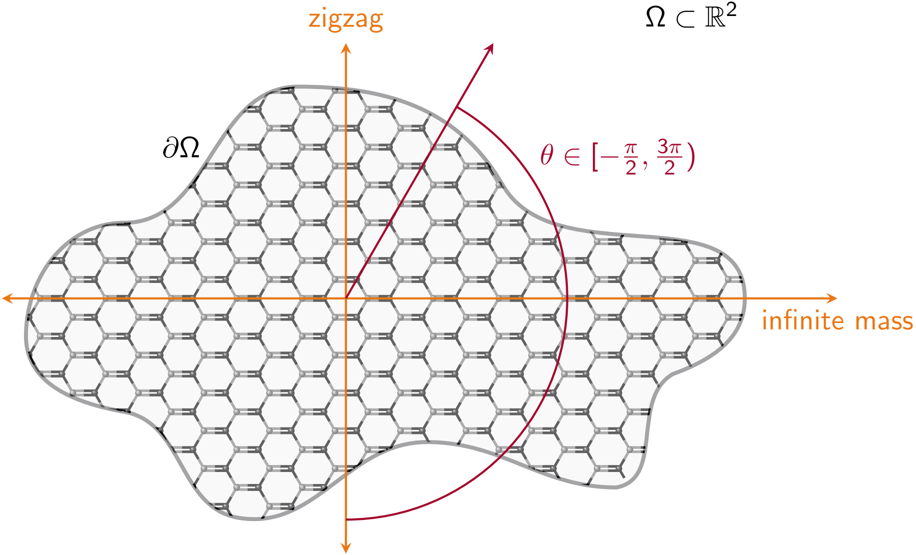


Graphene quantum dot



Graphene quantum dot



Physical model

From now on, we shall assume that $\Omega \subset \mathbb{R}^2$ is a bounded $\mathcal{C}^2$ domain,

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The energies of an electron of mass $m \geq 0$ in a graphene quantum dot Ω are modeled by the eigenvalues λ of

$$\begin{cases} \begin{pmatrix} m & -i(\partial_1 - i\partial_2) \\ -i(\partial_1 + i\partial_2) & -m \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix} & \text{in } \Omega, \\ v = i\dfrac{1 - \sin\theta}{\cos\theta}(\nu_1 + i\nu_2)u & \text{on } \partial\Omega, \end{cases}$$

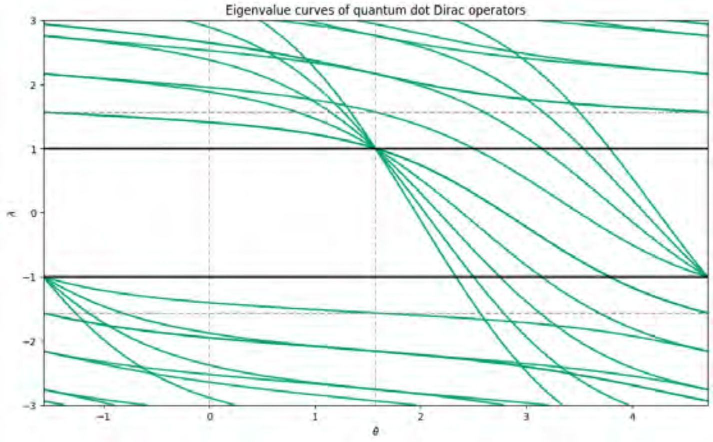
where $u, v : \Omega \to \mathbb{C}$ and $\theta \in [-\frac{\pi}{2}, \frac{3\pi}{2})$.

Qualitative behavior when the domain is a disk

$$\begin{cases} \begin{pmatrix} m & -i(\partial_1 - i\partial_2) \\ -i(\partial_1 + i\partial_2) & -m \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix} & \text{in } D_R, \\ v = i\dfrac{1-\sin\theta}{\cos\theta}(\nu_1 + i\nu_2)u & \text{on } \partial D_R. \end{cases}$$

$m \geq 0$

$\theta \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right)$



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$$\begin{cases} \begin{pmatrix} m & -i(\partial_1 - i\partial_2) \\ -i(\partial_1 + i\partial_2) & -m \end{pmatrix} \varphi = \lambda\varphi & \text{in } D_R, \\ v = i\dfrac{1-\sin\theta}{\cos\theta}(\nu_1 + i\nu_2)u, \quad \varphi = (u,v)^\intercal & \text{on } \partial D_R. \end{cases}$$

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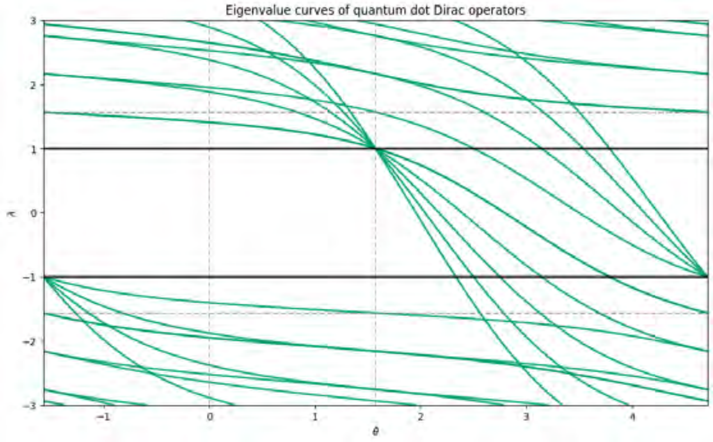

