Two-dimensional Schrödinger operators with non-local singular interactions: Self-adjointness and the non-relativistic limits

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AAMP 2025

Introduction

Summarize following articles and extend results:

- HHSST L. H., M. Holzmann, C. Stelzer-Landauer, G. Stenzel, M. Tušek,
  Two-dimensional Schrödinger operators with non-local singular potentials. (2025)
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We investigate self-adjoint realizations of differential operators on

$$L^2(\mathbb{R}^2) = L^2(\Omega_+) \oplus L^2(\Omega_-)$$

with suitable transmission/boundary conditions along Σ.

Framework: Boundary triples

Generalized Boundary Triple

Let A be a densely defined, symmetric, closed operator on a Hilbert space $\mathcal{H}$, and let $\overline{T} = A^*$.

Definition

Let $\Gamma_0, \Gamma_1 : \operatorname{Dom} T \to \mathcal{G}$ be linear mappings into a Hilbert space $\mathcal{G}$ such that:

1. For all $f, g \in \operatorname{Dom} T$,
$$\langle Tf, g\rangle_{\mathcal{H}} - \langle f, Tg\rangle_{\mathcal{H}} = \langle \Gamma_1 f, \Gamma_0 g\rangle_{\mathcal{G}} - \langle \Gamma_0 f, \Gamma_1 g\rangle_{\mathcal{G}}.$$
2. $\operatorname{Ran} \Gamma_0 = \mathcal{G}$.
3. The operator $T_0 := T|_{\operatorname{Ker} \Gamma_0}$ is self-adjoint on $\mathcal{H}$.

Then $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is called a *generalized boundary triple*.

Auxiliary mappings:

$$\gamma(z) := \left(\Gamma_0|_{\operatorname{Ker}(T-z)}\right)^{-1}, \quad M(z) := \Gamma_1 \gamma(z), \qquad z \in \rho(T_0).$$

Extensions via Boundary Triples

We study

$$T_{L,R} := T \upharpoonright \mathrm{Ker}(L\Gamma_0 - R\Gamma_1),$$

where L, R are bounded operators on $\mathcal{G}$

[Behrndt–Langer '06], [Behrndt–Holzmann–Stelzer-Landauer–Stenzel '24]

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Let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a generalized boundary triple for $\overline{T} = A^*$, and $T_0 = T|_{\mathrm{Ker}\,\Gamma_0}$. Then:

1. (**Abstract Birman-Schwinger**) For all $z \in \rho(T_0)$,

$$\mathrm{Ker}(T_{L,R} - z) = \gamma(z)\,\mathrm{Ker}(L - RM(z)).$$

   In particular, $z \in \sigma_p(T_{L,R}) \;\Leftrightarrow\; 0 \in \sigma_p(L - RM(z))$.

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$$f \in \operatorname{Ran}(T_{L,R} - z) \Leftrightarrow R\gamma(\overline{z})^* f \in \operatorname{Ran}(L - RM(z)).$$
3. (**Krein resolvent formula**) For all such z and $f \in \operatorname{Ran}(T_{L,R} - z)$:
$$(T_{L,R} - z)^{-1} f = (T_0 - z)^{-1} f + \gamma(z)(L - RM(z))^{-1} R\gamma(\overline{z})^* f.$$

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$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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- $H^p_\Delta(\mathbb{R}^2 \setminus \Sigma) := \left\{ f = f_+ \oplus f_- \in H^p(\Omega_+) \oplus H^p(\Omega_-) \;\middle|\; \Delta f_\pm \in L^2(\Omega_\pm) \right\}$

Boundary Triple for Schrödinger Operator

[HHSST]

Consider the operator

$$\begin{aligned}\operatorname{Dom} S &= \left\{ f \in H^{1/2}_{\Delta}(\mathbb{R}^2 \setminus \Sigma) \;\middle|\; \overline{\partial} f_{\pm} \in H^{1/2}(\Omega_{\pm}) \right\},\\ Sf &= (-\Delta f_+) \oplus (-\Delta f_-).\end{aligned}$$

