Orthogonality of Weyl orbit functions on lattices and the central splitting mechanism

Vojtěch Teska

joint work with
Jiří Hrivnák, Lenka Motlochová

Department of Physics
Faculty of Nuclear Sciences and Physical Engineering
Czech Technical University in Prague

29 August 2025

Outline

Introduction

- Weyl orbit functions generalize the periodicity and boudary behaviour of sine and cosine in dimensions > 1.

Introduction

- Weyl orbit functions generalize the periodicity and boudary behaviour of sine and cosine in dimensions > 1.
- Our goal is study Fourier–like transformations of functions on discrete sets in the most general setting

Introduction

- Weyl orbit functions generalize the periodicity and boudary behaviour of sine and cosine in dimensions > 1.
- Our goal is study Fourier–like transformations of functions on discrete sets in the most general setting
- Specific cases were already examined $\rightarrow$ find a more general formulation, look for new ones

Introduction

- Weyl orbit functions generalize the periodicity and boudary behaviour of sine and cosine in dimensions > 1.
- Our goal is study Fourier–like transformations of functions on discrete sets in the most general setting
- Specific cases were already examined $\rightarrow$ find a more general formulation, look for new ones
- We also want to find ways to make these calculations more efficient

Introduction

- Weyl orbit functions generalize the periodicity and boudary behaviour of sine and cosine in dimensions > 1.
- Our goal is study Fourier–like transformations of functions on discrete sets in the most general setting
- Specific cases were already examined $\rightarrow$ find a more general formulation, look for new ones
- We also want to find ways to make these calculations more efficient
- Special cases: discrete sine and cosine transforms – used in image processing

Introduction

- Weyl orbit functions generalize the periodicity and boudary behaviour of sine and cosine in dimensions > 1.
- Our goal is study Fourier–like transformations of functions on discrete sets in the most general setting
- Specific cases were already examined $\rightarrow$ find a more general formulation, look for new ones
- We also want to find ways to make these calculations more efficient
- Special cases: discrete sine and cosine transforms – used in image processing
- In physics - graphene, quantum dots

Weyl groups

- $\mathbb{R}^n$ with standard inner product $\langle \cdot, \cdot \rangle$.

Weyl groups

- $\mathbb{R}^n$ with standard inner product $\langle \cdot, \cdot \rangle$.
- We examine an irreducible essential crystallographic root system Π with simple roots $\{\alpha_i\}_{i=1}^n$.

Weyl groups

- $\mathbb{R}^n$ with standard inner product $\langle\cdot,\cdot\rangle$.
- We examine an irreducible essential crystallographic root system Π with simple roots $\{\alpha_i\}_{i=1}^n$.
- Dual root to $\alpha \in \Pi$ is defined as $\alpha^\vee = \frac{2}{\langle\alpha,\alpha\rangle}\alpha$.

Weyl groups

- $\mathbb{R}^n$ with standard inner product $\langle \cdot, \cdot \rangle$.
- We examine an irreducible essential crystallographic root system Π with simple roots $\{\alpha_i\}_{i=1}^n$.
- Dual root to $\alpha \in \Pi$ is defined as $\alpha^\vee = \frac{2}{\langle \alpha,\alpha \rangle}\alpha$.
- Weyl group W is generated by the reflections

$$r_i v = v - \left\langle v, \alpha_i^\vee \right\rangle \alpha_i = v - \langle v, \alpha_i \rangle \alpha_i^\vee, \quad v \in \mathbb{R}^n. \tag{1}$$

Weyl groups

- $\mathbb{R}^n$ with standard inner product $\langle \cdot, \cdot \rangle$.
- We examine an irreducible essential crystallographic root system Π with simple roots $\{\alpha_i\}_{i=1}^n$.
- Dual root to $\alpha \in \Pi$ is defined as $\alpha^\vee = \frac{2}{\langle \alpha, \alpha \rangle} \alpha$.
- Weyl group W is generated by the reflections

$$r_i v = v - \left\langle v, \alpha_i^\vee \right\rangle \alpha_i = v - \langle v, \alpha_i \rangle \alpha_i^\vee, \quad v \in \mathbb{R}^n. \tag{1}$$

- Reflections are orthogonal maps with $\det(r_i) = -1$, therefore $W \subseteq \mathrm{O}(n)$.

Invariant lattices

- A lattice L in $\mathbb{R}^n$ is the $\mathbb{Z}$–span of some basis of $\mathbb{R}^n$,

$$L = \mathbb{Z}\ell_1 \oplus \cdots \oplus \mathbb{Z}\ell_n. \tag{2}$$

Invariant lattices

- A lattice L in $\mathbb{R}^n$ is the $\mathbb{Z}$–span of some basis of $\mathbb{R}^n$,

$$L = \mathbb{Z}\ell_1 \oplus \cdots \oplus \mathbb{Z}\ell_n. \tag{2}$$

- The dual lattice of L is

$$L^{\perp} = \{v \in \mathbb{R}^n | \langle v, l\rangle \in \mathbb{Z}, \forall l \in L\}. \tag{3}$$

Invariant lattices

- A lattice L in $\mathbb{R}^n$ is the $\mathbb{Z}$–span of some basis of $\mathbb{R}^n$,

$$L = \mathbb{Z}\ell_1 \oplus \cdots \oplus \mathbb{Z}\ell_n. \tag{2}$$

- The dual lattice of L is

$$L^{\perp} = \{v \in \mathbb{R}^n | \langle v, l \rangle \in \mathbb{Z}, \forall l \in L\}. \tag{3}$$

- Lattices are commutative groups.

