On the incompleteness of the classification of quadratically integrable Hamiltonian systems in the three–dimensional Euclidean space

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Introduction

In their seminal paper *A systematic search for nonrelativistic systems with dynamical symmetries, Nuovo Cimento A Series 10 (1967) 1061–1084* Makarov, Smorodinsky, Valiev and Winternitz presented their classification of quadratically integrable natural Hamiltonian systems in the three–dimensional Euclidean space and identified them with systems separable in ortogonal coordinate systems, found earlier by Eisenhart. Their result is one of the standard references in the theory of integrable and superintegrable systems and lead to numerous further developments. It was widely accepted as a proof of the equivalence of quadratic integrability and separability in Euclidean 3D space.

Introduction

However, it became forgotten that the original derivation of the list by Makarov et al. was based on one technical assumption which limits the universality of the above mentioned equivalence. We shall demonstrate that without that assumption a quadratically integrable however nonseparable 3D natural Hamiltonian system exists and the original derivation needs to be revisited.

Review of the original argument and its loophole

Let us review the original argument and indicate the point where the analysis becomes incomplete.

We consider the classical natural Hamiltonian for a particle of unit mass moving in the 3D Euclidean space under the influence of the potential $V(\vec{x})$,

$$H = \frac{1}{2}\vec{p}^2 + V(\vec{x}), \tag{1}$$

and assume that it is integrable with a pair of integrals of motion X_1, X_2 in involution which are quadratic polynomials in the momenta, a.k.a. quadratic integrals.

Notation:
Cartesian position $\vec{x} = (x, y, z)$,
conjugated momenta p_x, p_y, p_z,
angular momenta $l_x = yp_z - zp_y$, $l_y = zp_y - xp_z$, $l_z = xp_y - yp_x$.

Review of the original argument and its loophole

As even and odd order terms in the Poisson brackets commute independently, the left hand sides of the involutivity conditions

$$\begin{aligned} \{H, X_1\}_{P.B} &= 0, \quad \{H, X_2\}_{P.B} = 0, \qquad (2) \\ \{X_1, X_2\}_{P.B} &= 0 \qquad (3) \end{aligned}$$

can be assumed to be third order odd polynomials in the momenta. As the momenta are arbitrary, all their coefficients must vanish.

The third order terms in (2–3) are easily solved and imply that the second order terms in X_1 and X_2 must be commuting elements in the universal enveloping algebra of the Euclidean algebra $\mathfrak{U}(\mathfrak{e}_3)$, i.e. quadratic polynomials in the linear and angular momenta.

Review of the original argument and its loophole

It remains to solve the conditions from linear terms in (2–3), i.e. to find the scalar terms in the integrals X_1 and X_2, denoted by $m_1(\vec{x})$ and $m_2(\vec{x})$ below, and find the restrictions on the potential $V(\vec{x})$ implied by their existence.

Eliminating $m_1(\vec{x})$ and $m_2(\vec{x})$, one arrives at three homogeneous linear first order PDEs for the potential $V(\vec{x})$, cf. (6) below. As the coefficients of $\partial_a V$, $a = x, y, z$ form an antisymmetric 3×3 matrix R, it either has rank 2 or vanishes identically.

Review of the original argument and its loophole

At this point Makarov et al. stated

> "Thus the potential V either satisfies three first-order equations – a case which will be considered separately – or the consistency conditions (39) are satisfied trivially."

and proceeded assuming that the condition (3) vanishes identically.

Only under this assumption they arrived at their list of quadratically integrable natural Hamiltonian systems and showed that one by one they precisely match with the separable systems of Eisenhart.

Review of the original argument and its loophole

Makarov et al. left several problems to be resolved in the planned Part II of their paper and we can assume that that's where they intended to address the case of the matrix R of rank 2. However,

Soviets came to save communism in Czechoslovakia!

Figure: Military occupation of Czechoslovakia by the forces of Soviet Union and its communist satellites in 1968, leading to emigration of P. Winternitz to the other side of Iron Curtain

Review of the original argument and its loophole

Thus the sequel was never written.

The long forgotten assumption on the rank of the matrix R came back to light recently, when we discussed with P. Winternitz the modification of the classification of quadratically integrable systems when linear terms in the momenta are present in the Hamiltonian.

We decided to investigate the problem of $\mathrm{rank}\, R = 2$ from the perspective of algebraic classification of leading order terms obtained with A. Marchesiello in *J. Phys. A 55 (2022) 145203* and, as we shall elucidate in the next section, arrived at the conclusion that a quadratically integrable nonseparable system does exist.

