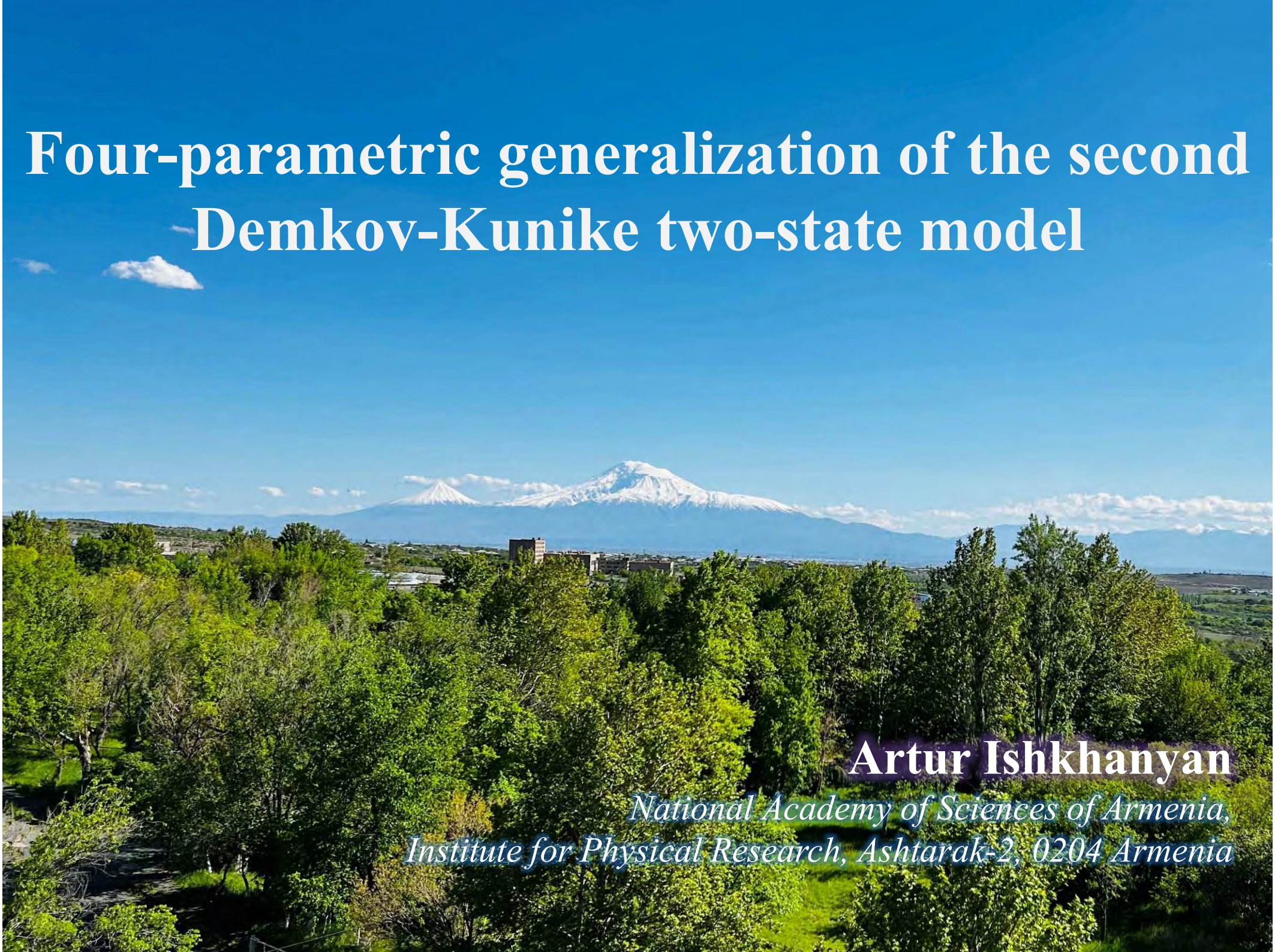
Four-parametric generalization of the second Demkov-Kunike two-state model



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Content

- Analytic framework for the new model
- The new model and its connection to the standard DK2
- Final transition probability at $t \to \infty$
- Conclusions

- **General Heun Classes of Exactly Solvable Models**

Time-dependent Schrödinger equations

$$i\frac{da_1}{dt} = U(t)\,e^{-i\,\delta(t)}a_2 \qquad\qquad i\frac{da_2}{dt} = U(t)\,e^{+i\,\delta(t)}a_1$$

detuning modulation function

$$\delta(t) = \int \delta_t(t)dt$$

Eliminating $a_1(t)$ *from the system yields a second-order linear differential equation for* $a_2(t)$

$$\frac{d^2a_2}{dt^2} + \left(-i\,\delta_t(t) - \frac{U_t(t)}{U(t)}\right)\frac{da_2}{dt} + U^2(t)\,a_2 = 0$$

$$\frac{d^2 a_2}{dt^2} + \left(-i\,\delta_t(t) - \frac{U_t(t)}{U(t)} \right) \frac{da_2}{dt} + U^2(t)\, a_2 = 0 \qquad (1)$$

Solutions of this equation generate infinite classes of solvable models through arbitrary real or complex transformations of the time variable.

