Dirac operators with critical shell interactions in a finite box

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Given a **bounded smooth** domain (Manifold) $\Omega \subset \mathbb{R}^n$, find a **self-adjoint local** operator $\mathcal{H}$ in $L^2(\Omega)$ that produces an **interval** of essential spectra.

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Local operators with at least one point in the essential spectrum:

(i) Sign-changing Laplacians

- A.-S. Bonnet-Ben Dhia, M. Dauge, K. Ramdani: *Analyse spectrale et singularités d'un problème de transmission non-coercive.* C. R. Acad. Sci. Paris **328** (1999) 717–720.
- C. Cacciapuoti, K. Pankrashkin, A. Posilicano: *Self-adjoint indefinite Laplacians*. J. Anal. Math. **139** (2018) 155–177.

(ii) The Harmonic operator on smooth domain has an eigenvalue with infinite multiplicity:

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Non-local operators: The Dirichlet-Neumann operator on the boundary of a bounded domain with a peak has an interval of essential spectrum:

- S. A. Nazarov, J. Taskinen, *On the Spectrum of the Steklov Problem in a Domain with a Peak*. Vestnik St.Petersb. Univ.Math. **41**, 45–52 (2008)

The free Dirac operator

Given $m \in \mathbb{R}$, we consider the free Dirac operator in $\mathbb{R}^3$

$$\boxed{H\psi = (-i\alpha \cdot \nabla + m\beta)\psi = -i\sum_{j=1}^{3} \alpha_j \partial_j \psi + m\beta\psi,}$$

for $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T \in \mathrm{dom}(H) = H^1(\mathbb{R}^3, \mathbb{C}^4)$,

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for $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T \in \mathrm{dom}(H) = H^1(\mathbb{R}^3, \mathbb{C}^4)$, where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and β are the Dirac matrices:

$$\beta = \begin{pmatrix} \mathbf{I}_2 & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_2 \end{pmatrix} \quad \text{and} \quad \alpha_j = \begin{pmatrix} \mathbf{0} & \sigma_j \\ \sigma_j & \mathbf{0} \end{pmatrix} \quad j = 1, 2, 3,$$

and $\sigma := (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and satisfy the anticommutation relations:

$$\alpha_i \alpha_j + \alpha_j \alpha_i = 2\delta_{ij} I_2, \quad \beta\alpha_i = -\alpha_i \beta.$$

For $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ we use the notations

$$\alpha \cdot x := x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3 \quad \text{and} \quad \sigma \cdot x := x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3.$$

Some facts

$$H = \begin{pmatrix} m & 0 & -i\partial_3 & -i(\partial_1 - i\partial_2) \\ 0 & m & -i(\partial_1 + i\partial_2) & i\partial_3 \\ -i\partial_3 & -i(\partial_1 - i\partial_2) & -m & 0 \\ -i(\partial_1 + i\partial_2) & i\partial_3 & 0 & -m \end{pmatrix}.$$

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We have that

- $(H, H^1(\mathbb{R}^3, \mathbb{C}^4))$ is self-adjoint and its spectrum is

$$\mathrm{spec}(H) = \mathrm{spec}_{\mathrm{cont}}(H) = (-\infty, -|m|] \cup [+|m|, +\infty).$$

- $(H - z)(H + z) = (-\Delta + m^2 - z^2)I_4$
- $(H - z)^{-1} f(x) = \int_{\mathbb{R}^3} \phi_z(x - y) f(y) \mathrm{d}y$

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where the kernel ϕ_z is given by

$$\phi_z(x) = \frac{e^{i\sqrt{z^2 - m^2}|x|}}{4\pi|x|} \left(z + m\beta + (1 - i\sqrt{z^2 - m^2}|x|) i\alpha \cdot \frac{x}{|x|^2} \right)$$

for all $x \in \mathbb{R}^3 \setminus \{0\}$ and $z \in \rho(H)$.

