Complex masses in lattice models at finite density

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I discuss an emergence of complex masses and an oscillating decay of correlations in some lattice models at finite density. The models are Z(N) LGT's with static quarks and SU(N) Polyakov loop models both for N=3 and in the 't Hooft-Veneziano limit. As a mathematical tool used to study the correlation functions, I review different approaches to construction of dual formulations of various Abelian and non-Abelian LGTs.

Based on papers in collaboration with A. Papa, V. Chelnokov, S. Voloshyn, E. Mendicelli

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1. Complex masses in lattice spin and gauge models
2. Dual formulations of LGT: general review
3. Duals of lattice models and sign problem
4. Abelian models with static quarks
5. Polyakov loop model: the ’t Hooft-Veneziano limit
6. Polyakov loop model: Monte-Carlo simulations
7. Summary

I. Complex masses in lattice spin and gauge models

A. Bazavov and J.H. Weber, Color Screening in Quantum Chromodynamics, Progress in Particle and Nuclear Physics, **116** (2021) 103823;
80 pages of text in [arXiv:2010.01873 [hep-lat]];
only 2 pages devoted to screening masses at finite density

Simulations at imaginary chemical potential:

M. Andreoli, C. Bonati, M. D'Elia, M. Mesiti, F. Negro, A. Rucci, F. Sanfilippo, Phys.Rev. D **97** 054515 (2018)

Phase with oscillating decay of correlations was shown to exist in:

- $(1+1)$-dimensional $SU(3)$ QCD with static quark determinant and non-zero chemical potential
  H. Nishimura, M. Ogilvie, K. Pangeni, Phys.Rev. D93 (2016) 094501

- Three-dimensional $Z(3)$ spin model in a complex external field
  O. Akerlund, P. de Forcrand, T. Rindlisbacher, JHEP 10 (2016) 055

Can similar phase be realized in QCD at finite baryon density
and, if yes, how to study it?

II. Dual formulations of LGT: general review

Dual representations and their significance:

- Map strong coupling (disordered) phase to weak coupling (ordered) phase

- Allow to establish in many cases relevant configurations responsible for such phenomena like confinement, mass gap generation, etc.

- Have been used to establish many rigorous results in spin and gauge models

Conventional dual transformations can be defined as a sequence of transformations consisting of the following steps:

1. Fourier expansion of the Boltzmann weight $\exp(H)$. This is an essentially character expansion on the group.
2. Exact integration over original degrees of freedom. As a result one obtains a set of constraints on the summation variables (which label representations of the group G and matrix elements of $U(x)$) in the character expansion.
3. Solution of the constraints in terms of new dual variables.

This is a generalization of the Kramers–Wannier dual transformations of the two-dimensional Ising model and works perfectly well for many Abelian spin and lattice gauge models.

- Dual representations based on the plaquette formulation. Dual variables are introduced as variables conjugate to local Bianchi identities. The dual model is non-local due to the presence of connectors C. (Batrouni, Halpern, '82-84; Borisenko, Voloshin, Faber, '09)

$$Z = \sum_{r_c} \int \prod_p dU_p e^{\beta \mathrm{ReTr} U_p} \prod_c d(r_c) \chi_{r_c}(U_c = U_A C U_B c^\dagger) \quad (1)$$

- Dual representations based on 1) the character expansion of the Boltzmann weight and 2) the integration over link variables using Clebsch-Gordan expansion (Anishetty et.al., '93)

$$Z = \sum_{r_p, r_l} \prod_p C_{r_p}(\beta_{\mu\nu}) \prod_x (6j \text{ links}) \prod_{\mathrm{c}} (\text{6j cubes}) \quad (2)$$

- In the strong coupling limit, $\beta = 0$, the $SU(N)$ model can be mapped onto monomer-dimer-closed baryon loop model (Karsch, Mütter'89)

- Recent approaches:
  - n-link action (Vairinhos, de Forcrand, '14);
  - Abelian color cycles (Gattringer, C. Marchis, '17-18);
  - gauge integration via Weingarten calculus $Wg^N(\sigma)$ (Borisenko, Chelnokov, Voloshin, '17; Gagliardi, Unger, '20)