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We define the boundary mappings:

$$\begin{aligned} \Gamma_0^S f &:= 2 \begin{pmatrix} \overline{\mathsf{n}}\big(\gamma_D^{+}\overline{\partial} f_{+} - \gamma_D^{-}\overline{\partial} f_{-}\big) \\ \mathsf{n}\big(\gamma_D^{+} f_{+} - \gamma_D^{-} f_{-}\big) \end{pmatrix}, \\ \Gamma_1^S f &:= \frac{1}{2} \begin{pmatrix} \gamma_D^{+} f_{+} + \gamma_D^{-} f_{-} \\ -\gamma_D^{+}\overline{\partial} f_{+} - \gamma_D^{-}\overline{\partial} f_{-} \end{pmatrix}. \end{aligned}$$

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Result: $\{L^2(\Sigma; \mathbb{C}^2), \Gamma_0^S, \Gamma_1^S\}$ is a generalized boundary triple for $\overline{S}$.

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Result: $\{L^2(\Sigma; \mathbb{C}^2), \Gamma_0^S, \Gamma_1^S\}$ is a generalized boundary triple for $\overline{S}$.

We study self-adjoint realizations

$$S \restriction \operatorname{Ker}(L\Gamma_0^S - R\Gamma_1^S).$$

Non-local δ-shell interaction

$S_B := S \upharpoonright \mathrm{Ker}\big(\Gamma_0^S + B\Gamma_1^S\big)$, where $B : L^2(\Sigma;\mathbb{C}^2) \to L^2(\Sigma;\mathbb{C}^2)$ is a compact self-adjoint operator.

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Main properties (self-adjointness, Birman–Schwinger principle, Kreĭn resolvent formula)

S_B is self-adjoint on $L^2(\mathbb{R}^2)$.

1. For all $z \in \mathbb{C} \setminus [0, \infty)$,

$$\mathrm{Ker}(S_B - z) = \gamma(z)\,\mathrm{Ker}(I + BM(z)).$$

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2. For all $z \in \rho(S_B) \cap (\mathbb{C} \setminus [0, \infty))$, $I + BM(z)$ is boundedly invertible on $L^2(\Sigma; \mathbb{C}^2)$ and

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Open problem: Is there a regular approximation of this model?

Spectrum of S_B

Let B compact, self-adjoint on $L^2(\Sigma; \mathbb{C}^2)$ with block structure

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{pmatrix}.$$

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3. Let $B_{12} = B_{22} = 0$. Then the operator S_B satisfies

$$\mathrm{Dom}(S_B) = \left\{ f \in H^{3/2}_{\Delta}(\mathbb{R}^2 \setminus \Sigma) \cap H^1(\mathbb{R}^2) \,\middle|\, \gamma_N^+ f_+ - \gamma_N^- f_- = -B_{11}\gamma_D f \right\},$$

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4. Let Σ be C^2-boundary and $B \geq 0$ such that B_{22} has infinitely many positive eigenvalues. Then $\sigma_{\text{disc}}(S_B)$ is **infinite** and **unbounded** from below.

Non-local boundary conditions

Let B be a compact self-adjoint operator on $L^2(\Sigma)$.

On $L^2(\Omega_+)$

$$\operatorname{Dom} S_B^+ = \left\{ f \in H^{1/2}_{\Delta}(\Omega_+) \,\middle|\, \overline{\partial} f \in H^{1/2}(\Omega_+),\ \gamma_D^+ f = B\,\overline{\mathrm{n}}\,\gamma_D^+ \overline{\partial} f \right\},$$
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On $L^2(\Omega_-)$

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$$S_B^- f = -\Delta f.$$

$$\mathbb{S}_B := S_B^+ \oplus S_B^-$$

Relation to Boundary Triple Framework

It can be shown that

$$\mathbb{S}_B = S \restriction \mathrm{Ker}(L\Gamma_0^S - R\Gamma_1^S), \qquad L = \begin{pmatrix} B & 0 \\ 0 & \overline{\mathsf{n}} \end{pmatrix}, \quad R = \begin{pmatrix} 4 & 0 \\ 0 & -4B\overline{\mathsf{n}} \end{pmatrix}.$$

