Determination of the Green's function for the one-dimensional Dirac operator with regular potential using the SPPS method

Jocelyn Guadalupe Hernández García, Advisor: Dr. Víctor Barrera Figueroa

UPIITA-IPN

Outline

- Determination of the Green function for the one-dimensional Dirac operator with a singular potential
- Spectral parameter power series method (SPPS) for Dirac equation
- Green's funcion associated with the Dirac operator written in terms of the SPPS method

Outline

Consider the one-dimensional Dirac equation free of units for particles spin-$\frac{1}{2}$-with mass m expressed as

$$\mathfrak{D}_Q\psi := J\frac{d\psi}{dx} + Q\psi = \lambda\psi. \tag{1}$$

Where J and Q are 2×2 matrices

$$J = \frac{1}{i}\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} p_1(x) & q(x) \\ q(x) & p_2(x) \end{pmatrix}$$

and $q(x), p_1(x), p_2(x)$ are functions that define electromagnetic and scalar interactions

[1] V.Barrera-Figueroa, V.S Rabinovich, S-A.C. Loredo-Ramirez On the calculation of the discrete spectra of one-dimensional Dirac operators. Math Meth Appl Sci. 2022

$$p_1(x) = q_{el}(x) + (m + q_{sc}(x)),$$
$$p_2(x) = q_{el}(x) - (m + q_{sc}(x)), \quad q(x) = q_{am}(x).$$

Where q_{el} is an electrostatic potential, q_{am} is anomalous magnetic moment, q_{sc} is a scalar potential and $m > 0$ is the mass for spin-$\frac{1}{2}$. The solutions to the Dirac equation ar of the form $\psi = \begin{pmatrix} \psi_1 & \psi_2 \end{pmatrix}^\top$ which represents a 2-spinor, and λ has the physical meaning of the particle's energy.

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Now we are going to consider a rotation in two dimensions. The transformation matrix is

$$A = \begin{pmatrix} \cos\varphi(x) & \sin\varphi(x) \\ -\sin\varphi(x) & \cos\varphi(x) \end{pmatrix}$$

Note that the product of the matrices A and J is commutative , if we write a $\psi(x) = Az$ where z is $z(x) = \begin{pmatrix} z_1, & z_2 \end{pmatrix}^T$ and substituting in the equation 1 We obtain the following expression

$$J\frac{d}{dx}Az + QAz = \lambda Az$$

By doing some operations we obtain the following expression

$$J\frac{d}{dx}z + \left(A^{-1}J\frac{dA}{dx} + A^{-1}QA\right)z = \lambda z$$

The expression placed in parentheses is called $\mathcal{Q}$

$$J\frac{d}{dx}z + \mathcal{Q}z = \lambda z$$

such that we can write a new form of the Dirac equation. Where

$$\mathcal{Q} = \begin{pmatrix} \mathrm{P}_1(\mathrm{x}) & \mathrm{q}(x) \\ \mathrm{q}(x) & \mathrm{P}_2(x) \end{pmatrix}$$

$$\mathrm{P}_1(x) = p_1(x)\cos^2\varphi(x) - q(x)\sin 2\varphi(x) + p_2(x)\sin^2\varphi(x)$$
$$\mathrm{P}_2(x) = p_1(x)\sin^2\varphi(x) + p_2(x)\cos^2\varphi(x) + q(x)\sin 2\varphi(x)$$
$$\mathrm{q}(x) = q(x)\cos 2\varphi(x) + \tfrac{1}{2}(p_1(x) - p_2(x))\sin 2\varphi(x)$$

[2] V.Barrera-Figueroa, V.S Rabinovich, S-A.C. Loredo-Ram rez On the calculation of the discrete spectra of one-dimensional Dirac operators. Math Meth Appl Sci. 2022

We can find a function $\varphi(x)$ such that the trace of the matrix is equal zero such that we can arrive at what is known as the canonical form of the operator

$$\varphi(x) = -\frac{1}{2}\int (p_1 + p_2)\, dx$$

$$J\frac{d}{dx}z + \begin{pmatrix} \mathrm{P}_1(x) & \mathrm{q}(x) \\ \mathrm{q}(x) & -\mathrm{P}_1(x) \end{pmatrix} z = \lambda z$$

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Let us consider the eigenvalue equation considering a surce term for the one-Dimensional Dirac operator with a regular potential

$$\left(\mathfrak{D}_Q - \lambda \mathbb{I}\right) u = f$$

formally the solution to the adove equation can be written as

$$u = \left(\mathfrak{D}_Q - \lambda \mathbb{I}\right)^{-1} f = \mathfrak{R}_\lambda f$$

Where

$$\mathfrak{R}_\lambda = \left(\mathfrak{D}_Q - \lambda \mathbb{I}\right)^{-1}$$

is the resolvent operator associeted with the one-dimensional Dirac operator with regular potential.