Invariant lattices

- A lattice L in $\mathbb{R}^n$ is the $\mathbb{Z}$–span of some basis of $\mathbb{R}^n$,

$$L = \mathbb{Z}\ell_1 \oplus \cdots \oplus \mathbb{Z}\ell_n. \tag{2}$$

- The dual lattice of L is

$$L^{\perp} = \{v \in \mathbb{R}^n | \langle v, l \rangle \in \mathbb{Z}, \forall l \in L\}. \tag{3}$$

- Lattices are commutative groups.
- Considering a sublattice $K \subseteq L$ with basis $\Bbbk_1, \ldots \Bbbk_n$, it holds that

$$|L/K| = |\det(X)|, \quad \Bbbk_i = \sum_{j=1}^{n} X_{ij}\ell_j. \tag{4}$$

Invariant lattices

- A lattice L in $\mathbb{R}^n$ is the $\mathbb{Z}$–span of some basis of $\mathbb{R}^n$,

$$L = \mathbb{Z}\ell_1 \oplus \cdots \oplus \mathbb{Z}\ell_n. \tag{2}$$

- The dual lattice of L is

$$L^{\perp} = \{v \in \mathbb{R}^n | \langle v, l \rangle \in \mathbb{Z}, \forall l \in L\}. \tag{3}$$

- Lattices are commutative groups.
- Considering a sublattice $K \subseteq L$ with basis $\Bbbk_1, \dots \Bbbk_n$, it holds that

$$|L/K| = |\det(X)|, \quad \Bbbk_i = \sum_{j=1}^{n} X_{ij}\ell_j. \tag{4}$$

- Lattice is W–invariant if

$$wL \subseteq L, \quad \forall w \in W. \tag{5}$$

Invariant lattices

- Given a W–invariant lattice L, $L^{\perp}$ is also W–invariant.

Invariant lattices

- Given a W–invariant lattice L, $L^{\perp}$ is also W–invariant.
- $(L^{\perp})^{\perp} = L$

Invariant lattices

- Given a W–invariant lattice L, $L^{\perp}$ is also W–invariant.
- $(L^{\perp})^{\perp} = L$
- Root and coroot lattice:

$$Q = \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_n, \quad Q^{\vee} = \mathbb{Z}\alpha_1^{\vee} \oplus \cdots \oplus \mathbb{Z}\alpha_n^{\vee} \tag{6}$$

Invariant lattices

- Given a W–invariant lattice L, $L^\perp$ is also W–invariant.
- $(L^\perp)^\perp = L$
- Root and coroot lattice:

$$Q = \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_n, \quad Q^\vee = \mathbb{Z}\alpha_1^\vee \oplus \cdots \oplus \mathbb{Z}\alpha_n^\vee \tag{6}$$

- Weight and coweight lattice:

$$P = \mathbb{Z}\omega_1 \oplus \cdots \oplus \mathbb{Z}\omega_n, \quad P^\vee = \mathbb{Z}\omega_1^\vee \oplus \cdots \oplus \mathbb{Z}\omega_n^\vee \tag{7}$$

Invariant lattices

- Given a W–invariant lattice L, $L^\perp$ is also W–invariant.
- $(L^\perp)^\perp = L$
- Root and coroot lattice:

$$Q = \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_n, \quad Q^\vee = \mathbb{Z}\alpha_1^\vee \oplus \cdots \oplus \mathbb{Z}\alpha_n^\vee \tag{6}$$

- Weight and coweight lattice:

$$P = \mathbb{Z}\omega_1 \oplus \cdots \oplus \mathbb{Z}\omega_n, \quad P^\vee = \mathbb{Z}\omega_1^\vee \oplus \cdots \oplus \mathbb{Z}\omega_n^\vee \tag{7}$$

- $Q^\perp = P^\vee, (Q^\vee)^\perp = P$

Admissible shifts

- Admissible shift of a W–invariant lattice L is $\zeta \in \mathbb{R}^n$ such that

$$w(\zeta + L) \subseteq \zeta + L, \quad \forall w \in W. \tag{8}$$

Admissible shifts

- Admissible shift of a W–invariant lattice L is $\zeta \in \mathbb{R}^n$ such that

$$w(\zeta + L) \subseteq \zeta + L, \quad \forall w \in W. \tag{8}$$

- ζ is admissible if and only if $\zeta - r_i\zeta \in L$ for $i = 1, \ldots, n$.

Admissible shifts

- Admissible shift of a W–invariant lattice L is $\zeta \in \mathbb{R}^n$ such that

$$w(\zeta + L) \subseteq \zeta + L, \quad \forall w \in W. \tag{8}$$

- ζ is admissible if and only if $\zeta - r_i\zeta \in L$ for $i = 1, \dots, n$.
- Admissible shifts ζ_1 and ζ_2 are equivalent if

$$\zeta_1 + L = \zeta_2 + L. \tag{9}$$

Admissible shifts

- Admissible shift of a W–invariant lattice L is $\zeta \in \mathbb{R}^n$ such that

$$w(\zeta + L) \subseteq \zeta + L, \quad \forall w \in W. \tag{8}$$

- ζ is admissible if and only if $\zeta - r_i\zeta \in L$ for $i = 1, \ldots, n$.
- Admissible shifts ζ_1 and ζ_2 are equivalent if

$$\zeta_1 + L = \zeta_2 + L. \tag{9}$$

- An admissible shift ζ is trivial if it is equivalent to 0,

$$\zeta + L = L. \tag{10}$$

Admissible shifts

- Admissible shift of a W–invariant lattice L is $\zeta \in \mathbb{R}^n$ such that

$$w(\zeta + L) \subseteq \zeta + L, \quad \forall w \in W. \tag{8}$$

- ζ is admissible if and only if $\zeta - r_i\zeta \in L$ for $i = 1, \dots, n$.
- Admissible shifts ζ_1 and ζ_2 are equivalent if

$$\zeta_1 + L = \zeta_2 + L. \tag{9}$$

- An admissible shift ζ is trivial if it is equivalent to 0,

$$\zeta + L = L. \tag{10}$$

- From this point onward, we will use the notation $L_\zeta \equiv \zeta + L$.

