Local form subordination without a power decay and the Riesz property of spectral projections

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Based on the joint works with B. Mityagin (OSU)

[1] B. Mityagin and P. Siegl [2016]. “Root system of singular perturbations of the harmonic oscillator type operators”. In: *Lett. Math. Phys.* 106, pp. 147–167

[2] B. Mityagin and P. Siegl [2019]. “Local form-subordination condition and Riesz basisness of root systems”. In: *J. Anal. Math.* 139, pp. 83–119

[3] B. Mityagin and P. Siegl [2024]. *Local form subordination without a power decay and a criterion of Riesz basesness.* arXiv:2411.10553 [math.SP]

$A = A^*$ with $(\mathrm{i} - A)^{-1}$ compact (in a separable Hilbert space $\mathcal{H}$)

- $\sigma(A) = \{\mu_n\}_n$ is discrete and eigenvectors form an orthonormal basis of $\mathcal{H}$
- properties of spectral projections ($P_n^2 = P_n$)

$$P_n = \frac{1}{2\pi\mathrm{i}} \int_{\Gamma_{\mu_n}} (z - A)^{-1} \, \mathrm{d}z$$

  - orthogonal $P_n^* = P_n$ and with finite rank
  - disjoint

$$P_j P_k = \delta_{jk} P_j, \quad \forall j, k$$

  - complete

$$(\forall x \ \forall j \quad \langle P_j x, y \rangle = 0) \quad \Longrightarrow \quad y = 0$$

    (equivalently, every $x \in \mathcal{H}$ can be approximated with an arbitrary accuracy by linear combinations of $x_k \in \mathrm{Ran}(P_k)$)
  - expansion and Parseval identity

$$x = \sum_n P_n x, \quad \|x\|^2 = \sum_n \|P_n x\|^2$$

Harmonic oscillator in $L^2(\mathbb{R}^d)$

$$\boxed{A = -\Delta + |x|^2} \quad \mu_n = 2n + d,\ n \in \mathbb{N}_0, \quad \mathrm{rank} P_n = \binom{n+d-1}{n}$$

Anharmonic oscillators in $L^2(\mathbb{R})$

$$\boxed{A = -\partial_x^2 + |x|^\beta} \quad \beta > 0 \quad \mu_n = C_\beta n^{\frac{2\beta}{\beta+2}}(1 + o(1)),\ n \to \infty, \quad \mathrm{rank} P_n = 1$$

Laplace-Beltrami on $\mathbb{S}^d$

$$\boxed{A = -\Delta_{\mathbb{S}^d}} \quad \mu_n = n(n+d-1),\ n \in \mathbb{N}_0, \quad \mathrm{rank} P_n = \frac{(2n+d-1)(n+d-2)!}{n!(d-1)!}$$

Landau Hamiltonian (twisted Laplacian) in $L^2(\mathbb{R}^d)$ with $d \in 2\mathbb{N}$

$$\boxed{A = \sum_{j=1}^{\frac{d}{2}} \left(-i\partial_{x_j} + \frac{y_j}{2}\right)^2 + \left(-i\partial_{y_j} + \frac{x_j}{2}\right)^2} \quad \mu_n = 2n + \tfrac{d}{2},\ n \in \mathbb{N}_0, \quad \mathrm{rank} P_n = \infty$$

Damped wave equation

$$\begin{pmatrix} 0 & I \\ \Delta & \boxed{-2a} \end{pmatrix} \quad \rightsquigarrow \quad -\Delta + \boxed{2\lambda\, a(x)} + \lambda^2, \quad \lambda \in \mathbb{C}, \qquad \text{in} \quad L^2(\mathbb{R}^d)$$

Superconductivity (Ginzburg-Landau equation) [Almog and Helffer, 2015]

$$-\partial_x^2 + \left(\mathrm{i}\partial_y - x^2\right)^2 + \boxed{\mathrm{i}y} \qquad \text{in} \quad L^2(\mathbb{R}^2)$$