If $a_2^*(z)$ *is a solution of equation* (1)*, then the function* $a_2(t) = a_2^*(z(t))$ *is the solution of that equation for the field-configuration defined as*

$$U(t) = U^*(z)\frac{dz}{dt}, \qquad \delta_t(t) = \delta_z^*(z)\frac{dz}{dt}.$$

Applying transformations of both the independent and dependent variables

$$z = z(t)$$

$$a_2 = \varphi(z)\, u(z)$$

$\varphi(z)$ Suitably chosen prefactor

$u(z)$ New unknown function

We reduce the equation (1) to an equation with known solutions

$$u_{zz} + \left(2\frac{\varphi_z}{\varphi} - i\delta_z^* - \frac{U_z^*}{U^*} \right) u_z + \left(\frac{\varphi_{zz}}{\varphi} + \left(-i\delta_z^* - \frac{U_z^*}{U^*} \right) \frac{\varphi_z}{\varphi} + U^{*2} \right) u = 0$$

$$\downarrow \qquad\qquad \downarrow$$

$$u_{zz} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\varepsilon}{z-a} \right) u_z + \frac{\alpha\beta z - q}{z(z-1)(z-a)} u = 0$$

General Heun equation

Then one obtains 35 classes of general Heun models $\{U(t),\ \delta_t(t)\}$ generated by a set of 35 basic functions $\{U^*,\ \delta_z^*\}$

A.M. Ishkhanyan, T.A. Shahverdyan, T.A. Ishkhanyan, Eur. Phys. J. D **69**, 10 (2015)

35 general-Heun classes are defined by class-generator pairs of the form

$$U^* = U_0^* z^{k_1} (z-1)^{k_2} (z-a)^{k_3}$$

$$\delta_z^* = \frac{\delta_1}{z} + \frac{\delta_2}{z-1} + \frac{\delta_3}{z-a}$$

$$-1 \le k_{1,2,3} \qquad k_1 + k_2 + k_3 \le -1$$

Only 11 of these classes are algebraically independent.

$$a_{2tt} + \left(-i\,\delta_t(t) - \frac{U_t(t)}{U(t)} \right) a_{2t} + U^2(t)\, a_2 = 0 \quad (1)$$

A fundamental solution of equation (1)

$$a_2 = z^{\alpha_1} (z-1)^{\alpha_2} (z-a)^{\alpha_3} u(z)$$

When:

$$\gamma = -1$$

$$q^2 + q\left(a\delta + \varepsilon - 1 - a\right) + a\alpha\beta = 0$$

The general-Heun function can be reduced to the Clausen generalized hypergeometric function ${}_3F_2$

Checking the thirty-five general Heun classes of the two-state models, we find that the only class for which $\gamma = -1$ and the algebraic constraint on the accessory parameter is *unconditionally* satisfied is the class defined by the triad

$$(k_1, k_2, k_3) = (1, -1, -1)$$

$$\delta_1 = 0$$

$$U(t) = \frac{U_0^* z}{(z-1)(z-a)} \frac{dz}{dt} \qquad \delta_t(t) = \left(\frac{\delta_2}{z-1} + \frac{\delta_3}{z-a} \right) \frac{dz}{dt}$$

The general solution of the two-state problem for this family can be written in terms of the Clausen general hypergeometric functions as shapes and the occupation probabilities

$$a_2 = \varphi(z)\left(C_1 \cdot {}_3F_2\left(\alpha,\beta,e_1+1;e_1,\delta;\frac{z-1}{a-1}\right) + C_2 \cdot {}_3F_2\left(\alpha,\beta,e_2+1;e_2,\varepsilon;\frac{a-z}{a-1}\right)\right)$$

$$\varphi(z) = (z-1)^{\alpha_2}(z-a)^{\alpha_3}, \qquad e_1 = -\frac{\alpha\beta}{q}, \qquad e_2 = -\frac{a\alpha\beta}{q}$$