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Self-adjointness of $\mathbb{S}_B$

If Σ is a $\mathbb{C}^3$ boundary and $B : L^2(\Sigma) \to H^1(\Sigma)$ is compact and self-adjoint, then

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Open problems:

- Prove self-adjointness for a larger class of self-adjoint operators B.
- Prove self-adjointness for boundary conditions of the type

$$B\gamma_D^{\pm} f \;=\; \pm\,\overline{\mathrm{n}}\,\gamma_D^{\pm}\overline{\partial} f.$$

- Prove self-adjointness under weaker boundary regularity assumptions.

Dirac operator

Define the free Dirac operator in $\mathbb{R}^2$:

$$\mathcal{D} \;=\; -ic(\sigma \cdot \nabla) + \frac{c^2}{2}\sigma_3 = -ic\big(\sigma_1\partial_1 + \sigma_2\partial_2\big) + \frac{c^2}{2}\sigma_3.$$

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On $L^2(\mathbb{R}^2;\mathbb{C}^2)$ we set

$$\operatorname{Dom} D = H^{1/2}_\sigma(\mathbb{R}^2\setminus\Sigma;\mathbb{C}^2), \qquad Df = \mathcal{D}f_+ \oplus \mathcal{D}f_-,$$

where

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By [Behrndt, Holzmann, Stelzer-Landauer, Stenzel '24], the triple

$$\{\, L^2(\Sigma;\mathbb{C}^2),\ \Gamma^D_0,\ \Gamma^D_1 \,\}$$

is a generalized boundary triple for D, with

$$\Gamma^D_0 f := i(\sigma\cdot n)(\gamma^+_D f_+ - \gamma^-_D f_-), \qquad \Gamma^D_1 f := \tfrac{1}{2}(\gamma^+_D f_+ + \gamma^-_D f_-).$$

Relativistic non-local δ-shell interaction

Definition

The non-local relativistic δ-shell interaction is the operator

$$D_B : L^2(\mathbb{R}^2;\mathbb{C}^2) \to L^2(\mathbb{R}^2;\mathbb{C}^2),$$

given by

$$\begin{aligned} \operatorname{Dom} D_B &= \left\{ f \in H^{1/2}_\sigma(\mathbb{R}^2 \setminus \Sigma;\mathbb{C}^2) \;\middle|\; \Gamma^D_0 f + B\,\Gamma^D_1 f = 0 \right\}, \\ D_B f &= \mathcal{D} f_+ \oplus \mathcal{D} f_-, \end{aligned} \tag{1}$$

where $B : L^2(\Sigma;\mathbb{C}^2) \to L^2(\Sigma;\mathbb{C}^2)$ is linear, self-adjoint, and compact.

Relativistic non-local δ-shell interaction

Definition

The non-local relativistic δ-shell interaction is the operator

$$D_B : L^2(\mathbb{R}^2; \mathbb{C}^2) \to L^2(\mathbb{R}^2; \mathbb{C}^2),$$

given by

$$\begin{aligned} \operatorname{Dom} D_B &= \left\{ f \in H_\sigma^{1/2}(\mathbb{R}^2 \setminus \Sigma; \mathbb{C}^2) \,\middle|\, \Gamma_0^D f + B\,\Gamma_1^D f = 0 \right\}, \\ D_B f &= \mathcal{D} f_+ \oplus \mathcal{D} f_-, \end{aligned} \tag{1}$$

where $B : L^2(\Sigma; \mathbb{C}^2) \to L^2(\Sigma; \mathbb{C}^2)$ is linear, self-adjoint, and compact.

- D_B is self-adjoint on $L^2(\mathbb{R}^2; \mathbb{C}^2)$,
- the essential spectrum is

$$\sigma_{\text{ess}}(D_B) = (-\infty, -\tfrac{c^2}{2}] \cup [\tfrac{c^2}{2}, +\infty).$$

Non-relativistic limit

Let $A \in L^2(\Sigma; \mathbb{C}^{2,2})$ be a matrix-valued function. Define the operator $\Pi_A : L^2(\Sigma; \mathbb{C}^2) \to \mathbb{C}^2$ by $\Pi_A f = \int_\Sigma A^* f$.