Hydrodynamics (Oseen vortices) [Gallagher, Gallay, and Nier, 2009]

$$-\partial_x^2 + x^2 + \boxed{\frac{\mathrm{i}}{\varepsilon} f(x)} \qquad \text{in} \quad L^2(\mathbb{R})$$

MRI (Bloch-Torrey equation) [Grebenkov and Helffer, 2018]

$$\begin{pmatrix} -\Delta & \boxed{b} \\ \boxed{-b} & -\Delta \end{pmatrix} \quad \rightsquigarrow \quad -\Delta \pm \boxed{\mathrm{i}b(x)} \qquad \text{in} \quad L^2(\mathbb{R}^2)$$

Black-Scholes-Merton equation (mathematical finance) [Baaquie, 2020]

$$-\sigma^2\partial_x^2 + \boxed{(\sigma^2 - r)\partial_x} + r + \boxed{\mathrm{i}e^x} \qquad \text{in} \quad L^2(\mathbb{R})$$

Quantum resonances, open quantum systems, graphene, . . .

General non-self-adjoint or non-normal operators

- the properties of spectral projections can be lost
- examples $T = -\partial_x^2 + \mathrm{i}x$ in $L^2(\mathbb{R})$
    - T^{-1} is compact, but $\boxed{\sigma(T) = \emptyset}$ [Herbst, 1979; Helffer, 2013]

$\rightsquigarrow$ no non-trivial P_n (no completeness, expansion, Parseval, etc.)

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Small perturbations of self-adjoint operators

$$T = A + B, \qquad A = A^* \text{ and } B \text{ “small”}$$

- example of a differential operator in $L^2(\mathbb{R})$

$$T = \underbrace{-\partial_x^2 + |x|^\beta}_{A} + \underbrace{\mathrm{i}|x|^\gamma}_{B}, \quad \gamma \le \beta$$

Q: what properties of spectral projections can be expected?

Riesz property of projections $\{P_n\}_{n=0}^{\infty}$

$\{P_n\}$ has a Riesz property if it is disjoint, complete and

- $\exists W \in \mathcal{B}(\mathcal{H})$ with $W^{-1} \in \mathcal{B}(\mathcal{H})$ and
- $\exists$ disjoint and complete system of orthogonal projections $\{P_n^0\}$ such that

$$P_n = W^{-1} P_n^0 W, \quad \forall n.$$

$\rightsquigarrow$ expansion and (modified) Parseval hold $\quad [C_W = \|W\| \|W^{-1}\|]$

$$x = \sum_n P_n x, \quad \frac{1}{C_W^2} \|x\|^2 \leq \sum_n \|P_n x\|^2 \leq C_W^2 \|x\|^2$$

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Criterium for the Riesz property

[Gohberg and Krein, 1969, Chap. 6]

Let $\{P_n\}$ be a disjoint and complete system of projections. Then

$$\{P_n\} \text{ has the Riesz property} \quad \Longleftrightarrow \quad \forall x \in \mathcal{H} : \boxed{\sum_{n=0}^{\infty} |\langle P_n x, x\rangle| < \infty.}$$

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Completeness vs. Riesz property: $T = -\partial_x^2 + \mathrm{i}x^2$ in $L^2(\mathbb{R})$ [Davies and Kuijlaars, 2004]

- $\{P_n\}$ are complete, disjoint, $\operatorname{rank} P_n = 1$, but without the Riesz property

Theorem [Kato, 1995, Thm.V.4.15a]

Let $A = A^* \geq 0$ with compact resolvent and eigenvalues $\{\mu_n\}$ of A be simple. Assume that

1. $\mu_{n+1} - \mu_n \to \infty$ as $n \to \infty$ [size of EV gaps]
2. $\|B\| < \infty$. ["strength" of perturbation]