Dirac operators in bounded domains

Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz domain and let D_B **be a** **regular self-adjoint** Dirac operator in $L^2(\Omega, \mathbb{C}^4)$ acting as

$$D_B u = Hu,$$
$$\mathrm{dom} D_B = \left\{u \in H^{\frac{1}{2}}(\Omega; \mathbb{C}^4) : \alpha \cdot \nabla u \in L^2(\Omega; \mathbb{C}^4) \text{ and } Bu_{|\partial\Omega} = 0\right\},$$

where B is a **local boundary condition**. In particular, D_B has a compact resolvent, and one has

$$\mathrm{dom} D_B \subset H^1_{\mathrm{loc}}(\Omega; \mathbb{C}^4).$$

Example: Genaralized MIT boundary conditions $B = B(\tau) := 1 - i\cos(\tau)\beta(\alpha \cdot \nu^{\partial\Omega}) + \sin(\tau)\beta$ where $\tau \in [0, 2\pi) \setminus \{\frac{\pi}{2}, \frac{3\pi}{2}\}$.

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The special cases $\tau = 0$ and $\tau = \pi$ correspond to the MIT and anti-MIT boundary conditions, respectively:

$$\mathrm{spec}\, \mathbb{D}_{B(0)} \subset \mathbb{R} \setminus [-|m|, |m|] \quad \text{if } m \geq 0,$$

and

$$\mathrm{spec}\, \mathbb{D}_{B(\pi)} \subset \mathbb{R} \setminus [-|m|, |m|] \quad \text{if } m \leq 0.$$

Shell interaction in a finite box

Let $\Omega_+ \subset \mathbb{R}^3$ be a bounded smooth domain such that $\overline{\Omega_+} \subset \Omega$, and set

$$\Omega_- = \mathbb{R}^3 \setminus \overline{\Omega_+} \quad \text{and} \quad \Sigma = \partial\Omega_+ = \partial\Omega_-,$$

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Given $\epsilon, \mu \in \mathbb{R}$ such that $\epsilon^2 - \mu^2 = 4$, we consider in $L^2(\Omega, \mathbb{C}^4)$ the Dirac operator

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which is rigorously defined as follows

$$\begin{aligned} H_B u &= (Hu_+) \oplus (Hu_-), \\ \mathrm{dom} H_B &= \big\{ u = u_+ \oplus u_- \in L^2(\Omega_+; \mathbb{C}^4) \oplus L^2(\Omega_-; \mathbb{C}^4) : (\alpha \cdot \nabla) u_\pm \in L^2(\Omega_\pm; \mathbb{C}^4) \\ &\qquad \frac{1}{2}(\epsilon I_4 + \mu\beta)(u_+ + u_-) + i(\alpha \cdot \nu)(u_+ - u_-) = 0 \text{ holds in } H^{-1/2}(\Sigma, \mathbb{C}^4) \text{ and } Bu_{|\partial\Omega} = 0 \big\}, \end{aligned}$$

The main result

Theorem (Behrndt, Holzmann, B, Pankrashkin; in prep)

Let κ_1 *and* κ_2 *be the principal curvatures on* Σ *and set*

$$A_\Sigma := \max_{x\in\Sigma} \left|\kappa_1(x) - \kappa_2(x)\right|.$$

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Let D_B be a regular Dirac operator. Then H_B is self-adjoint and there holds

$$\mathrm{spec}_{\mathrm{ess}}(H_B) = \left[-\frac{m\mu}{\epsilon} - \frac{A_\Sigma}{2|\epsilon|}, -\frac{m\mu}{\epsilon} + \frac{A_\Sigma}{2|\epsilon|} \right]$$

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Remark: Note that $\mathrm{spec}_{\mathrm{ess}}(H_B) = \{ -\frac{m\mu}{\epsilon} \}$ if and only if Σ is the union of finitely many disjoint spheres.

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The main idea comes from:

B. Benhellal, K. Pankrashkin: *Curvature contribution to the essential spectrum of Dirac operators with critical shell interactions*. Pure Appl. Anal. **6** (2024) 237–252.

Some details: the Green function of D_B

Recall that the fundamental solution of H:

$$\phi_z(x) = \left(zI_4 + m\beta + \left(1 - i\sqrt{\lambda^2 - m^2}|x|\right) \frac{i(\alpha \cdot x)}{|x|^2} \right) \frac{e^{i\sqrt{z^2-m^2}|x|}}{4\pi|x|}, \quad z \in \mathbb{C}.$$

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Theorem

For any $f \in L^2(\Omega, \mathbb{C}^4)$ there holds

$$(D_B - z)^{-1} f(x) = \int_\Omega M_z(x,y)\varphi(y)\mathrm{d}y, \quad \forall x \in \Omega,$$

for any $z \in \rho(D_B)$, where $M_z(x,y) = \phi_z(x,y) + \psi_z^B(x,y)$ with a smoothing kernel $\psi_z^B \in C^\infty(\Omega \times \Omega, \mathcal{M}_4(\mathbb{C}))$.