Qualitative behavior when the domain is a disk

$$\begin{cases} \big(-i(\sigma_1\partial_1+\sigma_2\partial_2)+m\sigma_3\big)\varphi=\lambda\varphi & \text{in } D_R, \\ v=i\dfrac{1-\sin\theta}{\cos\theta}(\nu_1+i\nu_2)u, \quad \varphi=(u,v)^{\mathsf{T}} & \text{on } \partial D_R. \end{cases}$$

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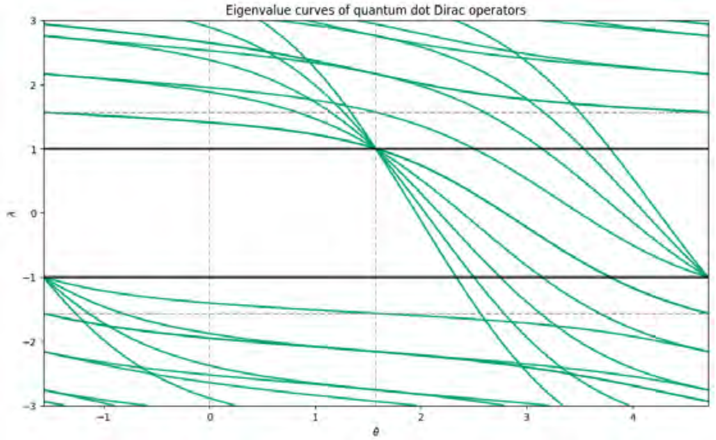

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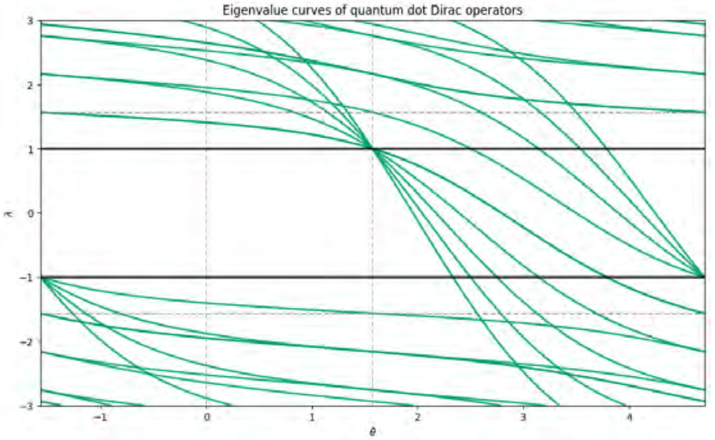

Quantum dot Dirac operators

For $\theta \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right) \setminus \left\{\frac{\pi}{2}\right\}$ (Benguria, Fournais, Stockmeyer, Van Den Bosch, 2017):

$$\mathrm{Dom}(\mathcal{D}_\theta) := \left\{\varphi \in H^1(\Omega)^2 : \varphi = (i\cos\theta(\sigma_1\nu_1 + \sigma_2\nu_2) + \sin\theta)\sigma_3\varphi \text{ in } H^{1/2}(\partial\Omega)\right\},$$
$$\mathcal{D}_\theta\varphi := \big(-i(\sigma_1\partial_1 + \sigma_2\partial_2) + m\sigma_3\big)\varphi \quad \text{for all } \varphi \in \mathrm{Dom}(\mathcal{D}_\theta).$$

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For $\theta \in \left\{-\frac{\pi}{2}, \frac{\pi}{2}\right\}$ (Schmidt, 1995):

$$\mathrm{Dom}(\mathcal{D}_{-\frac{\pi}{2}}) := \left\{\varphi = (u, v)^\intercal \in L^2(\Omega)^2 : (\sigma_1\partial_1 + \sigma_2\partial_2)\varphi \in L^2(\Omega)^2, u \in H^1_0(\Omega)\right\},$$
$$\mathrm{Dom}(\mathcal{D}_{+\frac{\pi}{2}}) := \left\{\varphi = (u, v)^\intercal \in L^2(\Omega)^2 : (\sigma_1\partial_1 + \sigma_2\partial_2)\varphi \in L^2(\Omega)^2, v \in H^1_0(\Omega)\right\},$$
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$$\sigma(D_{\pm\frac{\pi}{2}}) = \{\pm m\} \cup \left\{\pm\sqrt{\Lambda + m^2} : \Lambda \in \sigma(-\Delta_D)\right\}, \quad \text{with } \pm m \in \sigma_{\mathrm{ess}}(D_{\pm\frac{\pi}{2}}).$$