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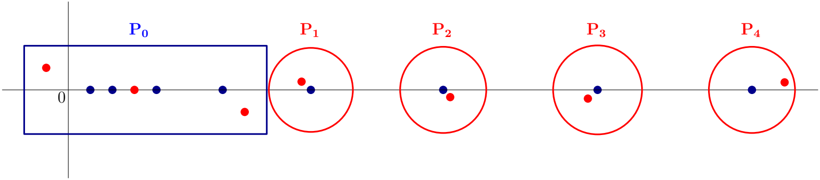
Then for $T := A + B$,

- the eigenvalues $\{\lambda_n\}$ of T are eventually simple
- the system of spectral projections $\{P_n\}_{n=0}^{\infty}$ of T has the Riesz property



- generalizations (relative boundedness/subordination)

  [Dunford and Schwartz, 1988, Chap. XIX.2], [Markus, 1988, Chap. I], [Clark, 1968; Agranovich, 1994; Wyss, 2010]

Theorem [Mityagin and Siegl, 2024]

Let $A = A^* \geq 1$ with A^{-1} compact with

- eigenvalues μ_n with gaps $r_n := \frac{1}{2}\operatorname{dist}(\mu_n, \sigma(A) \setminus \{\mu_n\})$
- spectral projections P_n^0

Assume that B is locally form-subordinated

$$\boxed{|\langle B P_j^0 f, P_k^0 g\rangle| \leq \omega_j \omega_k \|P_j^0 f\| \|P_k^0 g\|,} \quad j,k \in \mathbb{N}, \ f, g \in \mathcal{H} \quad \text{and}$$

$$\boxed{\sum_{j \neq n}^{\infty} \frac{\omega_j^2}{|\mu_n - \mu_j|} + \frac{\omega_n^2}{r_n} = o(1),} \quad n \to \infty.$$

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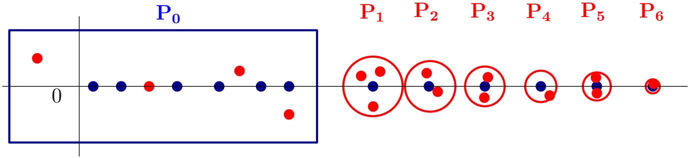
Then

- the eigenvalues of $T = A + B$ are eventually localized at μ_n,
- the system of spectral projections $\{P_n\}_{n=0}^{\infty}$ of T has the Riesz property.



$$|\langle BP_j^0 f, P_k^0 g\rangle| \leq \omega_j \omega_k \|P_j^0 f\| \|P_k^0 g\| \qquad \text{and} \qquad \sum_{j\neq n}^{\infty} \frac{\omega_j^2}{|\mu_n - \mu_j|} + \frac{\omega_n^2}{r_n} = o(1), \quad n \to \infty$$

Simple eigenvalues and power-behavior of gaps

- $\operatorname{rank} P_n^0 = 1$: $A\psi_n = \mu_n \psi_n$, $\quad \|\psi_n\| = 1 \quad \rightsquigarrow \quad |\langle B\psi_j, \psi_k\rangle| \leq \omega_j \omega_k$
- power-like gaps: with some $\kappa > 0$ $\quad \mu_{n+1} - \mu_n \gtrsim n^{\kappa-1}$
- local form-subordination satisfied if [Mityagin and Siegl, 2019]

$$\omega_j \lesssim j^{-\alpha} \qquad \text{and} \qquad 2\alpha + \kappa > 1$$

[cannot we relaxed to $2\alpha + \kappa = 1$]

$$|\langle BP_j^0 f, P_k^0 g\rangle| \leq \omega_j \omega_k \|P_j^0 f\| \|P_k^0 g\| \qquad \text{and} \qquad \sum_{j \neq n}^{\infty} \frac{\omega_j^2}{|\mu_n - \mu_j|} + \frac{\omega_n^2}{r_n} = o(1), \quad n \to \infty$$

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- power behavior of ω_j not needed, e.g. for $\kappa = 1$ [above $\alpha > 0$]
  - log-decay

$$\omega_j = \frac{1}{(\log j)^{\frac{1}{2}} (\log \log j)^a}, \quad j \geq 2, \ a > \frac{1}{2}$$

  - arbitrarily slow decay of ω_j and gaps, e.g.