Generalized affine Weyl groups

- Generalized affine Weyl group W_L^{aff} is generated by the reflections r_i and the shifts ℓ_i, $i = 1, \dots, n$, where L is W-invariant

Generalized affine Weyl groups

- Generalized affine Weyl group W_L^{aff} is generated by the reflections r_i and the shifts ℓ_i, $i = 1, \dots, n$, where L is W-invariant
- $W_L^{\mathrm{aff}} = L \rtimes W$, the action of $z \in W_L^{\mathrm{aff}}$ on $v \in \mathbb{R}^n$ is given by

$$z \cdot v = wv + l, \quad w \in W, l \in L. \tag{11}$$

Generalized affine Weyl groups

- Generalized affine Weyl group W_L^{aff} is generated by the reflections r_i and the shifts ℓ_i, $i = 1, \ldots, n$, where L is W-invariant
- $W_L^{\mathrm{aff}} = L \rtimes W$, the action of $z \in W_L^{\mathrm{aff}}$ on $v \in \mathbb{R}^n$ is given by

$$z \cdot v = wv + l, \quad w \in W, l \in L. \tag{11}$$

- Projections $\psi_L : W_L^{\mathrm{aff}} \to W,\ \tau_L : W_L^{\mathrm{aff}} \to L.$

Generalized affine Weyl groups

- Generalized affine Weyl group W_L^{aff} is generated by the reflections r_i and the shifts ℓ_i, $i = 1, \dots, n$, where L is W-invariant
- $W_L^{\text{aff}} = L \rtimes W$, the action of $z \in W_L^{\text{aff}}$ on $v \in \mathbb{R}^n$ is given by

$$z \cdot v = wv + l, \quad w \in W, l \in L. \tag{11}$$

- Projections $\psi_L : W_L^{\text{aff}} \to W,\ \tau_L : W_L^{\text{aff}} \to L.$
- There exist only 4 homomorphisms $\sigma : W \to \{\pm 1\}$:

$$\begin{aligned} &\mathbb{1}(r_i) = 1, && \sigma^e(r_i) = \det(r_i) = -1, \\ &\sigma^s(r_i) = \begin{cases} -1, & \alpha_i \in \Delta_s \\ 1, & \alpha_i \in \Delta_l \end{cases}, && \sigma^l(r_i) = \begin{cases} 1, & \alpha_i \in \Delta_s \\ -1, & \alpha_i \in \Delta_l \end{cases}, \end{aligned} \tag{12}$$

$\Delta_{s,l}$ are the sets of short and long simple roots, respectively.

Weyl orbit functions

- Each sign homomorphism σ corresponds to a family of orbit functions $\varphi^\sigma : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$,

$$\varphi_u^\sigma(v) = \sum_{w \in W} \sigma(w) e^{2\pi \mathrm{i} \langle wu, v \rangle}. \tag{13}$$

Weyl orbit functions

- Each sign homomorphism σ corresponds to a family of orbit functions $\varphi^\sigma : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$,

$$\varphi_u^\sigma(v) = \sum_{w \in W} \sigma(w) e^{2\pi \mathrm{i} \langle wu, v \rangle}. \tag{13}$$

- Define the homomorphism $\gamma^\sigma_{L,\varkappa} : W_L^{\mathrm{aff}} \to U(1)$ by

$$\gamma^\sigma_{L,\varkappa}(z) = \sigma \circ \psi_L(z) e^{2\pi \mathrm{i} \langle \varkappa, \tau_L(z) \rangle}, \tag{14}$$

where $\varkappa$ is an admissible shift of $L^\perp$.

Weyl orbit functions

- Each sign homomorphism σ corresponds to a family of orbit functions $\varphi^\sigma : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$,

$$\varphi^\sigma_u(v) = \sum_{w \in W} \sigma(w) e^{2\pi \mathrm{i} \langle wu, v \rangle}. \tag{13}$$

- Define the homomorphism $\gamma^\sigma_{L,\varkappa} : W^{\mathrm{aff}}_L \to U(1)$ by

$$\gamma^\sigma_{L,\varkappa}(z) = \sigma \circ \psi_L(z) e^{2\pi \mathrm{i} \langle \varkappa, \tau_L(z) \rangle}, \tag{14}$$

where $\varkappa$ is an admissible shift of $L^\perp$.

- For $z \in W^{\mathrm{aff}}_{L^\perp}$, ζ an admissible shift of L, $\lambda \in L_\zeta$, $v \in \mathbb{R}^n$, it holds that

$$\varphi^\sigma_\lambda(z \cdot v) = \gamma^\sigma_{L^\perp,\zeta}(z) \varphi^\sigma_\lambda(v). \tag{15}$$

Weyl orbit functions

- Denote F_L the fundamental domain of W_L^{aff}.

Weyl orbit functions

- Denote F_L the fundamental domain of W_L^{aff}.
- Define $F_L^\sigma(\varkappa) \subseteq F_L$ by

$$F_L^\sigma(\varkappa) = \left\{ v \in F_L | \gamma_{L,\varkappa}^\sigma (\mathrm{Stab}_L(v)) = \{1\} \right\}. \tag{16}$$

- Denote F_L the fundamental domain of W_L^{aff}.
- Define $F_L^\sigma(\varkappa) \subseteq F_L$ by

$$F_L^\sigma(\varkappa) = \left\{ v \in F_L | \gamma_{L,\varkappa}^\sigma(\mathrm{Stab}_L(v)) = \{1\} \right\}. \tag{16}$$

- The symmetry relation implies that

$$\varphi_\lambda^\sigma(v) = 0, \quad v \in F_{L^\perp} \setminus F_{L^\perp}^\sigma(\zeta). \tag{17}$$

for $\lambda \in L_\zeta$

Discrete orthogonality and transforms

- In the remaining part:

Discrete orthogonality and transforms

- In the remaining part:
  - W–invariant lattices A, B such that $B^{\perp} \subseteq A$,

Discrete orthogonality and transforms

- In the remaining part:
  - W–invariant lattices A, B such that $B^{\perp} \subseteq A$,
  - ϱ an admissible shift of A,

Discrete orthogonality and transforms

- In the remaining part:
    - W–invariant lattices A, B such that $B^\perp \subseteq A$,
    - ϱ an admissible shift of A,
    - χ an admissible shift of B,

Discrete orthogonality and transforms

- In the remaining part:
  - W–invariant lattices A, B such that $B^{\perp} \subseteq A$,
  - ϱ an admissible shift of A,
  - χ an admissible shift of B,
  - $M \in \mathbb{N}$.