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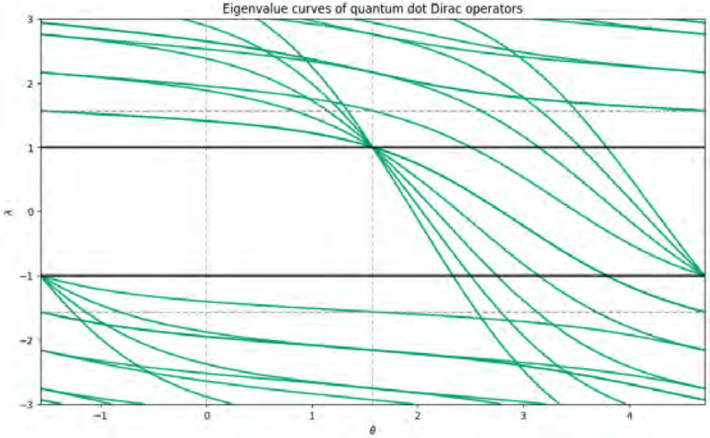
$$m \geq 0$$

$$\theta \in \left[-\tfrac{\pi}{2}, \tfrac{3\pi}{2}\right)$$



Invariances of quantum dot Dirac operators

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$m \leq 0$

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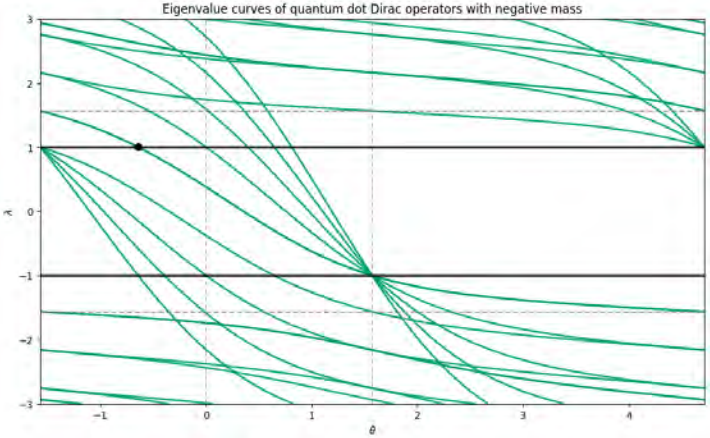

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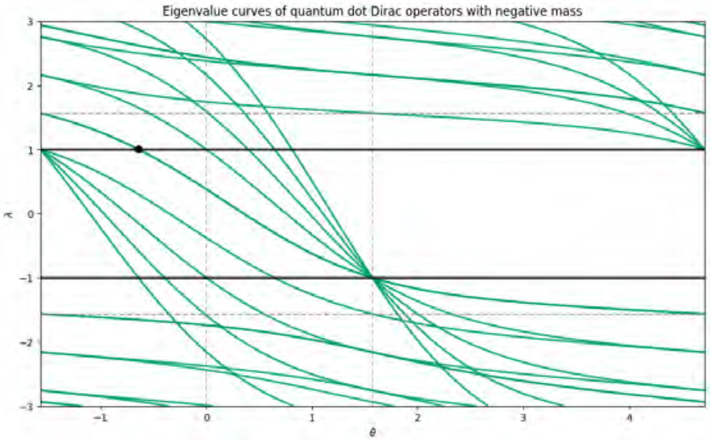
$m \in \mathbb{R}$

$\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



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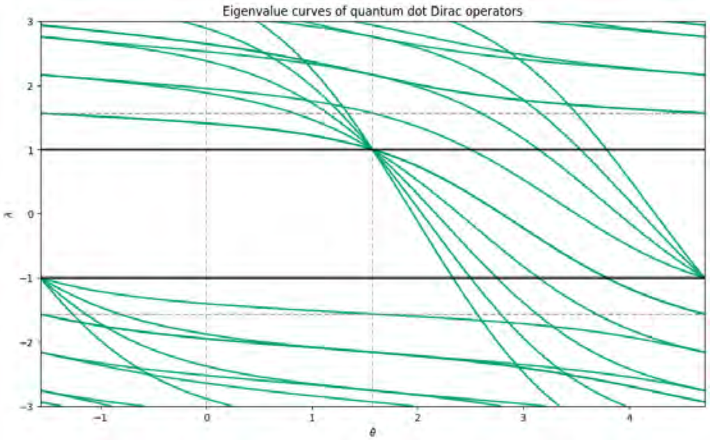

The fundamental energy

For $m \in \mathbb{R}$ and $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the fundamental energy is

$$\lambda_\Omega(\theta) := \min\Big(\sigma(\mathcal{D}_\theta) \cap [0, +\infty)\Big).$$

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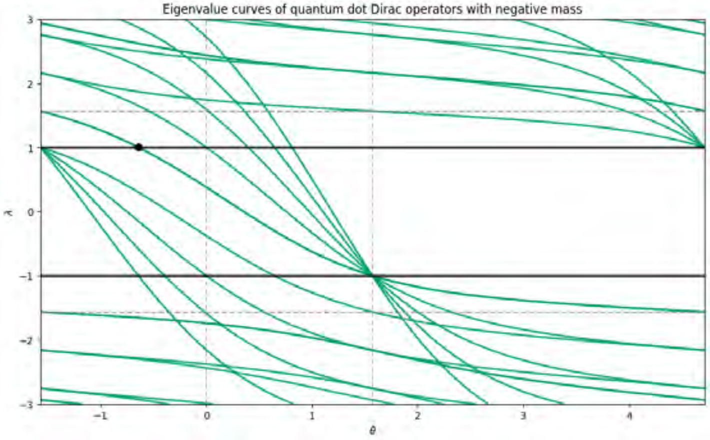