Where the involved parameters are given as

$$\{\delta,\varepsilon\} = \{2\alpha_2 - i\delta_2 + 1, 2\alpha_3 - i\delta_3 + 1\}$$

$$\{\alpha\beta, q\} = \left\{\alpha_3(\alpha_2 - i\delta_2) + \alpha_2(\alpha_3 - i\delta_3) - \frac{2aU_{0s}^2}{(a-1)^2}, -a\alpha_2 - \alpha_3\right\}$$

$$\{\alpha_2,\alpha_3\} = \left\{\frac{i}{2}\left(\delta_2 \pm \sqrt{\frac{4U_{0s}^2}{(a-1)^2} + \delta_2^2}\right), \frac{i}{2}\left(\delta_3 \pm \sqrt{\frac{4a^2U_{0s}^2}{(a-1)^2} + \delta_3^2}\right)\right\}$$

- **The new model and its connection to the standard DK2**

Constant-amplitude model - 1

$$U_0^* z^{k_1} (z-1)^{k_2} (z-a)^{k_3} \frac{dz}{dt} = \text{const} \qquad (2)$$

$$\delta_t(t) = \left(\frac{\delta_1}{z} + \frac{\delta_2}{z-1} + \frac{\delta_3}{z-a} \right) \frac{dz}{dt} \qquad (3)$$

Integration of (2) **yields**

$$\frac{t-t_0}{\tau} = \log|z-1| - a\log|z-a|$$

$$\tau = U_0^* / ((1-a)U_0)$$

Constant-amplitude model

$$U(t) = U_0 = \text{const}$$

$$\delta_t(t) = \Delta_1 + \frac{\Delta_0 - \Delta_1}{z}$$

$z(t)$ obeys the algebraic equation $\dfrac{z-1}{(z-a)^a} = e^{(t-t_0)/\tau}$,

$a = 0$: $$z = 1 + e^{(t-t_0)/\tau}$$

$a = -1$: $$z = \sqrt{1 + e^{(t-t_0)/\tau}}$$

$a = -2$: $$z = -1 + \frac{1}{\left(e^{t/2\tau} + \sqrt{1+e^{t/\tau}}\right)^{2/3}} + \left(e^{t/2\tau} + \sqrt{1+e^{t/\tau}}\right)^{2/3}$$

$a = -3$: $$z = -2 + \sqrt{1 - \frac{3T}{2A} + \frac{3A}{2}} + \sqrt{2 + \frac{3T}{2A} - \frac{3A}{2} + \frac{2}{\sqrt{1 - \frac{3T}{2A} + \frac{3A}{2}}}}$$

$$A = T^{1/3}\left(-1 + \sqrt{1+T}\right)^{1/3}, \quad T = e^{(t-t_0)/\tau} / 27$$

The resonance condition $\delta_t = 0$ *is satisfied at the critical point* $z_0 = 1 - \Delta_0 / \Delta_1$