Discrete orthogonality and transforms

- In the remaining part:
  - W–invariant lattices A, B such that $B^{\perp} \subseteq A$,
  - ϱ an admissible shift of A,
  - χ an admissible shift of B,
  - $M \in \mathbb{N}$.
- Define:

Discrete orthogonality and transforms

- In the remaining part:
  - W–invariant lattices A, B such that $B^{\perp} \subseteq A$,
  - ϱ an admissible shift of A,
  - χ an admissible shift of B,
  - $M \in \mathbb{N}$.
- Define:
  - the point set $F_M^{\sigma}(A_{\varrho}, B_{\chi}) = \frac{1}{M} B_{\chi} \cap F_{A^{\perp}}^{\sigma}(\varrho)$,

Discrete orthogonality and transforms

- In the remaining part:
  - W–invariant lattices A, B such that $B^{\perp} \subseteq A$,
  - ϱ an admissible shift of A,
  - χ an admissible shift of B,
  - $M \in \mathbb{N}$.
- Define:
  - the point set $F^{\sigma}_{M}(A_{\varrho}, B_{\chi}) = \frac{1}{M} B_{\chi} \cap F^{\sigma}_{A^{\perp}}(\varrho)$,
  - the label set $\Lambda^{\sigma}_{M}(A_{\varrho}, B_{\chi}) = A_{\varrho} \cap M F^{\sigma}_{B^{\perp}}(\chi)$.

Discrete orthogonality and transforms

- In the remaining part:
  - W–invariant lattices A, B such that $B^{\perp} \subseteq A$,
  - ϱ an admissible shift of A,
  - χ an admissible shift of B,
  - $M \in \mathbb{N}$.
- Define:
  - the point set $F_M^{\sigma}(A_{\varrho}, B_{\chi}) = \frac{1}{M} B_{\chi} \cap F_{A^{\perp}}^{\sigma}(\varrho)$,
  - the label set $\Lambda_M^{\sigma}(A_{\varrho}, B_{\chi}) = A_{\varrho} \cap M F_{B^{\perp}}^{\sigma}(\chi)$.
- The scalar product of complex-valued functions f, g on $F_M^{\sigma}(A_{\varrho}, B_{\chi})$ is chosen as

$$\langle f, g \rangle_{A_{\varrho}, B_{\chi}}^{\sigma, M} = \sum_{s \in F_M^{\sigma}(A_{\varrho}, B_{\chi})} \varepsilon_{A^{\perp}}(s) f(s) \overline{g(s)}, \tag{18}$$

$$\varepsilon_{A^{\perp}}(v) = |W| / \left|\mathrm{Stab}_{A^{\perp}}(v)\right|. \tag{19}$$

Discrete orthogonality and transforms

- In the remaining part:
  - W–invariant lattices A, B such that $B^\perp \subseteq A$,
  - ϱ an admissible shift of A,
  - χ an admissible shift of B,
  - $M \in \mathbb{N}$.
- Define:
  - the point set $F^\sigma_M(A_\varrho, B_\chi) = \frac{1}{M} B_\chi \cap F^\sigma_{A^\perp}(\varrho)$,
  - the label set $\Lambda^\sigma_M(A_\varrho, B_\chi) = A_\varrho \cap M F^\sigma_{B^\perp}(\chi)$.
- The scalar product of complex-valued functions f, g on $F^\sigma_M(A_\varrho, B_\chi)$ is chosen as

$$\langle f, g \rangle^{\sigma, M}_{A_\varrho, B_\chi} = \sum_{s \in F^\sigma_M(A_\varrho, B_\chi)} \varepsilon_{A^\perp}(s) f(s) \overline{g(s)}, \tag{18}$$

$$\varepsilon_{A^\perp}(v) = |W| / \left|\mathrm{Stab}_{A^\perp}(v)\right|. \tag{19}$$

- Corresponding Hilbert space will be denoted $\mathcal{H}^\sigma_M(A_\varrho, B_\chi)$.

Discrete orthogonality and transforms

- For any $\lambda, \nu \in \Lambda^\sigma_M(A_\varrho, B_\chi)$, it holds that

$$\langle \varphi^\sigma_\lambda, \varphi^\sigma_\nu \rangle^{\sigma,M}_{A_\varrho,B_\chi} = M^n |W| |B/A^\perp| \, |\mathrm{Stab}_{B^\perp}(\lambda/M)| \, \delta_{\lambda\nu}. \tag{20}$$

Discrete orthogonality and transforms

- For any $\lambda, \nu \in \Lambda_M^\sigma(A_\varrho, B_\chi)$, it holds that

$$\langle \varphi_\lambda^\sigma, \varphi_\nu^\sigma \rangle_{A_\varrho, B_\chi}^{\sigma, M} = M^n |W| |B/A^\perp| \, |\mathrm{Stab}_{B^\perp}(\lambda/M)| \, \delta_{\lambda\nu}. \tag{20}$$

- Moreover, $|F_M^\sigma(A_\varrho, B_\chi)| = |\Lambda_M^\sigma(A_\varrho, B_\chi)|$, so the corresponding orbit functions form a basis of $\mathcal{H}_M^\sigma(A_\varrho, B_\chi)$.