The set of point $\lambda \in \mathbb{C}$ where the operator exist is called the resovent set and is denoted by ρ_λ besides

$$\sigma \cup \rho = \mathbb{C}$$

For this reason, the study of the resolvent operator is relevant, as it gives us information outside of sigma. It is also associated with the Green's function with this expression.

$$\mathfrak{R}_\lambda = \int G(x,y) f(y)\, dy$$

To this system of equations we place boundary conditions in an interval $[a, b]$ and with α and β arbitrary numbers

$$\begin{aligned} z_1(a)\sin\alpha + z_2(a)\cos\alpha &= 0 \\ z_1(b)\sin\beta + z_2(b)\cos\beta &= 0 \end{aligned}$$

Now let $\varphi(x,\lambda)$ be a solution of the homogeneous system that satisfies boundary conditions at the point a

$$\varphi_1(a,\lambda) = \cos\alpha, \qquad \varphi_2(a,\lambda) = -\sin\alpha$$

and let $\psi(x,\lambda)$be a solution of the homegeneous system but now it meets boundary conditions at point b

$$\psi_1(b,\lambda) = \cos\beta, \qquad \psi_2(b,\lambda) = -\sin\beta$$

It can be shown that $\varphi(x,\lambda)$and $\psi(x,\lambda)$ are linearly independent solutions, i.e. the Wronskian does not vanish on the interval$[a,b]$

$$W\{\varphi,\psi\} = \begin{vmatrix} \varphi_1(x,\lambda) & \psi_1(x,\lambda) \\ \varphi_2(x,\lambda) & \psi_2(x,\lambda) \end{vmatrix} \neq 0, \qquad x\epsilon(a,b)$$

Therefore we can write the Green function in the following way

$$G(x,y;\lambda) = \frac{1}{w(\lambda)} \begin{cases} \psi(x,\lambda) \quad \varphi^{\top}(x,\lambda) & x<y \\ \varphi(x,\lambda) \quad \psi^{\top}(y,\lambda) & y<x \end{cases}$$

[3] A Svinarenko, AV Glushkov, AV Loboda, DE Sukharev, Yu. V. Dubrovskaya, NV Mudraya. Green's Function of the Dirac Equation whit Complex Energy and Non-singular Central Nuclear Potential. AIP Conference Proceedings 1232,259 (2010)

Outline

- The spectral parameter power series (SPPS) representation for solutions to second-order linear differential equations has proven to be an effective tool for the (analytical and numerical) resolution and study of various problems.
- The SPPS method starts from a nonzero solution of the equation for a fixed value of the spectral parameter and, through a series of recursive integrations, generates Taylor series coefficients of the solution with respect to the spectral parameter.
- Dr. Kravchenko's result was subsequently extended to solve systems such as the Dirac system.

[4] Vladislav V. Kravchenko . Dirac and Inverse Sturn- Liouville Problems. Birkhäuser Frontiers in Mathematics.

The power series method of the spectral parameter for Dirac systems considers an expression

$$\mathfrak{D}_Q Y := J\frac{dY}{dx} + QY = \lambda Y$$

Let us suppose that the homogeneous system

$$J\frac{dY}{dx} + QY = 0$$

has a solution Y$= (f_1, f_2)^T$such that f_1 and $f_2$do not vanish in the interval $[a, b]$. Let x_0 be a point on the segment $[a, b]$ Let us consider the following system of functions defined by the recursive relations

$$X^{(0)}(x) = f_1(x_0)\, f_2(x_0) \int_{x_0}^{x} \frac{p_2(s)}{f^2(s)} ds$$

$$Y^{(0)}(x) = 1 + f_1(x_0)\, f_2(x_0) \int_{x_0}^{x} \frac{p_1(s)}{g^2(s)} ds$$

$$X^{(n+1)}(x) = (n+1) \int_{x_0}^{x} \left(-Y^{(n)}(s) + \frac{p_2(s)}{f_1^2(s)} Z^{(n)}(s) \right) ds$$

$$Y^{(n+1)} = (n+1) \int_{x_0}^{x} \left(\frac{f_1(s)}{g(s)} X^{(n)}(s) + \frac{p_1(s)}{f_2^2(s)} Z^{(n)}(s) \right) ds$$

Let us observe that from the seed equations we know all the terms. And from it, we can find the rest of the recursive equations similarly

$$\widetilde{X}^{(0)}(x) = 1 - f_1(x_0) f_2(x_0) \int_{x_0}^{x} \frac{p_2(s)}{f_1^2(s)} ds$$

$$\widetilde{Y}^{(0)}(x) = -f_1(x_0) f_2(x_0) \int_{x_0}^{x} \frac{p_1(s)}{f_2^2(s)} ds$$

$$\widetilde{Z}^{(n)}(x) = \int_{x_0}^{x} \left(\widetilde{X}^{(n)}(s) f_1^2(s) + \widetilde{Y}^{(n)}(s) f_2^2(s) \right) ds$$

$$\widetilde{X}^{(n+1)}(x) = (n+1) \int_{x_0}^{x} \left(-\widetilde{Y}^{(n)}(s) + \frac{p_2(s)}{f_1^2(s)} \widetilde{Z}^{(n)}(s) \right) ds$$

$$\widetilde{Y}^{(n+1)} = (n+1) \int_{x_0}^{x} \left(\frac{f(s)}{g(s)} \widetilde{X}^{(n)}(s) + \frac{p_1(s)}{f_2^2(s)} \widetilde{Z}^{(n)}(s) \right) ds$$