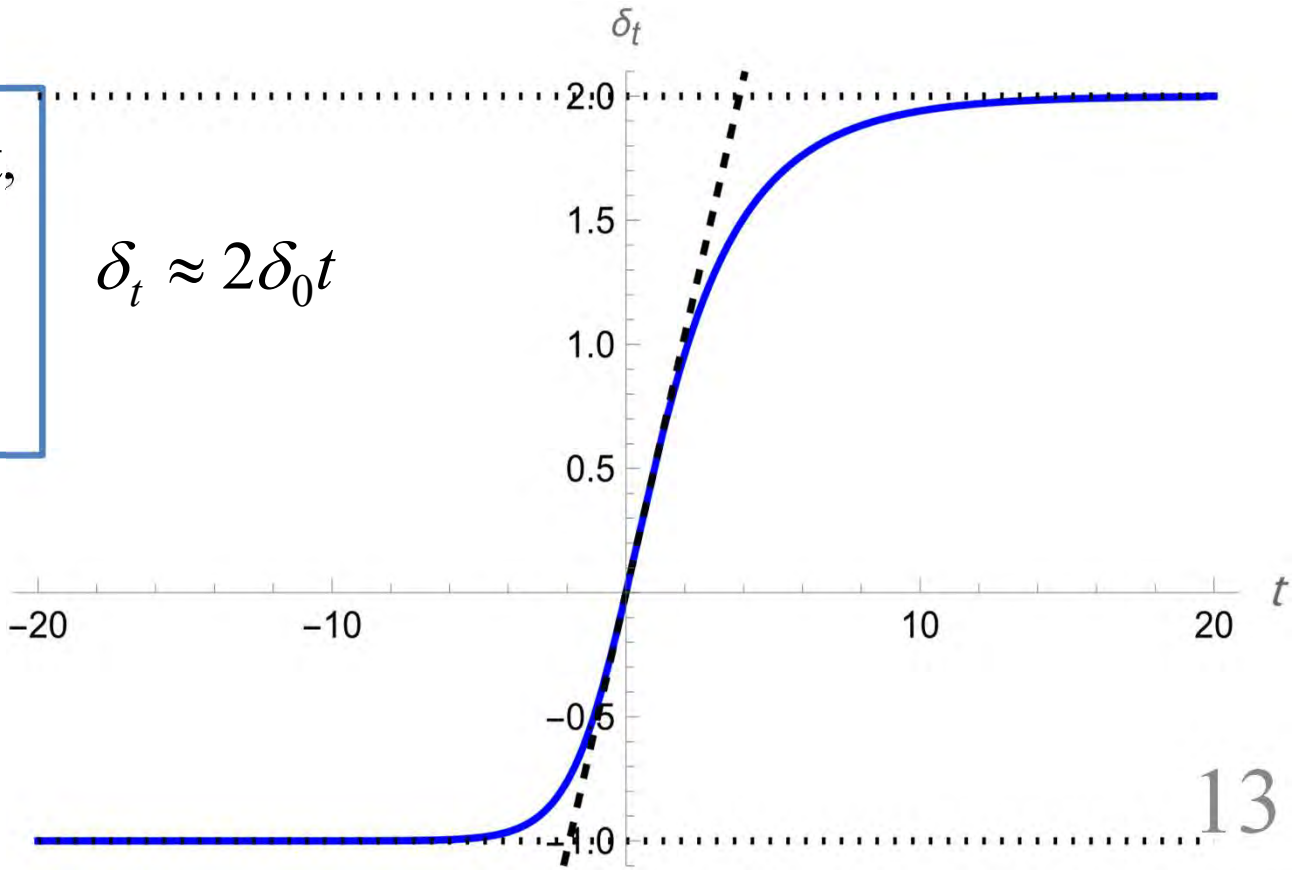
By choosing

$$t_0 = \tau \log \frac{(z_0 - a)^a}{z_0 - 1}$$

We ensure that the resonance occurs at the time point $t = 0$

Close to this resonance point, the detuning varies approximately linearly with time.

$$\delta_t \approx 2\delta_0 t$$



The slope

$$\delta_0 = \frac{\Delta_0 \Delta_1 \left(\Delta_0 + (a-1)\Delta_1\right)}{2(1-a)\left(\Delta_0 - \Delta_1\right)^2 \tau} > 0$$

Effective Landau-Zener parameter

$$\Lambda = U_0^2 / \delta_0$$

$\Lambda \ll 1$ Low excitation probability

$\Lambda \gg 1$ Higher probability of transition

- **Final Transition Probability**

at $t \rightarrow \infty$

The general solution of the two-state problem

$$a_2 = \varphi(z)\left(C_1 \cdot {}_3F_2\left(\alpha, \beta, e_1+1; e_1, \delta; \frac{z-1}{a-1}\right) + C_2 \cdot {}_3F_2\left(\alpha, \beta, e_2+1; e_2, \varepsilon; \frac{a-z}{a-1}\right)\right)$$

The asymptotic form of the general solution at $t \to -\infty$ can generally be written as a linear combination of two Floquet modes:

$$a_2 \sim c_1 e^{i\lambda_L^- t} + c_2 e^{i\lambda_L^+ t}$$

where $\lambda_L^{\mp}$ are the Floquet (quasienergy) exponents at $t \to -\infty$, given by

$$\lambda_L^{\mp} = \frac{\Delta_0}{2} \mp \sqrt{\frac{\Delta_0^2}{4} + U_0^2}$$

Let us fix $\Delta_0 < 0$

The system is initially prepared in the *lower* quasienergy state by adiabatically switching on the interaction from the bare state $a_1(-\infty) = 1$

$C_2 = 0: \rightarrow$ The solution becomes

$$a_2 = \varphi(z) \cdot C_1 \cdot {}_3F_2\left(\alpha, \beta, e_1 + 1; e_1, \delta; \frac{z-1}{a-1}\right)$$

The asymptotic behavior of the amplitudes at $t \rightarrow -\infty$

$$a_1 \sim e^{i(-\Delta_0 + \lambda_L)t}$$

$$a_2 \sim e^{i\lambda_L t}$$

$$t \rightarrow +\infty: \qquad \lambda_R^{\pm} = \frac{\Delta_1}{2} \pm \sqrt{\frac{\Delta_1^2}{4} + U_0^2}$$