Discrete orthogonality and transforms

- For any $\lambda, \nu \in \Lambda^\sigma_M(A_\varrho, B_\chi)$, it holds that

$$\langle \varphi^\sigma_\lambda, \varphi^\sigma_\nu \rangle^{\sigma,M}_{A_\varrho,B_\chi} = M^n |W| |B/A^\perp| \, |\mathrm{Stab}_{B^\perp}(\lambda/M)| \, \delta_{\lambda\nu}. \tag{20}$$

- Moreover, $|F^\sigma_M(A_\varrho, B_\chi)| = |\Lambda^\sigma_M(A_\varrho, B_\chi)|$, so the corresponding orbit functions form a basis of $\mathcal{H}^\sigma_M(A_\varrho, B_\chi)$.
- At this point, we can calculate the Fourier coefficients and write the Plancherel formulas

Discrete orthogonality and transforms

- For any $\lambda, \nu \in \Lambda_M^\sigma(A_\varrho, B_\chi)$, it holds that

$$\langle \varphi_\lambda^\sigma, \varphi_\nu^\sigma \rangle_{A_\varrho, B_\chi}^{\sigma, M} = M^n |W| |B/A^\perp| \, |\mathrm{Stab}_{B^\perp}(\lambda/M)| \, \delta_{\lambda\nu}. \tag{20}$$

- Moreover, $|F_M^\sigma(A_\varrho, B_\chi)| = |\Lambda_M^\sigma(A_\varrho, B_\chi)|$, so the corresponding orbit functions form a basis of $\mathcal{H}_M^\sigma(A_\varrho, B_\chi)$.
- At this point, we can calculate the Fourier coefficients and write the Plancherel formulas
- The unitary transform matrix $\mathbb{I}_M^\sigma(A_\varrho, B_\chi)$ can be defined as

$$[\mathbb{I}_M^\sigma(A_\varrho, B_\chi)]_{\lambda s} = \sqrt{\frac{\varepsilon_{A^\perp}(s)}{M^n |W| |B/A^\perp| \, |\mathrm{Stab}_{B^\perp}(\lambda/M)|}} \, \overline{\varphi_\lambda^\sigma(s)}, \tag{21}$$

$$\lambda \in \Lambda_M^\sigma(A_\varrho, B_\chi), \; s \in F_M^\sigma(A_\varrho, B_\chi). \tag{22}$$

Example - a shifted lattice in A_3

- All roots have the same length, the angles between the simple roots $\alpha_{1,2,3}$ are

$$\measuredangle(\alpha_1, \alpha_2) = \tfrac{2}{3}\pi, \quad \measuredangle(\alpha_1, \alpha_3) = \tfrac{1}{2}\pi, \quad \measuredangle(\alpha_2, \alpha_3) = \tfrac{2}{3}\pi. \tag{23}$$

Example - a shifted lattice in A_3

- All roots have the same length, the angles between the simple roots $\alpha_{1,2,3}$ are

$$\measuredangle(\alpha_1, \alpha_2) = \tfrac{2}{3}\pi, \quad \measuredangle(\alpha_1, \alpha_3) = \tfrac{1}{2}\pi, \quad \measuredangle(\alpha_2, \alpha_3) = \tfrac{2}{3}\pi. \tag{23}$$

- Simple roots expressed in terms of fundamental weights:

$$\alpha_1 = 2\omega_1 - \omega_2, \quad \alpha_2 = -\omega_1 + 2\omega_2 - \omega_3, \quad \alpha_3 = -\omega_2 + 2\omega_3. \tag{24}$$

Example - a shifted lattice in A_3

- All roots have the same length, the angles between the simple roots $\alpha_{1,2,3}$ are

$$\measuredangle(\alpha_1, \alpha_2) = \tfrac{2}{3}\pi, \quad \measuredangle(\alpha_1, \alpha_3) = \tfrac{1}{2}\pi, \quad \measuredangle(\alpha_2, \alpha_3) = \tfrac{2}{3}\pi. \tag{23}$$

- Simple roots expressed in terms of fundamental weights:

$$\alpha_1 = 2\omega_1 - \omega_2, \quad \alpha_2 = -\omega_1 + 2\omega_2 - \omega_3, \quad \alpha_3 = -\omega_2 + 2\omega_3. \tag{24}$$

- We will use the normalization condition $\langle \alpha, \alpha \rangle = 2$ for all $\alpha \in \Pi$, implying $Q^\vee = Q$, $P^\vee = P$.

Example - a shifted lattice in A_3

- All roots have the same length, the angles between the simple roots $\alpha_{1,2,3}$ are

$$\measuredangle(\alpha_1, \alpha_2) = \tfrac{2}{3}\pi, \quad \measuredangle(\alpha_1, \alpha_3) = \tfrac{1}{2}\pi, \quad \measuredangle(\alpha_2, \alpha_3) = \tfrac{2}{3}\pi. \tag{23}$$

- Simple roots expressed in terms of fundamental weights:

$$\alpha_1 = 2\omega_1 - \omega_2, \quad \alpha_2 = -\omega_1 + 2\omega_2 - \omega_3, \quad \alpha_3 = -\omega_2 + 2\omega_3. \tag{24}$$

- We will use the normalization condition $\langle \alpha, \alpha \rangle = 2$ for all $\alpha \in \Pi$, implying $Q^\vee = Q$, $P^\vee = P$.
- The disjoint decomposition of P into cosets of Q is

$$P = Q \cup (\omega_1 + Q) \cup (\omega_2 + Q) \cup (\omega_3 + Q). \tag{25}$$

Example - a shifted lattice in A_3

- We look for W–invariant lattices S such that $Q \subseteq S \subseteq P$.