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Define the operators $\mathcal{C}_z^B : L^2(\Sigma, \mathbb{C}^4) \longrightarrow L^2(\Sigma, \mathbb{C}^4)$

$$\mathcal{C}_z^B g(x) = \lim_{\rho \searrow 0} \int_{|x-y|>\rho} M_z(x,y) g(y) \mathrm{d}\sigma(y), \quad \forall x \in \Sigma.$$

Then, $\mathcal{C}_z^B$ is a bounded operators from $H^s(\Sigma, \mathbb{C}^4)$ into itself for any $s \in \mathbb{R}$.

Let $\Lambda : H^s(\Sigma) \to H^{s-1/2}(\Sigma)$ be an isomorphism $\forall s \in \mathbb{R}$, with $\Lambda = \Lambda^*$ in $L^2(\Sigma)$.
For $z \in \rho(D_B)$ consider the operator

$$\mathcal{L}_z = \Lambda \left(\frac{1}{4}(\epsilon - \mu\beta) + \mathcal{C}_z^B \right) \Lambda$$

acting on the maximal domain $\mathrm{dom}(\mathcal{L}_z) = \{ g \in L^2(\Sigma, \mathbb{C}^4) : \mathcal{L}_z g \in L^2(\Sigma, \mathbb{C}^4) \}$.

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Theorem

Let D_B be a regular Dirac operator. Then

- *For any $z \in \mathbb{R} \cap \rho(D_B)$: H_B is self-adjoint $\Leftrightarrow$ $\mathcal{L}_z$ is self-adjoint.*
- *For any $z \in \rho(D_B)$:* $z \in \mathrm{spec}_{\mathrm{ess}}(H_B) \quad \Leftrightarrow \quad 0 \in \mathrm{spec}_{\mathrm{ess}}(\mathcal{L}_z)$.
- $z \in \rho(H_B) \cap \rho(D_B)$ *if and only if $\mathcal{L}_z$ is boundedly invertible in $L^2(\Sigma; \mathbb{C}^4)$, and in this case*

$$(H_B - z)^{-1} = (D_B - z)^{-1} - \Phi_{B,z} (\mathcal{L}_z)^{-1} \Phi_{B,\bar{z}}^*$$

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Theorem

let B and $\tilde{B}$ be two boundary conditions such that $B \neq \tilde{B}$. Then, for any $z \in \mathbb{C} \setminus \mathbb{R}$ the operator $(H_B - z)^{-1} - (H_{\tilde{B}} - z)^{-1}$ is compact in $L^2(\Omega; \mathbb{C}^4)$

$$M_z = \phi_z + \psi_z^B = \begin{pmatrix} (z+m)k_z I_2 & w_z \\ w_z & (z-m)k_z I_2 \end{pmatrix} + \psi_z^B,$$

with

$$k_z(x) = \frac{e^{i\sqrt{z^2-m^2}|x|}}{4\pi|x|}, \qquad w_z(x) = \frac{e^{i\sqrt{z^2-m^2}|x|}}{4\pi|x|}\left(1 - i\sqrt{\lambda^2-m^2}|x|\right) i\sigma \cdot \frac{x}{|x|^2}$$

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For any $z \in \mathbb{C}$, K_z and W_z are ΨDO of order -1 and 0, resp. Moreover, For any $|z| \leq |m|$, $K_z : H^s(\Sigma) \to H^{s-1}(\Sigma)$ is an isomorphism for any $s \in \mathbb{R}$.

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and thus for any $z \in \rho(D_B)$ we have

$$\mathcal{L}_z = \Lambda \begin{pmatrix} \frac{\epsilon-\mu}{4} + (z+m)K_z \otimes I_2 & W_z \\ W_z & \frac{\epsilon+\mu}{4} + (z-m)K_z \otimes I_2 \end{pmatrix} \Lambda + \underbrace{\Lambda \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} \Lambda}_{\text{smoothing operator}},$$