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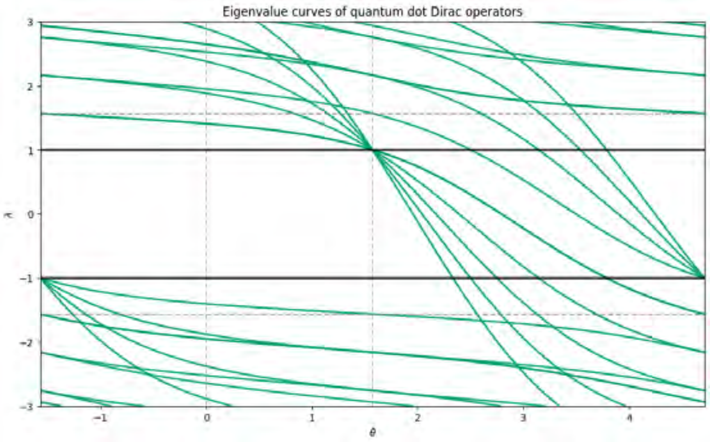
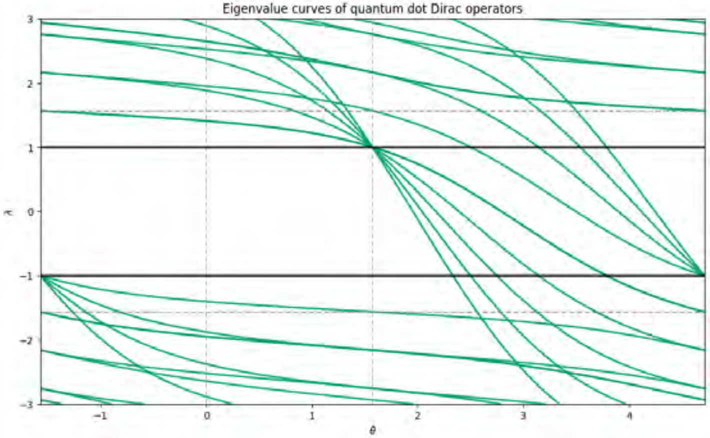


A shape optimization problem

Conjecture

Assume that $m \geq 0$. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with $\mathcal{C}^2$ boundary and let $D \subset \mathbb{R}^2$ be a disk with the same area as Ω. If Ω is not a disk, then $\lambda_\Omega(\theta) > \lambda_D(\theta)$ for all $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$.



Antunes, Benguria, Lotoreichik, Ourmières-Bonafos, 2021

Quantum dot Dirac operator $\mathcal{D}_\theta$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $m \geq 0$.

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$$\begin{cases} -i(\partial_1 - i\partial_2)v = (\lambda - m)u & \text{in } \Omega, \\ -i(\partial_1 + i\partial_2)u = (\lambda + m)v & \text{in } \Omega, \\ v = i\dfrac{1-\sin\theta}{\cos\theta}(\nu_1 + i\nu_2)u & \text{on } \partial\Omega. \end{cases}$$

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$$\begin{cases} -\Delta u = (\lambda^2 - m^2)u & \text{in } \Omega, \\ & \end{cases}$$

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Assume $\lambda > m$ $\quad (\theta, \lambda) \overset{T}{\mapsto} (a, \mu) := \left((\lambda + m)\dfrac{1-\sin\theta}{\cos\theta}, \lambda^2 - m^2 \right)$, $\mu, a > 0$

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$$\begin{cases} -\Delta u = \mu u & \text{in } \Omega, \\ (\nu_1 - i\nu_2)(\partial_1 + i\partial_2)u + au = 0 & \text{on } \partial\Omega. \end{cases}$$

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The auxiliary problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ 2\overline{\nu}\partial_{\bar{z}} u + au = 0 & \text{on } \partial\Omega, \end{cases} \quad \text{where}$$

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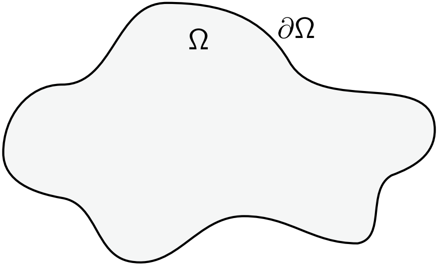

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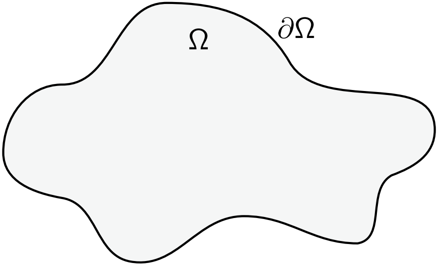

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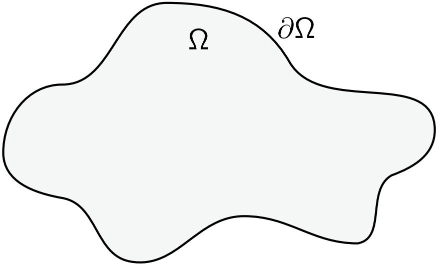
- $a > 0$, and