Example - a shifted lattice in A_3

- We look for W–invariant lattices S such that $Q \subseteq S \subseteq P$.
- The transition matrix from α–basis to ω–basis is the Cartan matrix C

Example - a shifted lattice in A_3

- We look for W–invariant lattices S such that $Q \subseteq S \subseteq P$.
- The transition matrix from α–basis to ω–basis is the Cartan matrix C
- Lattices are commutative groups, Lagrange's theorem tells us that $|P/S|$ must divide $|\det C|$

Example - a shifted lattice in A_3

- We look for W–invariant lattices S such that $Q \subseteq S \subseteq P$.
- The transition matrix from α–basis to ω–basis is the Cartan matrix C
- Lattices are commutative groups, Lagrange's theorem tells us that $|P/S|$ must divide $|\det C|$
- For A_3, $\det C = 4$.

Example - a shifted lattice in A_3

- We look for W–invariant lattices S such that $Q \subseteq S \subseteq P$.
- The transition matrix from α–basis to ω–basis is the Cartan matrix C
- Lattices are commutative groups, Lagrange's theorem tells us that $|P/S|$ must divide $|\det C|$
- For A_3, $\det C = 4$.
- By Langrange's theorem, any sublattice $Q \subsetneq S \subsetneq P$ must satisfy $|P/S| = |S/Q| = 2$.

Example - a shifted lattice in A_3

- We look for W–invariant lattices S such that $Q \subseteq S \subseteq P$.
- The transition matrix from α–basis to ω–basis is the Cartan matrix C
- Lattices are commutative groups, Lagrange's theorem tells us that $|P/S|$ must divide $|\det C|$
- For A_3, $\det C = 4$.
- By Langrange's theorem, any sublattice $Q \subsetneq S \subsetneq P$ must satisfy $|P/S| = |S/Q| = 2$.
- Does it exist???

Example - a shifted lattice in A_3

- We look for W–invariant lattices S such that $Q \subseteq S \subseteq P$.
- The transition matrix from α–basis to ω–basis is the Cartan matrix C
- Lattices are commutative groups, Lagrange's theorem tells us that $|P/S|$ must divide $|\det C|$
- For A_3, $\det C = 4$.
- By Langrange's theorem, any sublattice $Q \subsetneq S \subsetneq P$ must satisfy $|P/S| = |S/Q| = 2$.
- Does it exist???
- If it does, it must be of the form

$$S = Q \cup (\omega_i + Q), \quad i \in \{1, 2, 3\}. \tag{26}$$

Example - a shifted lattice in A_3

- The only possibilty is

$$S = Q \cup (\omega_2 + Q) \tag{27}$$

because the other choices are not closed with respect to addition.

Example - a shifted lattice in A_3

- The only possibilty is

$$S = Q \cup (\omega_2 + Q) \tag{27}$$

  because the other choices are not closed with respect to addition.
- Duality $Q^{\perp} = P$ implies $S^{\perp} = S$.

Example - a shifted lattice in A_3

- The only possibilty is

$$S = Q \cup (\omega_2 + Q) \tag{27}$$

because the other choices are not closed with respect to addition.

- Duality $Q^{\perp} = P$ implies $S^{\perp} = S$.
- The basis of S is given by

$$\mathfrak{s}_1 = -\omega_1 - \omega_2 + \omega_3, \quad \mathfrak{s}_2 = -\omega_1 + \omega_3, \quad \mathfrak{s}_3 = \omega_1 - \omega_2 + \omega_3. \tag{28}$$

Example - a shifted lattice in A_3

- The only possibilty is

$$S = Q \cup (\omega_2 + Q) \tag{27}$$

because the other choices are not closed with respect to addition.

- Duality $Q^\perp = P$ implies $S^\perp = S$.
- The basis of S is given by

$$\mathfrak{s}_1 = -\omega_1 - \omega_2 + \omega_3, \quad \mathfrak{s}_2 = -\omega_1 + \omega_3, \quad \mathfrak{s}_3 = \omega_1 - \omega_2 + \omega_3. \tag{28}$$

- The only non-trivial admissible shift of S is ω_1.

Example - a shifted lattice in A_3

- The only possibilty is

$$S = Q \cup (\omega_2 + Q) \tag{27}$$

because the other choices are not closed with respect to addition.
- Duality $Q^{\perp} = P$ implies $S^{\perp} = S$.
- The basis of S is given by

$$\mathfrak{s}_1 = -\omega_1 - \omega_2 + \omega_3, \quad \mathfrak{s}_2 = -\omega_1 + \omega_3, \quad \mathfrak{s}_3 = \omega_1 - \omega_2 + \omega_3. \tag{28}$$

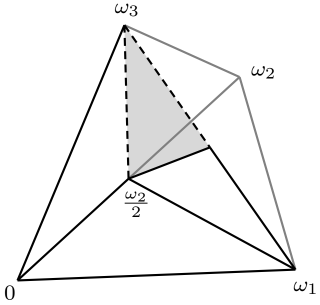
- The only non-trivial admissible shift of S is ω_1.
- The canonical fundamental domain of W_Q^{aff} is given by

$$F_Q = \left\{ u_1\omega_1 + u_2\omega_2 + u_3\omega_3 \middle| u_0 + u_1 + u_2 + u_3 = 1, u_i \geq 0 \right\}. \tag{29}$$

Example - a shifted lattice in A_3

- The only possibilty is
$$S = Q \cup (\omega_2 + Q) \tag{27}$$
because the other choices are not closed with respect to addition.
- Duality $Q^\perp = P$ implies $S^\perp = S$.
- The basis of S is given by
$$\mathfrak{s}_1 = -\omega_1 - \omega_2 + \omega_3, \quad \mathfrak{s}_2 = -\omega_1 + \omega_3, \quad \mathfrak{s}_3 = \omega_1 - \omega_2 + \omega_3. \tag{28}$$
- The only non-trivial admissible shift of S is ω_1.
- The canonical fundamental domain of W_Q^{aff} is given by
$$F_Q = \left\{ u_1\omega_1 + u_2\omega_2 + u_3\omega_3 \middle| u_0 + u_1 + u_2 + u_3 = 1, u_i \geq 0 \right\}. \tag{29}$$
- The fundamental domain of W_S^{aff} is
$$F_S = \left\{ u \in F_Q \middle| (u_0 > u_2) \vee (u_0 = u_2, u_1 \geq u_3) \right\}. \tag{30}$$

The fundamental domain of F_S



Example - a shifted lattice in A_3

- Using the semidirect product decomposition

$$\mathrm{Stab}_{W_S^{\mathrm{aff}}}(u) = \mathrm{Stab}_{W_Q^{\mathrm{aff}}}(u) \rtimes \mathrm{Stab}_{\Gamma_2}(u), \quad u \in F_Q, \tag{31}$$

where $S/Q \cong \Gamma_2 = \{1, g_2\}$.