Now, choosing $\Lambda := K_m^{-\frac{1}{2}}$ and using some ΨDO properties of K_z and W_z, we get

$$\mathcal{L}_z = \underbrace{\begin{pmatrix} \frac{\epsilon-\mu}{4} K_m^{-1} + (z+m) \otimes I_2 & K_m^{-\frac{1}{2}} W_m K_m^{-\frac{1}{2}} \\ K_m^{-\frac{1}{2}} W_m K_m^{-\frac{1}{2}} & \frac{\epsilon+\mu}{4} K_m^{-1} + (z-m) \otimes I_2 \end{pmatrix}}_{zI+\mathcal{A}} + \underbrace{\begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}}_{\textbf{compact operator}} ,$$

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Theorem

For any $z \in \mathbb{C}$, one has $-z \in \mathrm{spec}_{\mathrm{ess}}(\mathcal{A}) \Leftrightarrow z \in J$. In particular, the operator-valued function $z \mapsto (zI + \mathcal{A})^{-1}$ is meromorphic in $\mathbb{C} \setminus J$.

Sketch of the proof. We write $z + \mathcal{A} = \begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix}$,

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$$S_z = l_{22} + l_{21}(l_{11})^{-1} l_{12}.$$

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$$0 \in \mathrm{spec}_{\mathrm{ess}}(\overline{S_z}) \Longleftrightarrow -z \in \mathrm{spec}_{\mathrm{ess}}(\mathcal{A}) \Longleftrightarrow 0 \in \mathrm{spec}_{\mathrm{ess}}(\mathcal{L}_z) \Longleftrightarrow z \in \mathrm{spec}_{\mathrm{ess}}(H_B).$$

The expression for S_z can be simplified as follows

$$S_z = \left[(\epsilon + \mu) K_m^{-1/2} \left(\frac{1}{4} - W_m^2 \right) K_m^{-1/2} + \frac{2}{\epsilon - \mu} (z\epsilon + m\mu) \right] + L$$

for some compact self-adjoint operator L.

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Note that

$$W_m^2 - \frac{1}{4} = (\sigma\cdot\nu)\underbrace{\left[(\sigma\cdot\nu)W_m + W_m(\sigma\cdot\nu)\right]}_{\Theta} W_m.$$

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Then, for any $z\in\rho(D_B)$, S_z is **bounnded** in $L^2(\Sigma;\mathbb{C}^2)$, and for any $z\in\rho(D_B)$ one has

$$z\in \mathrm{spec}_{\mathrm{ess}}(H_B) \Longleftrightarrow -z\in \mathrm{spec}_{\mathrm{ess}}(\mathcal{A}) \Longleftrightarrow 0\in \mathrm{spec}_{\mathrm{ess}}(S_z) \Longleftrightarrow z\in \mathrm{spec}_{\mathrm{ess}}\underbrace{\Big[\frac{2}{\epsilon}K_m^{-\frac{1}{2}}(\sigma\cdot\nu)\Theta W_m K_m^{-\frac{1}{2}} - \frac{m\mu}{\epsilon}\Big]}_{\mathcal{P}}.$$

Theorem (B.B & Konstantin Pankrashkin; PAA 24')

We have $\mathrm{spec}_{\mathrm{ess}}(\mathcal{P}) = \Big[-\frac{m\mu}{\epsilon} - \frac{A_\Sigma}{2|\epsilon|}, -\frac{m\mu}{\epsilon} + \frac{A_\Sigma}{2|\epsilon|}\Big].$

This implies that

$$\forall z \notin \mathrm{spec}(D_B): \quad z \in \mathrm{spec}_{\mathrm{ess}}(H_B) \Leftrightarrow z \in J.$$

As $\mathrm{spec}(D_B)$ is discrete, we get $J \subset \mathrm{spec}_{\mathrm{ess}}(H_B)$.

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$$(H_B - z)^{-1} = (D_B - z)^{-1} - \Phi_{B,z} (\mathcal{L}_z)^{-1} \Phi^*_{B,\bar{z}}$$

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> There is a discrete set $\mathcal{O} \subset \mathbb{C} \setminus J$ without accumulation points in $\mathbb{C} \setminus J$, such that the inverse $(\mathcal{L}_z)^{-1}$ exists and is bounded for $z \in \mathbb{C} \setminus (J \cup \mathcal{O} \cup \mathrm{spec}(D_B))$ and has a meromorphic extension to $\mathbb{C} \setminus (J \cup \mathcal{O})$ such that the coefficient in the Laurent series at the point of $\mathcal{O}$ are finite rank operators.