$$\omega_j = o(1) \quad \text{and} \quad \omega_j = 0, \quad j \neq m^2, \ m \in \mathbb{N}$$

Remarks

- some optimality: cannot be relaxed to $2\alpha + \kappa = 1$ or $\{\omega_j^2/\mu_j\} \in \ell^1$
- operator version of the local subordination earlier

$$\|B\psi_n\| \leq o(n^{\kappa-1}), n \to \infty$$

[Adduci and Mityagin, 2012a; Shkalikov, 2010; Adduci and Mityagin, 2012b]

- form version (power behavior) [Mityagin and Siegl, 2016; Mityagin and Siegl, 2019]
- survey of earlier results and further generalizations

[Shkalikov, 2016; Motovilov and Shkalikov, 2019]

- $r_n \geq d$ and $\|B\| < d$ also works (even with a general $\sigma(A)$)

[Motovilov and Shkalikov, 2017]

- further results on differential operators

[Cox & Zuazua], [Djakov & Mityagin], [Gesztesy & Tkachenko], [Lunyov & Malamud], [Motovilov & Shkalikov]

Theorem [Mityagin and Siegl, 2019]

Let

$$T = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + |x|^\beta + V(x), \quad \beta \geq 2, \quad \text{in } L^2(\mathbb{R})$$

where $V = V_1 + V_2 + V_3 + V_4$ satisfies

- $|V_1(x)| \lesssim \langle x \rangle^\gamma$ with $\gamma < \frac{\beta}{2} - 1$
- $V_2 \in L^p(\mathbb{R})$ with $p \in [1, \infty)$
- $V_3 \in W^{-s,2}(\mathbb{R})$ with $s \in [0, \frac{\beta-1}{2\beta})$
- $V_4(x) = \sum_{k \in \mathbb{Z}} \nu_k \, \delta(x - x_k)$ with $\{\nu_k\} \in \ell^1(\mathbb{Z})$

Then

- the eigenvalues of T are eventually simple,
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Remarks

- essential ingredient: asymptotics of eigenfunctions of A (weighted L^p-norms)
- indication of optimality for $V(x) = \mathrm{i} \operatorname{sgn} x |x|^\gamma$ with $\gamma > \frac{\beta}{2} - 1$

$$\|(T - (a + \mathrm{i}b))^{-1}\| \gtrsim a^N, \qquad a \to +\infty \quad (a + \mathrm{i}b \in \rho(T)),$$

for any fixed $b \in \mathbb{R}$ and $N \in \mathbb{N}$ [Krejčiřík and Siegl, 2019]

- harmonic oscillator / Landau Hamiltonian / Laplace-Beltrami on $\mathbb{S}^d$

$$A_1 = -\Delta + |x|^2, \quad A_2 = \sum_{j=1}^{\frac{d}{2}} \left(-i\partial_{x_j} + \frac{y_j}{2}\right)^2 + \left(-i\partial_{y_j} + \frac{x_j}{2}\right)^2, \quad A_3 = -\Delta_{\mathbb{S}^d}$$

Theorem

The spectrum of $T_j = A_j + B$ eventually localizes at eigenvalues of A_j and the spectral projections of T have the Riesz property if

- for $j = 1, 2$

$$B = V_1 + V_2 \quad \text{with} \quad V_1 \in L^r(\mathbb{R}^d), \; r \in (\tfrac{d}{2}, \infty), \quad \|V_2\|_{L^\infty} < 1$$

- for $j = 3$

$$B = V + W\delta_\Sigma \quad \text{with} \quad V \in L^r(\mathbb{S}^d), \; r \in (\tfrac{d}{2}, \infty], \quad W \in L^s(\Sigma), \; s \in (d-1, \infty]$$

- essential ingredient in the proof: L^p-norms of spectral projections of A_j

[Koch and Tataru, 2005; Koch, Tataru, and Zworski, 2007; Koch and Ricci, 2007; Sogge, 1988; Burq, Gérard, and Tzvetkov, 2007]