Example - a shifted lattice in A_3

- Using the semidirect product decomposition

$$\mathrm{Stab}_{W_S^{\mathrm{aff}}}(u) = \mathrm{Stab}_{W_Q^{\mathrm{aff}}}(u) \rtimes \mathrm{Stab}_{\Gamma_2}(u), \quad u \in F_Q, \tag{31}$$

  where $S/Q \cong \Gamma_2 = \{1, g_2\}$.
- The value of $\gamma^{\mathbb{1}}_{S,\omega_1}$ is calculated as

$$\gamma^{\mathbb{1}}_{S,\omega_1}(g_2) = -1. \tag{32}$$

Example - a shifted lattice in A_3

- Using the semidirect product decomposition

$$\mathrm{Stab}_{W_S^{\mathrm{aff}}}(u) = \mathrm{Stab}_{W_Q^{\mathrm{aff}}}(u) \rtimes \mathrm{Stab}_{\Gamma_2}(u), \quad u \in F_Q, \tag{31}$$

  where $S/Q \cong \Gamma_2 = \{1, g_2\}$.

- The value of $\gamma^{\mathbb{1}}_{S,\omega_1}$ is calculated as

$$\gamma^{\mathbb{1}}_{S,\omega_1}(g_2) = -1. \tag{32}$$

- Therefore, the reduced fundamental domain $F^{\mathbb{1}}_S(\omega_1)$ is

$$F^{\mathbb{1}}_S(\omega_1) = \{u \in F_Q | (u_0 > u_2) \vee (u_0 = u_2, u_1 > u_3)\}. \tag{33}$$

Examples: the root system A_3

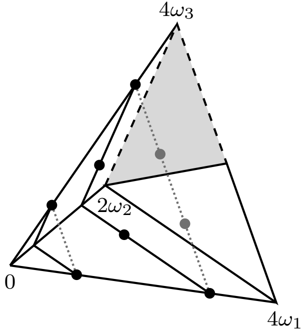
$$\Lambda_M^{\mathbb{1}}(S_{\omega_1}, S) =$$

$$\left\{ \sum_{i=1}^{3} s_i \omega_i \middle| s_1 + s_3 \equiv 1 \mod 2,\ (s_0 > s_2) \vee (s_0 = s_2, s_1 \geq s_3) \right\} \quad (34)$$

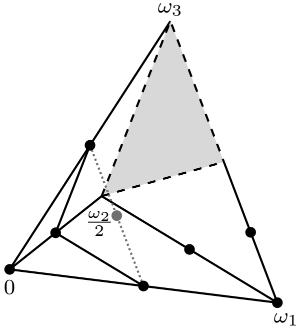
$$F_M^{\mathbb{1}}(S_{\omega_1}, S) =$$

$$\left\{ \sum_{i=1}^{3} \frac{s_i}{M} \omega_i \middle| s_1 + s_3 \equiv 0 \mod 2,\ (s_0 > s_2) \vee (s_0 = s_2, s_1 > s_3) \right\} \quad (35)$$

The label set $\Lambda_4^{\mathbb{1}}(S_{\omega_1}, S)$



The point set $\Lambda_4^{\mathbb{1}}(S_{\omega_1}, S)$



The matrix $\mathbb{I}_4^{\mathbb{1}}(S_{\omega_1}, S)$

$$
\mathbb{I}_4^{\mathbb{1}}(S_{\omega_1}, S) = \begin{pmatrix}
0.25 & 0.177 & 0 & 0.433 & 0 & 0.177 & 0.433 & 0 \\
0.25 & -0.177 & 0 & -0.433 & 0 & -0.177 & 0.433 & 0 \\
0.433 & -0.306 & 0 & 0.25 & 0 & -0.306 & -0.25 & 0 \\
0.25 & -0.177 & 0 & -0.433 & 0 & -0.177 & 0.433 & 0 \\
0.433 & -0.306 & 0 & 0.25 & 0 & -0.306 & -0.25 & 0 \\
0.25 & 0.177 & 0 & 0.433 & 0 & 0.177 & 0.433 & 0 \\
0.433 & 0.306 & 0 & -0.25 & 0 & 0.306 & -0.25 & 0 \\
0.433 & 0.306 & 0 & -0.25 & 0 & 0.306 & -0.25 & 0
\end{pmatrix}
+ \mathrm{i} \begin{pmatrix}
0 & 0.177 & -0.25 & 0 & -0.433 & -0.177 & 0 & -0.433 \\
0 & 0.177 & 0.25 & 0 & 0.433 & -0.177 & 0 & -0.433 \\
0 & 0.306 & 0.433 & 0 & -0.25 & -0.306 & 0 & 0.25 \\
0 & -0.177 & -0.25 & 0 & -0.433 & 0.177 & 0 & 0.433 \\
0 & -0.306 & -0.433 & 0 & 0.25 & 0.306 & 0 & -0.25 \\
0 & -0.177 & 0.25 & 0 & 0.433 & 0.177 & 0 & 0.433 \\
0 & 0.306 & -0.433 & 0 & 0.25 & -0.306 & 0 & 0.25 \\
0 & -0.306 & 0.433 & 0 & -0.25 & 0.306 & 0 & -0.25
\end{pmatrix} \tag{36}
$$