Quadratically integrable nonseparable system

We look for quadratically integrable Hamiltonian systems (1) of the form of the class (c) of *J. Phys. A 55 (2022) 145203*,

$$
\begin{aligned}
X_1 &= l_x^2 + l_y^2 + l_z^2 + 2b(l_x p_x - (3a-1) l_y p_y - 2 l_z p_z) + \\
&\quad +3b^2((1-4a)p_x^2 - (3a^2-2a-1)p_y^2 + 2(a-1)p_z^2) + m_1(\vec{x}), \\
X_2 &= a l_y^2 + l_z^2 + 6ab l_x p_x + 9ab^2(a p_z^2 + p_y^2) + m_2(\vec{x}), \qquad (4) \\
&\quad 0 < a \leq \frac{1}{2}, b \in \mathbb{R} - \{0\}.
\end{aligned}
$$

The allowed transformations, i.e. linear combinations of the integrals (and the Hamiltonian) and Euclidean transformations were all used in fixing the form of (4), thus we have no available transformations left and the assumed form of the integrals (4) is not equivalent to any other considered one, e.g. to any related to the separation of variables.

Quadratically integrable nonseparable system

Now $\{H, X_1\}_{P.B}$, $\{H, X_2\}_{P.B}$ and $\{X_1, X_2\}_{P.B}$ reduce to first order polynomials in the momenta, without zeroth order terms.
Separating the conditions (2) and solving them with respect to the first order derivatives of $m_a(\vec{x})$ we find:

$$
\begin{aligned}
\partial_x m_1(\vec{x}) &= 2(3(1-4a)b^2+y^2+z^2)\partial_x V(\vec{x}) - 2(3abz+xy)\partial_y V(\vec{x}) \\
&\quad +2(3by-xz)\partial_z V(\vec{x}), \\
\partial_y m_1(\vec{x}) &= -2(3abz+xy)\partial_x V(\vec{x}) + 2(3(1+2a-3a^2)b^2+x^2+z^2)\partial_y V(\vec{x}) \\
&\quad -2(3b(1-a)x+yz)\partial_z V(\vec{x}), \\
\partial_z m_1(\vec{x}) &= 2(3by-xz)\partial_x V(\vec{x}) - 2(3b(1-a)x+yz)\partial_y V(\vec{x}) \\
&\quad +2(6(a-1)b^2+x^2+y^2)\partial_z V(\vec{x}), \\
\partial_x m_2(\vec{x}) &= 2(az^2+y^2)\partial_x V(\vec{x}) - 2(3abz+xy)\partial_y V(\vec{x}) \\
&\quad +2a(3by-xz)\partial_z V(\vec{x}), \\
\partial_y m_2(\vec{x}) &= -2(3abz+xy)\partial_x V(\vec{x}) + 2(9ab^2+x^2)\partial_y V(\vec{x}), \\
\partial_z m_2(\vec{x}) &= 2a(3by-xz)\partial_x V(\vec{x}) + 2a(9ab^2+x^2)\partial_z V(\vec{x}).
\end{aligned} \tag{5}
$$

Quadratically integrable nonseparable system

Their compatibility implies 2nd order linear PDEs for the potential $V(\vec{x})$. On the other hand, substituting (5) into (3) we obtain first order linear homogeneous PDEs for the potential $V(\vec{x})$,

$$R \cdot (\partial_x V(\vec{x}), \partial_y V(\vec{x}), \partial_z V(\vec{x}))^T = 0 \tag{6}$$

where the matrix R is antisymmetric, has $\operatorname{rank} R = 2$ for all allowed values of a, b, and its independent elements read

$$\begin{aligned} R_{12} &= (1-a)azx^2 - 6(1-a)abyx - zy^2a - a^2z^3 - 9a^2b^2(1-a)z, \\ R_{13} &= (1-a)yx^2 + 6(1-a)abzx + y^3 + a(9(a-1)b^2 + z^2)y, \\ R_{23} &= -(1-a)^2x^3 + (1-a)(9a(a-1)b^2 + az^2 - y^2)x - 6(1-a)abyz. \end{aligned} \tag{7}$$

Quadratically integrable nonseparable system

Solving (6) using the method of characteristics we find $V(\vec{x})$ as arbitrary function $v(u)$ of the invariant coordinate u

$$\begin{aligned} u &= (a-1)^2x^4 + (az^2+y^2)^2 + 2(1-a)x^2(y^2-az^2) + \\ &\quad 6ab(a-1)\left(3\left((x^2-z^2)a - x^2 + y^2\right)b - 4xyz\right) + 81a^2(1-a)^2b^4. \end{aligned} \tag{8}$$

Substituting $V(\vec{x}) = v(u)$ into the compatibility conditions for (5) we get a system of ODEs which reduces to the single equation

$$2u\frac{\mathrm{d}^2 v(u)}{\mathrm{d}u^2} = -3\frac{\mathrm{d}v(u)}{\mathrm{d}u} \tag{9}$$

and we find the potential determined up to a multiplicative constant w_0:

$$V(\vec{x}) = v(u) = \frac{w_0}{\sqrt{u}}. \tag{10}$$

Quadratically integrable nonseparable system

Next, the equations (5) determine the scalar terms m_1 and m_2 in the integrals up to irrelevant additive constants.