Final-transition probability

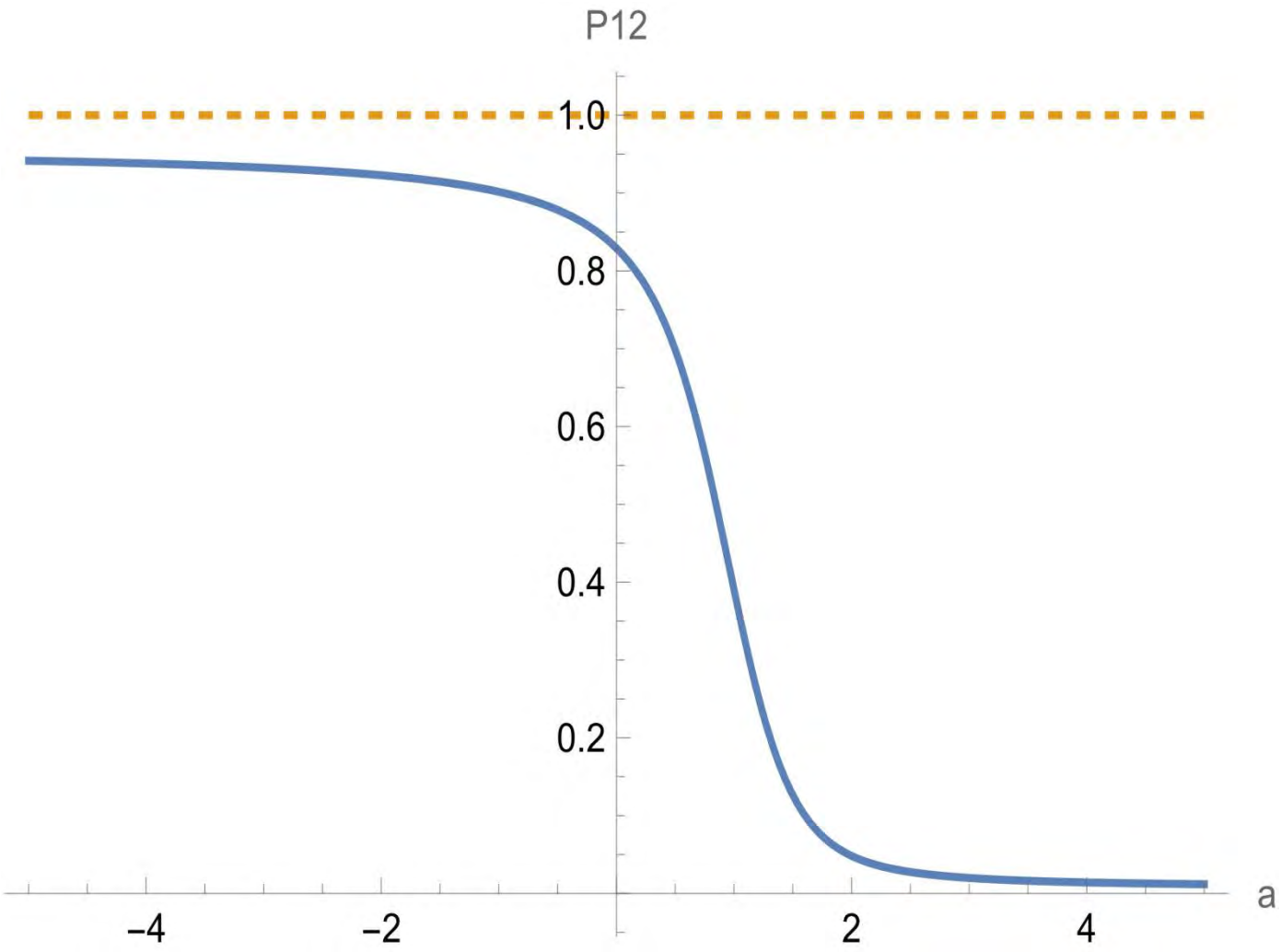
$$P_{I\to II} = 1 - \frac{\sinh\left(\pi\tau\left(R_L + (a-1)R_R - R\right)\right)\sinh\left(\pi\tau\left(R_L + (a-1)R_R + R\right)\right)}{\sinh\left(2\pi\tau R_L\right)\sinh\left(2(a-1)\pi\tau R_R\right)}$$

Generalized Rabi frequencies:

$$R_L = \sqrt{\frac{\Delta_0^2}{4} + U_0^2} \qquad R_R = \sqrt{\frac{\Delta_1^2}{4} + U_0^2}$$

$$R = \sqrt{\frac{(\Delta_0 + (a-1)\Delta_1)^2}{4} + a^2 U_0^2}$$

Dependence of the final transition probability on the asymmetry parameter a



- Discussion
- ……

Thank you !
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