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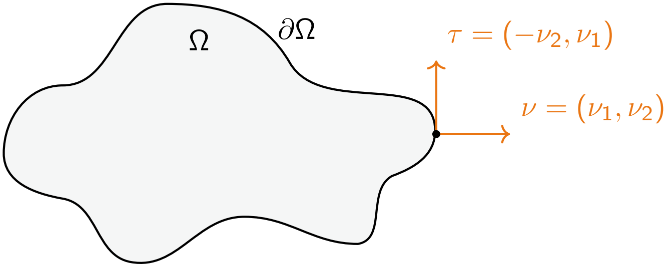

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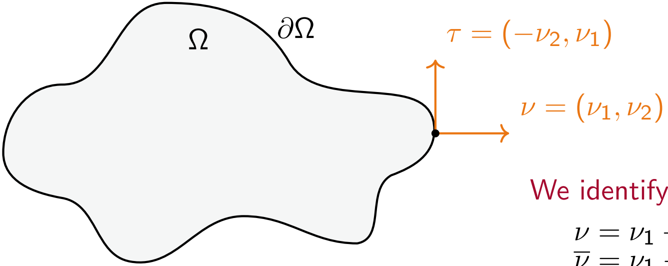

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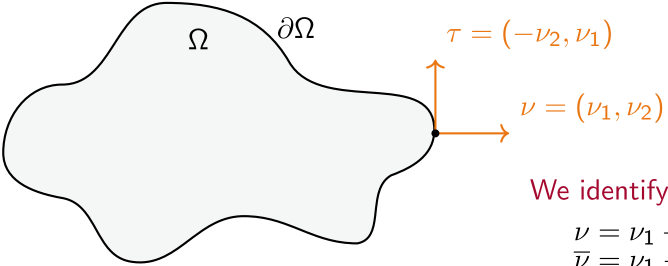

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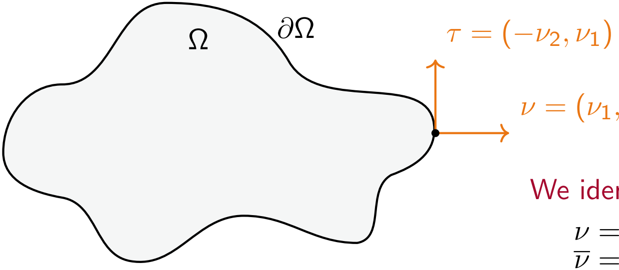

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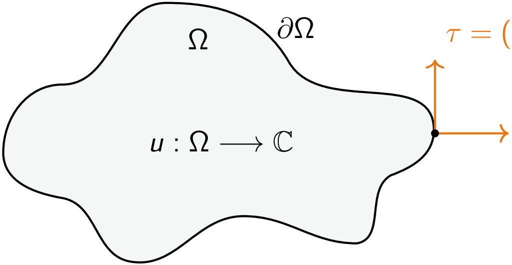

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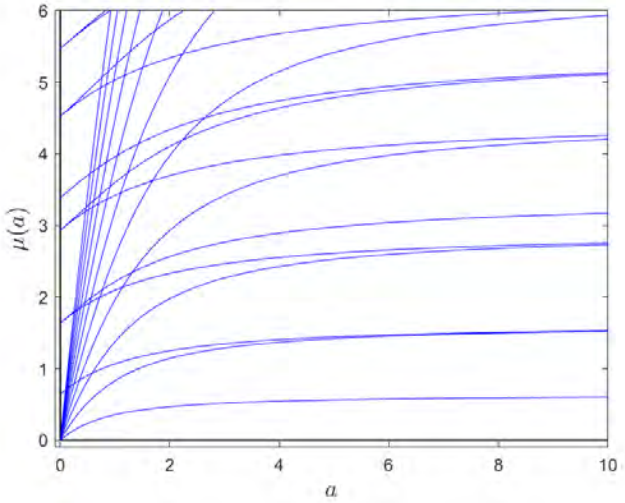
Decomposing, instead, $\Delta = 4\partial_z\partial_{\bar z}$ and integrating by parts, we get

$$\int_\Omega f\,\overline{v} = -4\int_\Omega \partial_z\partial_{\bar z} u\,\overline{v} = 4\int_\Omega \partial_{\bar z} u\,\overline{\partial_{\bar z} v} - \int_{\partial\Omega} 2\overline{\nu}\partial_{\bar z} u\,\overline{v}.$$

Qualitative behavior when the domain is a disk

$$\begin{cases} -\Delta u = \mu u & \text{in } D_R, \\ 2\overline{\nu}\partial_{\bar{z}} u + a u = 0 & \text{on } \partial D_R. \end{cases}$$



The $\overline{\partial}$-Robin Laplacian

Theorem (D., 2025)

For every $a > 0$, the operator

$$\mathrm{Dom}(\mathcal{R}_a) := \left\{ u \in H^1(\Omega) : \partial_{\bar{z}} u \in H^1(\Omega),\ 2\bar{\nu}\partial_{\bar{z}} u + au = 0 \text{ in } H^{1/2}(\partial\Omega) \right\},$$
$$\mathcal{R}_a u := -\Delta u \quad \text{for all } u \in \mathrm{Dom}(\mathcal{R}_a),$$

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in the weak sense.