Theorem

Suppose that the homogeneous system has a solution $\psi_0 = (f_1, f_2)^\top$ where f_1 and f_2 do not vanish on the interval $[a, b]$. A general solution of the system 1 has the form

$$\psi = c_1\phi_1 + c_2\phi_2 = c_1 \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} + c_2 \begin{pmatrix} u_2 \\ v_2 \end{pmatrix},$$

where c_1 and c_2 are arbitrary complex constants. and

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \begin{pmatrix} f_1 \widetilde{X}^{(n)} \\ f_2 \widetilde{Y}^{(n)} \end{pmatrix}, \qquad \begin{pmatrix} u_1 \\ v_2 \end{pmatrix} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \begin{pmatrix} f_1 X^{(n)} \\ f_2 Y^{(n)} \end{pmatrix}$$

Here the formal powers $X^{(n)}, Y^{(n)}, \widetilde{X}^{(n)}, \widetilde{Y}^{(n)}$

[5] Gutíerrez Jiménez Nelson, Torba Sergii M. Spectral parameter power series represen- tation for solutions of linear system of two rst order di erential equations. Applied Mathematics and Computation. 2020;370:124911

In our work we show that these terms are those of the Green function,

$$\psi(x,\lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \begin{pmatrix} f_1 \widetilde{X}^{(n)} \\ f_2 \widetilde{Y}^{(n)} \end{pmatrix} \qquad \varphi(x,\lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \begin{pmatrix} f_1 X^{(n)} \\ f_2 Y^{(n)} \end{pmatrix}$$

therefore the Wronskian written in terms of these power series

$$\begin{aligned} W\{\varphi,\psi\} &= \begin{vmatrix} \varphi_1(x,\lambda) & \psi_1(x,\lambda) \\ \varphi_2(x,\lambda) & \psi_2(x,\lambda) \end{vmatrix} \\ &= \begin{vmatrix} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} f_1 X^{(n)} & \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} f_1 \widetilde{X}^{(n)} \\ \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} f_2 Y^{(n)} & \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} f_2 \widetilde{Y}^{(n)} \end{vmatrix} \end{aligned}$$

Outline

Theorem

Let $\mathfrak{D}$ be the one-dimensional Dirac operator in its canonical representation and $G(x, y; \lambda)$ be the Green's function associated to the operator $\mathfrak{D}$ on an interval $[a, b]$. Let $\varphi(x, \lambda)$ and $\psi(x, \lambda)$ be linearly independent solutions of the homogeneous equation of the Dirac operator where $\varphi(x, \lambda)$ satisfies boundary conditions at point a and $\psi(x, \lambda)$ satisfies the boundary conditions at point b and let $W\{\varphi, \psi\}$ be the Wronskian of these solutions. Then the Green's function admits the following representation in terms of power series

$$G(x, y; \lambda) = \frac{1}{W\{\varphi, \psi\}} \sum_{n=0}^{n} \lambda^n g_n(x, y; \lambda)$$

Where $\lambda \in \mathbb{C}$ is the spectral parameter and $g_n(x, y, \lambda)$ is determined by the following expression:

$$g_{n(x,y,\lambda)}=\sum_{k=0}^{n}\begin{cases}\begin{pmatrix}\left(f_1(x)f_1(y)\widetilde{X}^{(k)}(x)X^{(n-k)}(y)h_1(y)+f_1(x)f_2(y)\widetilde{X}^{(k)}(x)Y^{(n-k)}(y)h_2(y)\right)\\\left(f_2(x)f_1(y)\widetilde{Y}^{(k)}(x)X^{(n-k)}(y)h_1(y)+f_2(x)f_2(y)\widetilde{Y}^{(k)}(x)Y^{(n-k)}(y)h_2(y)\right)\end{pmatrix} & x<y\\\begin{pmatrix}\left(f_1(y)f_1(x)\widetilde{X}^{(k)}(y)X^{(n-k)}(x)h_1(y)+f_1(y)f_2(x)\widetilde{X}^{(k)}(y)Y^{(n-k)}(x)h_2(y)\right)\\\left(f_2(y)f_1(x)\widetilde{Y}^{(k)}(y)X^{(n-k)}(x)h_1(y)+f_2(x)f_2(y)\widetilde{Y}^{(k)}(y)Y^{(n-k)}(x)h_2(y)\right)\end{pmatrix} & y<x\end{cases}$$

which is calculated from the functions $X^{(n)}, \widetilde{X}^{(n)}, Y^{(n)}, \widetilde{Y}^{(n)}$ that satisfy the recurrence relations written in the equations.

Outline

- The explicit form of the Green's function for the one-dimensional Dirac operator was determined by associating it with the resolvent operator, which allows for the analysis of the operator's spectrum.
- The fundamental relationship between the Green's function of the Dirac operator and the Spectral Parameter Power Series (SPPS) method was established and proven.
- As the main contribution, a novel representation of the Green's function was derived, expressed in terms of the polynomials generated by the SPPS method.


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Thank you
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A mi Asesor,amigos, padres y Joseph