Central splitting

- The idea is to calculate the Fourier coefficients on smaller lattice fragments, reducing computation complexity

Central splitting

- The idea is to calculate the Fourier coefficients on smaller lattice fragments, reducing computation complexity
- To achieve this, we must define the splitting label sets

$$\Lambda_M^{\sigma,k}(P, B_\chi) = A_{\omega_k} \cap M F_{B^\perp}^{\sigma}(\chi), \quad P = \bigcup_{k \in J_A} (\omega_k + A). \tag{37}$$

Central splitting

- The idea is to calculate the Fourier coefficients on smaller lattice fragments, reducing computation complexity
- To achieve this, we must define the splitting label sets

$$\Lambda_M^{\sigma,k}(P,B_\chi) = A_{\omega_k} \cap M F_{B^\perp}^\sigma(\chi), \quad P = \bigcup_{k\in J_A} (\omega_k + A). \tag{37}$$

- The splitting point sets are chosen as

$$F_M^{\sigma,k}(P,B_\chi) = \frac{1}{M} B_\chi \cap F_{A^\perp}^\sigma(\omega_k), \quad k \in J_A. \tag{38}$$

Central splitting

- The idea is to calculate the Fourier coefficients on smaller lattice fragments, reducing computation complexity
- To achieve this, we must define the splitting label sets

$$\Lambda_M^{\sigma,k}(P,B_\chi) = A_{\omega_k} \cap M F_{B^\perp}^\sigma(\chi), \quad P = \bigcup_{k\in J_A} (\omega_k + A). \tag{37}$$

- The splitting point sets are chosen as

$$F_M^{\sigma,k}(P,B_\chi) = \frac{1}{M} B_\chi \cap F_{A^\perp}^\sigma(\omega_k), \quad k \in J_A. \tag{38}$$

- Analogously, we define the inner product

$$\langle f,g\rangle_{P,B_\chi}^{\sigma,M,k} = \sum_{F_M^{\sigma,k}(P,B_\chi)} \varepsilon_{A^\perp}(s) f(s) \overline{g(s)}. \tag{39}$$

Central splitting

- We need a decomposition

$$f = \sum_{k \in J_A} f_k^\sigma \tag{40}$$

such that the Fourier coefficients of the weight lattice transform of f coincide with the Fourier coefficients of the k-th splitting transform of the k-th central component of f (for every k)

Central splitting

- We need a decomposition

$$f = \sum_{k \in J_A} f_k^\sigma \tag{40}$$

  such that the Fourier coefficients of the weight lattice transform of f coincide with the Fourier coefficients of the k-th splitting transform of the k-th central component of f (for every k)
- Such f_k^σ can be found utilizing the group $\Gamma^\vee$, appearing in the semidirect decomposition

$$W_{P^\vee}^{\mathrm{aff}} = W_{Q^\vee}^{\mathrm{aff}} \rtimes \Gamma^\vee \tag{41}$$

Central splitting

- We need a decomposition

$$f = \sum_{k \in J_A} f_k^\sigma \tag{40}$$

  such that the Fourier coefficients of the weight lattice transform of f coincide with the Fourier coefficients of the k-th splitting transform of the k-th central component of f (for every k)
- Such f_k^σ can be found utilizing the group $\Gamma^\vee$, appearing in the semidirect decomposition

$$W_{P^\vee}^{\mathrm{aff}} = W_{Q^\vee}^{\mathrm{aff}} \rtimes \Gamma^\vee \tag{41}$$

- Splitting transform matrices $\mathbb{I}_M^{\sigma,k}(P, B_\chi)$ can be defined in complete analogy

Central splitting

- Finally, we sould like to find a decomposition of $\mathbb{I}_M^{\sigma}(P, B_\chi)$ into a matrix product of type

$$\mathbb{I}_M^{\sigma}(P, B_\chi) = \left(\bigoplus_{k \in J_{A^\perp}} \mathbb{I}_M^{\sigma,k}(P, B_\chi) \right) \mathbb{T}_M^{\sigma}(P, B_\chi), \tag{42}$$

where $\mathbb{T}_M^{\sigma}(P, B_\chi)$ arises from the symmetry relations of orbit functions.

References I

N. Bourbaki, *Groupes et algèbres de Lie*, Chapitres 4, 5 et 6, Hermann, Paris, 1968.

J. E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge Studies in Advanced Mathematics **29**, Cambridge University Press, Cambridge, 1990.

D. Micciancio, S. Goldwasser, *Complexity of Lattice Problems: A Cryptographic Perspective*, Springer Science & Business Media, New York, 2002.

J. Hrivnák, J. Patera, *On Discretization of Tori of Compact Simple Lie groups*, J. Phys. A: Math. Theor. **42** (2009).

J. Hrivnák, L. Motlochová, J. Patera, *On Discretization of Tori of Compact Simple Lie Groups II.*, J. Phys. A **45**(2012).

References II

- T. Czyżycki, J. Hrivnák, *Generalized Discrete Orbit Function Transforms of Affine Weyl Groups*, J. Math. Phys. **55** (2014).
- J. Hrivnák, M. A. Walton, *Weight-Lattice Discretization of Weyl-Orbit Functions,* J. Math. Phys. **57** (2016) 083512.
- J. Hrivnák, L. Motlochová, *Dual-root Lattice Discretization of Weyl Orbit Functions*, J. Fourier Anal. Appl. **25** (2019) 2521–2569.
- T. Czyżycki, J. Hrivnák, L. Motlochová, *Generalized Dual-Root Lattice Transforms of Affine Weyl Groups*, Symmetry **12** (2020) 1018.
- J. Hrivnák, M. Myronova, J. Patera *Central Splitting of A_2 Discrete Fourier-Weyl Transforms*, Symmetry **12** (2020) 1828.
- V. Teska, J. Hrivnák, L. Motlochová, *Discrete Orthogonality of Weyl Orbit Functions on Invariant Lattices* (to be published).

THE END