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The spectrum of $\overline{\partial}$-Robin Laplacians

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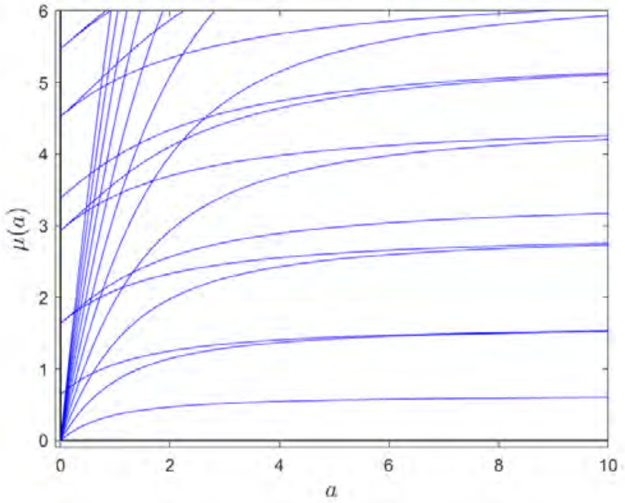

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- *The function $a \mapsto \mu_\Omega(a)$ is continuous, strictly increasing, and bijective from $(0, +\infty)$ to $(0, \Lambda_\Omega)$, where Λ_Ω is the first eigenvalue of the Dirichlet Laplacian in Ω.*

Relation with quantum dot Dirac operators

Quantum dot Dirac operator $\mathcal{D}_\theta$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $m \geq 0$.

$$\begin{cases} -i(\partial_1 - i\partial_2)v = (\lambda - m)u & \text{in } \Omega, \\ -i(\partial_1 + i\partial_2)u = (\lambda + m)v & \text{in } \Omega, \\ v = i\dfrac{1-\sin\theta}{\cos\theta}(\nu_1 + i\nu_2)u & \text{on } \partial\Omega. \end{cases}$$

$$\begin{cases} -\Delta u = (\lambda^2 - m^2)u & \text{in } \Omega, \\ -i(\partial_1 + i\partial_2)u = i(\lambda + m)\dfrac{1-\sin\theta}{\cos\theta}(\nu_1 + i\nu_2)u & \text{on } \partial\Omega. \end{cases}$$

Assume $\lambda > m$ $\quad (\theta, \lambda) \overset{T}{\mapsto} (a, \mu) := \left((\lambda + m)\dfrac{1-\sin\theta}{\cos\theta}, \lambda^2 - m^2\right)$, $\mu, a > 0$

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Fundamental energies are maped to each other

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Let $(\theta, \lambda) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \times (m, +\infty)$ *and* $(a, \mu) \in (0, +\infty) \times (0, +\infty)$ *be such that* $T(\theta, \lambda) = (a, \mu)$*. Then,* $\lambda = \lambda_\Omega(\theta)$ *if and only if* $\mu = \mu_\Omega(a)$*.*

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Assume that $m \geq 0$. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with $\mathcal{C}^2$ boundary and let $D \subset \mathbb{R}^2$ be a disk with the same area as Ω. If Ω is not a disk, then $\lambda_\Omega(\theta) > \lambda_D(\theta)$ for all $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

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Instead of tackling the conjecture for the quantum dot Dirac operators $\mathcal{D}_\theta$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $m \geq 0$:

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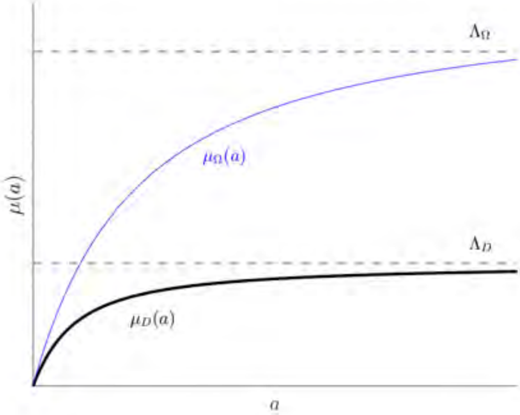
Validity of the $(\overline{\partial}\text{-R})$ conjecture in the regime $a \to +\infty$

Theorem (D., Mas, Sanz-Perela, 2025)

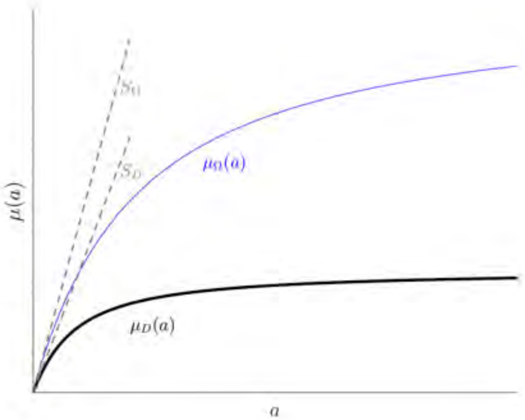
Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with $\mathcal{C}^2$ boundary and let $D \subset \mathbb{R}^2$ be a disk with the same area as Ω. If Ω is not a disk, then there exists $a_0 > 0$ (depending on Ω) such that $\mu_\Omega(a) > \mu_D(a)$ for all $a \in (a_0, +\infty)$.