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This implies that the set $(\mathbb{C} \setminus J) \cap \mathrm{spec}(H_B) \cap \rho(D_B) \subset \mathcal{O}$ has no accumulation point in $\mathbb{C} \setminus J$, and each point of this set is a discrete eigenvalue of H_B, and therefore $\mathrm{spec}_{\mathrm{ess}}(H_B) \subset J$.

The Kerzman-Stein operator

We have

$$\mathcal{A}g(x) = \lim_{\rho \searrow 0} \int_{\substack{y \in \Sigma \\ |x-y|>\rho}} \theta(x,y) g(y) \mathrm{d}\sigma(y)$$

with

$$\theta(x,y) = \frac{i}{4\pi|x-y|^3} \Big(\big(\sigma \cdot \nu(x)\big)\, \sigma \cdot (x-y) + \sigma \cdot (x-y) \big(\sigma \cdot \nu(y)\big) \Big).$$

Use the property $(\sigma \cdot x)(\sigma \cdot y) = \langle x, y \rangle I_2 + i\sigma \cdot (x \times y)$ for $x, y \in \mathbb{R}^3$. Then

$$\theta(x,y) = \frac{i}{4\pi|x-y|^3} \langle \nu(x) + \nu(y), x-y \rangle I_2 - \sigma \cdot \frac{\big(\nu(x) - \nu(y)\big) \times (x-y)}{4\pi|x-y|^3}.$$

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Thus

$$\Theta g(x) = \mathcal{N} g(x) - \mathcal{N}^* g(x) - \lim_{\rho \searrow 0} \int_{\substack{y \in \Sigma \\ |x-y| > \rho}} \sigma \cdot \frac{(\nu(x) - \nu(y)) \times (x-y)}{4\pi|x-y|^3} g(y) \mathrm{d}\sigma(y)$$

where $\mathcal{N}$ is the double layer operator (which is a ΨDO of order -1)

For $x \in \Sigma$ denote by $M_x := -\mathrm{d}\nu|_x : T_x\Sigma \to T_x\Sigma$ the respective Weingarten map with eigenvalues $\kappa_1(x)$ and $\kappa_2(x)$. We choose an orthonormal basis (e_1, e_2) in $T_x\Sigma$ such that $M_x e_j = \kappa_j(x) e_j$ for $j \in \{1, 2\}$ and the basis $\big(e_1, e_2, \nu(x)\big)$ of $\mathbb{R}^3$ is positively oriented. Then one can construct a local chart $\varphi : \mathbb{R}^2 \supset U \to V \subset \Sigma$ near x with $0 \in U$, $\varphi(0) = 0$, $\partial_j \varphi(0) = e_j$ for $j \in \{1, 2\}$.

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Theorem

The principal symbols q_Θ of Θ and $q_{\mathcal{P}}$ of $\mathcal{P}$ (in the above special local coordinates) near $x \in \Sigma$ are

$$q_\Theta(x, \xi) = \frac{1}{2}\big(\kappa_1(x) - \kappa_2(x)\big)(\sigma \cdot \nu(x))\,\frac{\xi_1\xi_2}{|\xi|^3},$$

$$q_{\mathcal{P}}(x, \xi) = \frac{\kappa_1(x) - \kappa_2(x)}{2\epsilon}\,\frac{\xi_1\xi_2}{|\xi|^3}\,\sigma \cdot (\xi_1 e_1 + \xi_2 e_2) - \frac{m\mu}{\epsilon}.$$

Moreover, $\mathrm{Sp}\big(q_{\mathcal{P}}(x, \xi)\big) = \left\{ \frac{\kappa_1(x) - \kappa_2(x)}{2\epsilon}\,\frac{\xi_1\xi_2}{|\xi|^2} - \frac{m\mu}{\epsilon}, -\frac{\kappa_1(x) - \kappa_2(x)}{2\epsilon}\,\frac{\xi_1\xi_2}{|\xi|^2} - \frac{m\mu}{\epsilon} \right\}$. *Therefore*

$$\mathrm{Sp}_{\mathrm{ess}}(\mathcal{P}) = \bigcup_{\xi \neq 0} \mathrm{Sp}(q_{\mathcal{P}}(x, \xi)) = \Big[-\frac{m\mu}{\epsilon} - \frac{A_\Sigma}{2|\epsilon|}, -\frac{m\mu}{\epsilon} + \frac{A_\Sigma}{2|\epsilon|} \Big] = J$$

Thank you for your attention