The polynomial (8) can of course have real roots and thus the potential (10) may blow up in the configuration space. The coordinate u vanishes and thus the potential $V(\vec{x})$ blows up along the two straight lines given by

$$x = -\epsilon\sqrt{\frac{a}{1-a}}z, \quad y = 3\epsilon\sqrt{a(1-a)}b, \qquad \epsilon = \pm 1. \qquad (11)$$

As these do not separate $\mathbb{R}^3$ into disconnected domains, everywhere else the potential (10) is a well–defined real function.

Quadratically integrable nonseparable system

If we assume that the parameter w_0 is positive, the singular lines (11) are not dynamically accessible for any initial condition with finite energy. Thus our Hamiltonian system is well–defined in the configuration space defined as $\mathbb{R}^3$ without the two singular lines (11). Whether the singularities are dynamically reachable for negative values of w_0 in finite time we don't know yet.

Quadratically integrable nonseparable system

In order to provide a more intuitive understanding of the potential (10) let us present several of its equipotential surfaces.

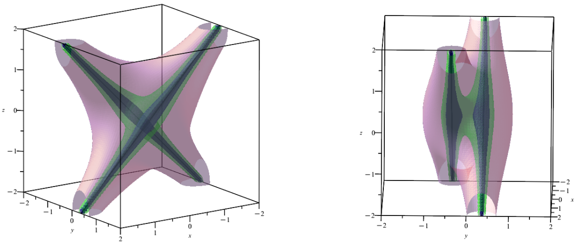

Figure: Equipotential surfaces of (10) with the parameters $a = \frac{1}{4}$, $b = 1$, $w_0 = 1$ for the values $V(x, y, z) = 8$, $V(x, y, z) = 4$ and $V(x, y, z) = 1$, viewed from two different directions.

Quadratically integrable nonseparable system

The potential (10) does not allow any other integrals of motion quadratic in the momenta. Thus it cannot be transformed into any of the separable potentials listed by Eisenhart / Makarov et al. by a coordinate change.

G. Rastelli observed also an alternative, more straightforward argument leading to the same conclusion, using a theorem due to Eisenhart.

That theorem implies that if our system was separable in an orthogonal coordinate system then the two Killing tensors K_1 and K_2 encoding the leading order terms of the involutive integrals (4) would commute as linear operators.

Quadratically integrable nonseparable system

These Killing tensors K_1 and K_2 corresponding to the integrals X_1 and X_2 read

$$K_1 = \begin{pmatrix} y^2 + z^2 + 3(1-4a)b^2 & -3abz - xy & 3by - xz \\ -3abz - xy & x^2 + z^2 + 3(1+2a-3a^2)b^2 & 3(a-1)bx - yz \\ 3by - xz & 3(a-1)bx - yz & x^2 + y^2 + 6(a-1)b^2 \end{pmatrix}$$

and

$$K_2 = \begin{pmatrix} az^2 + y^2 & -3abz - xy & a(3by - xz) \\ -3abz - xy & x^2 + 9ab^2 & 0 \\ a(3by - xz) & 0 & ax^2 + 9a^2b^2 \end{pmatrix}$$

which do not commute. As these are the only nontrivial Killing tensors, the system with the potential (10) cannot separate in any orthogonal coordinate system.

Conclusions

The purpose of this presentation was to bring the attention to the forgotten assumption in the paper by Makarov et al. *Nuovo Cimento A 10 (1967) 1061–84* and explicitly demonstrate that the statement on the equivalence of quadratic integrability and separability in 3D Euclidean space does not hold if that assumption is violated, arriving at a new quadratically integrable yet not separable system with the potential (10).

Conclusions

Let us recall that as the determining equations for the quadratic integrals of motion of natural Hamiltonian systems (1) in classical and quantum mechanics coincide, also the quantum system with the potential (10) is integrable with integrals of motion quadratic in momenta and its Schrödinger equation does not separate in any orthogonal coordinate system.

Conclusions

It is not yet known whether our system (10) is the sole exception or whether other quadratically integrable nonseparable systems in Euclidean 3D space do exist. Thus, a complete re–derivation of the list of quadratically integrable systems based on the classification of the leading order terms in *Marchesiello & Šnobl, J. Phys. A: Math. Theor. 55 (2022) 145203* is currently under way and we expect to report on it in not too distant future. We also intend to study the presented system (10) in more detail, e.g. to attempt to find its (generalized) action–angle variables (notice that the shape of the equipotential surfaces as in Fig. 2 implies noncompact level sets of the Hamiltonian), as well as to analyse its quantum counterpart.

Thank you for your attention!