Intuition for the regime $a \to 0^+$

$$\lim_{a\to 0^+} \frac{\mu_\Omega(a)}{a} = \lim_{a\to 0^+} \inf_{u\in E(\Omega)\setminus\{0\}} \frac{\frac{4}{a}\int_\Omega |\partial_{\bar z} u|^2 + \int_{\partial\Omega} |u|^2}{\int_\Omega |u|^2}$$

$$= \inf_{u\in E(\Omega)\setminus\{0\}:\, \partial_{\bar z} u = 0 \text{ in } \Omega} \frac{\int_{\partial\Omega} |u|^2}{\int_\Omega |u|^2} =: S_\Omega$$



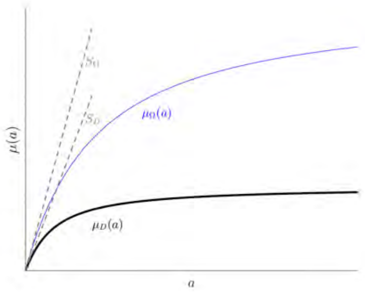
Validity of the ($\overline{\partial}$-R) conjecture in the regime $a \to 0^+$

Theorem (D., Mas, Sanz-Perela, 2025)

Let $\Omega \subset \mathbb{R}^2$ be a $\mathcal{C}^2$, bounded, simply connected domain, and let $D \subset \mathbb{R}^2$ be a disk with the same area as Ω. Then,

$$\lim_{a \to 0^+} \frac{\mu_\Omega(a)}{a} = \inf_{u \in E(\Omega) \setminus \{0\} \colon \partial_{\bar{z}} u = 0 \text{ in } \Omega} \frac{\int_{\partial\Omega} |u|^2}{\int_\Omega |u|^2} =: S_\Omega \geq S_D = 2\sqrt{\frac{\pi}{|\Omega|}},$$

with equality if and only if Ω is a disk. As a consequence, there exists $a_1 > 0$ (depending on Ω) such that $\mu_\Omega(a) > \mu_D(a)$ for all $a \in (0, a_1)$.



Translation to the quantum dot Dirac operators

Quantum dot Dirac operator $\mathcal{D}_\theta$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $m \geq 0$.

$$\begin{cases} -i(\partial_1 - i\partial_2)v = (\lambda - m)u & \text{in } \Omega, \\ -i(\partial_1 + i\partial_2)u = (\lambda + m)v & \text{in } \Omega, \\ v = i\dfrac{1-\sin\theta}{\cos\theta}(\nu_1 + i\nu_2)u & \text{on } \partial\Omega. \end{cases}$$

$$\begin{cases} -\Delta u = (\lambda^2 - m^2)u & \text{in } \Omega, \\ -i(\partial_1 + i\partial_2)u = i(\lambda + m)\dfrac{1-\sin\theta}{\cos\theta}(\nu_1 + i\nu_2)u & \text{on } \partial\Omega. \end{cases}$$

Assume $\lambda > m$ $\quad (\theta, \lambda) \overset{T}{\mapsto} (a, \mu) := \left((\lambda + m)\dfrac{1-\sin\theta}{\cos\theta}, \lambda^2 - m^2\right)$, $\mu, a > 0$

$$\begin{cases} -\Delta u = \mu u & \text{in } \Omega, \\ 2\overline{\nu}\partial_{\bar{z}} u + au = 0 & \text{on } \partial\Omega. \end{cases} \longrightarrow$$

Conjecture ($\overline{\partial}$-R) valid for $a \to 0^+, +\infty$.

Translation to the quantum dot Dirac operators

Theorem (D., Mas, Sanz-Perela, 2025)

Assume that $m \geq 0$. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with $\mathcal{C}^2$ boundary and let $D \subset \mathbb{R}^2$ be a disk with the same area as Ω. If Ω is not a disk, then:

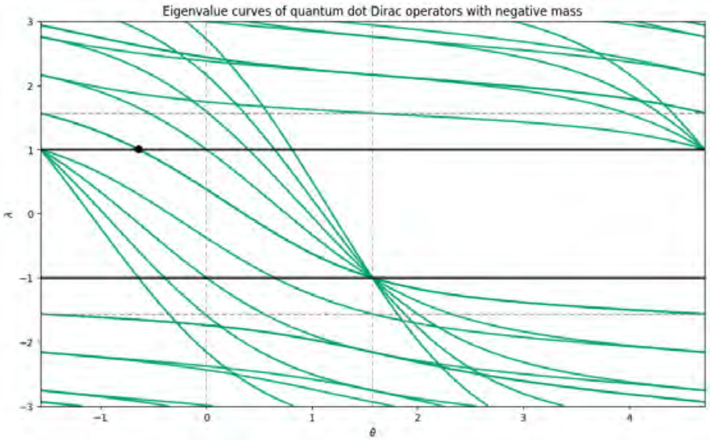
- *There exists $\theta_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$ (depending on Ω) such that $\lambda_\Omega(\theta) > \lambda_D(\theta)$ for all $\theta \in (-\frac{\pi}{2}, \theta_0)$.*
- *If in addition Ω is simply connected, then there exists $\theta_1 \in (-\frac{\pi}{2}, \frac{\pi}{2})$ (depending on Ω) such that $\lambda_\Omega(\theta) > \lambda_D(\theta)$ for all $\theta \in (\theta_1, \frac{\pi}{2})$.*