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The limit theorem

Let $F, G \in L^2(\Sigma; \mathbb{C}^{2,2})$, and set $B := \Pi_F^* \Pi_G = \Pi_G^* \Pi_F$. Define

$$V_c := \operatorname{diag}(1/\sqrt{c}, \sqrt{c}), \qquad V := \operatorname{diag}(1, -2i).$$

Then for every $z \in \mathbb{C} \setminus \mathbb{R}$ there exists $K > 0$ such that

$$\left\| \left(D_{V_c B V_c^*} - z - \tfrac{c^2}{2} \right)^{-1} - \left(S_{VBV^*} - z \right)^{-1} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\| \leq \frac{K}{c},$$

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for all sufficiently large $c > 0$.

Open problem: Does the statement hold for an arbitrary compact B?

Non-relativistic limit

The limit theorem

Let B be any compact, self-adjoint operator on $L^2(\Sigma; \mathbb{C}^2)$. Define

$$V_c := \operatorname{diag}(1/\sqrt{c}, \sqrt{c}), \qquad V := \operatorname{diag}(1, -2i).$$

Then for every $z \in \mathbb{C} \setminus \mathbb{R}$ there exists $K > 0$ such that

$$\left\| \left(D_{V_c B V_c^*} - z - \tfrac{c^2}{2}\right)^{-1} - \left(S_{VBV^*} - z\right)^{-1} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\| \le \frac{K}{c},$$

for all sufficiently large $c > 0$.

Answer: Yes

Generalized MIT model

Let B and C be bounded operators on $L^2(\Sigma)$. We define Dirac operator $D^{\pm}_{B,C}$ with generalized MIT boundary condition on $L^2(\Omega_\pm;\mathbb{C}^2)$ as

$$\operatorname{Dom} D^{\pm}_{B,C} = \{f \in H^{1/2}_{\sigma}(\Omega_\pm;\mathbb{C}^2) \mid \pm iC\mathrm{n}(\gamma^{\pm}_{D} f)_1 = B(\gamma^{\pm}_{D} f)_2\},$$
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$\mathbb{D}_{B,C} := D^{+}_{B,C} \oplus D^{-}_{B,C}$,where

$$\begin{aligned}\operatorname{Dom} \mathbb{D}_{B,C} &= \{H^{1/2}_{\sigma}(\mathbb{R}^2 \setminus \Sigma;\mathbb{C}^2) \mid L\Gamma^D_0 f - R\Gamma^D_1 f = 0\},\\ \mathbb{D}_{B,C} f &= \mathcal{D} f_+ \oplus \mathcal{D} f_-\end{aligned}$$

and

$$L = \begin{pmatrix} -B\mathrm{n} & 0 \\ 0 & C \end{pmatrix}, R = \begin{pmatrix} 2C\mathrm{n} & 0 \\ 0 & 2B \end{pmatrix}.$$

Self-adjointness of $D^{\pm}_{B,C}$

Theorem

Assume:

- Ω_+ has a C^3-boundary Σ,
- $B : L^2(\Sigma) \to H^1(\Sigma)$ is compact,
- $C : L^2(\Sigma) \to H^1(\Sigma)$ is bounded with bounded inverse,
- BC^* is self-adjoint.

Then:

- $D^{\pm}_{B,C}$ are self-adjoint on $L^2(\Omega_\pm)$,
- $\sigma(D^+_{B,C}) \setminus \{0\} = \sigma_{\mathrm{disc}}(D^+_{B,C})$,
- $\sigma_{\mathrm{ess}}(D^-_{B,C}) = (-\infty, -c^2/2] \cup [c^2/2, +\infty)$.

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Open problems: What happens if $C = I$? Can we extend self-adjointness to larger classes of operators B, C? Is it possible to weaken the boundary regularity assumptions?

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