The case of negative mass

$$\begin{cases} \begin{pmatrix} m & -i(\partial_1 - i\partial_2) \\ -i(\partial_1 + i\partial_2) & -m \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix} & \text{in } D_R, \\ v = i\dfrac{1-\sin\theta}{\cos\theta}(\nu_1 + i\nu_2)u & \text{on } \partial D_R. \end{cases}$$

$m < 0$

$\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



Same construction

Quantum dot Dirac operator $\mathcal{D}_\theta$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $m < 0$.

$$\begin{cases} -i(\partial_1 - i\partial_2)v = (\lambda - m)u & \text{in } \Omega, \\ -i(\partial_1 + i\partial_2)u = (\lambda + m)v & \text{in } \Omega, \\ v = i\dfrac{1-\sin\theta}{\cos\theta}(\nu_1 + i\nu_2)u & \text{on } \partial\Omega. \end{cases}$$

$$\downarrow$$

$$\begin{cases} -\Delta u = (\lambda^2 - m^2)u & \text{in } \Omega, \\ -i(\partial_1 + i\partial_2)u = i(\lambda + m)\dfrac{1-\sin\theta}{\cos\theta}(\nu_1 + i\nu_2)u & \text{on } \partial\Omega. \end{cases}$$

If $\lambda \to |m|^+$ $\Big\downarrow$ $(\theta, \lambda) \overset{T}{\mapsto} (a, \mu) := \left((\lambda + m)\dfrac{1-\sin\theta}{\cos\theta}, \lambda^2 - m^2\right) \to (0^+, 0^+)$

$$\begin{cases} -\Delta u = \mu u & \text{in } \Omega, \\ 2\overline{\nu}\partial_{\bar{z}} u + a u = 0 & \text{on } \partial\Omega. \end{cases} \longrightarrow$$

Conjecture ($\overline{\partial}$-R) valid for $a \to 0^+$.

Another shape optimization problem

$$
\begin{cases}
\begin{pmatrix} m & -i(\partial_1 - i\partial_2) \\ -i(\partial_1 + i\partial_2) & -m \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = |m| \begin{pmatrix} u \\ v \end{pmatrix} & \text{in } \Omega, \\
v = i\dfrac{1-\sin\theta}{\cos\theta}(\nu_1 + i\nu_2)u & \text{on } \partial\Omega.
\end{cases}
\tag{1}
$$

Theorem (D., Mas, Sanz-Perela, 2025)

Assume that $m < 0$, and set $\vartheta(\theta) := \frac{1-\sin\theta}{\cos\theta}$. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with $\mathcal{C}^2$ boundary. Then:

Another shape optimization problem

$$
\begin{cases}
\begin{pmatrix} m & -i(\partial_1 - i\partial_2) \\ -i(\partial_1 + i\partial_2) & -m \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = |m| \begin{pmatrix} u \\ v \end{pmatrix} & \text{in } \Omega, \\
v = i\dfrac{1 - \sin\theta}{\cos\theta}(\nu_1 + i\nu_2)u & \text{on } \partial\Omega.
\end{cases} \tag{1}
$$

Theorem (D., Mas, Sanz-Perela, 2025)

Assume that $m < 0$, *and set* $\vartheta(\theta) := \frac{1-\sin\theta}{\cos\theta}$. *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded domain with* $\mathcal{C}^2$ *boundary. Then:*

- $\min \left\{ \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) : (1) \text{ has a nonzero solution} \right\} = \vartheta^{-1}\left(\frac{2|m|}{S_\Omega}\right).$

Another shape optimization problem

$$\begin{cases} \begin{pmatrix} m & -i(\partial_1 - i\partial_2) \\ -i(\partial_1 + i\partial_2) & -m \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = |m| \begin{pmatrix} u \\ v \end{pmatrix} & \text{in } \Omega, \\ v = i\dfrac{1-\sin\theta}{\cos\theta}(\nu_1 + i\nu_2)u & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Theorem (D., Mas, Sanz-Perela, 2025)

Assume that $m < 0$, and set $\vartheta(\theta) := \frac{1-\sin\theta}{\cos\theta}$. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with C^2 boundary. Then:

- $\min\left\{\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) :\ (1) \textit{ has a nonzero solution}\right\} = \vartheta^{-1}\left(\frac{2|m|}{S_\Omega}\right)$.
- *If in addition Ω is simply connected, then*

$$\min\left\{\theta \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2}) :\ (1) \textit{ has a nonzero solution}\right\} \geq \vartheta^{-1}\left(|m|\sqrt{\frac{|\Omega|}{\pi}}\right)$$

and the equality holds if and only if Ω is a disk of the same area.

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