E.g., $2d\ U(N)$ *LGT with staggered fermions*

$$Z = \sum_{r(p)=-\infty}^{\infty} \sum_{t(p)=0}^{\infty} \sum_{k(l),n(l)=0}^{N} \sum_{s(x)=0}^{N} \sum_{\{\tau_l,\sigma_l\}} \prod_x m^{s(x)}\ N^{|\gamma(x)|}$$

$$\prod_p \frac{(\beta/2)^{2t(p)+|r(p)|}}{t_p!(t_p+|r(p)|)!} \prod_l \left[\frac{\eta_\nu(x)}{2}\right]^{k(l)+n(l)} Wg^N(\tau_l^{-1}\sigma_l) \prod_{\mathcal{L}} (-1)^{1+\frac{1}{2}|\mathcal{L}|}$$

$$\times\ (\text{constraints}) \tag{3}$$

III. Duals of lattice models and sign problem

Can dual formulations solve fully or partially the sign problem in spin and gauge models?

For an important class of classical spin models like $Z(N)$ model in an external complex magnetic field, $O(N)$ non-linear sigma model and principal chiral model at finite density the answer is "YES".

For the non-Abelian LGT with the action

$$S = \beta_t \sum_{p_t} \mathrm{ReTr} U_{p_t} + \beta_s \sum_{p_s} \mathrm{ReTr} U_{p_s} + \sum_{l_t} \bar{\Psi} V \Psi + \xi \sum_{l_s} \bar{\Psi} S \Psi \qquad (4)$$

the positive answer exists if $\beta_s = 0$ and $\xi = a_t/a_s \to 0$, *i.e.* in the static limit for the matter fields.

$$V_{tt'}(x) = 2a_t m_f \delta_{tt'} + e^{a_t \mu_f} U_0(x,t) \delta_{t,t'-1} - e^{-a_t \mu_f} U_0^{\dagger}(x,t') \delta_{t,t'+1} \qquad (5)$$

Model	Dual positive weight
$Z(N)$ and XY spin models, $\mu \neq 0$	Yes
$O(N)$ linear and non-linear sigma models, $\mu \neq 0$	Yes
Principal chiral models, $\mu \neq 0$	Yes
Polyakov loop spin models, $\mu \neq 0$	Yes
Pure Abelian LGT	Yes
Pure non-Abelian LGT	Yes, only for $n0$-link action
Abelian LGT, $\mu \neq 0$, static quarks	Yes
Abelian LGT, $\mu \neq 0$, $2d$	Yes, if $m = 0$
Abelian LGT, $\mu \neq 0$, $d > 2$	No
$Z(3)$ LGT with $Z(3)$ matter fields	Yes
Scalar Lattice QCD, $\mu \neq 0, \beta = 0$	Yes
Full scalar Lattice QCD, $\mu \neq 0$	No
Lattice QCD, $\mu \neq 0, \beta = 0$	Yes (partially)
Full lattice QCD, $\mu \neq 0$	No
Pure $SU(N)$ LGT with θ-term, $2d$	Yes
$O(3)$ non-linear sigma model, $\theta \neq 0$	No
Pure $SU(N)$ LGT with θ-term, $4d$	No

Duals of Polyakov loop models

Strategy

- Integration over fermion fields.

- Integration over spatial gauge fields (usually requires some approximation for fermion determinant and/or Wilson gauge action). Resulting theory is an effective d-dimensional spin model of interacting Polyakov loops, $W(x) = \prod_{t=1}^{N_t} U_0(x,t)$, in the external complex (if $\mu \neq 0$) field.

- Construction of a dual representation for the effective spin model.

Quark determinant is expanded as

$$\mathrm{Det}\,(V+\xi S) = \mathrm{Det}V \sum_{s=0}^{LN_tN/2} \frac{\xi^{2s}}{(2s)!} \sum_{\sigma\in S_{2s}} \mathrm{sgn}(\sigma) P_\sigma(V^{-1}S)\,. \qquad (6)$$

$P_\sigma(X)$ is power sum symmetric function of the matrix argument and the static quark determinant can be computed exactly

$$\mathrm{Det}\,V_{tt'}(x) \sim \prod_x \mathrm{Det}_c \left(1+h_+^f\,W(x)\right)\left(1+h_-^f W^\dagger(x)\right) \qquad (7)$$

$h_\pm = \exp(-N_t \sinh^{-1} a_t m \pm \beta\mu)$. By Cauchy identity

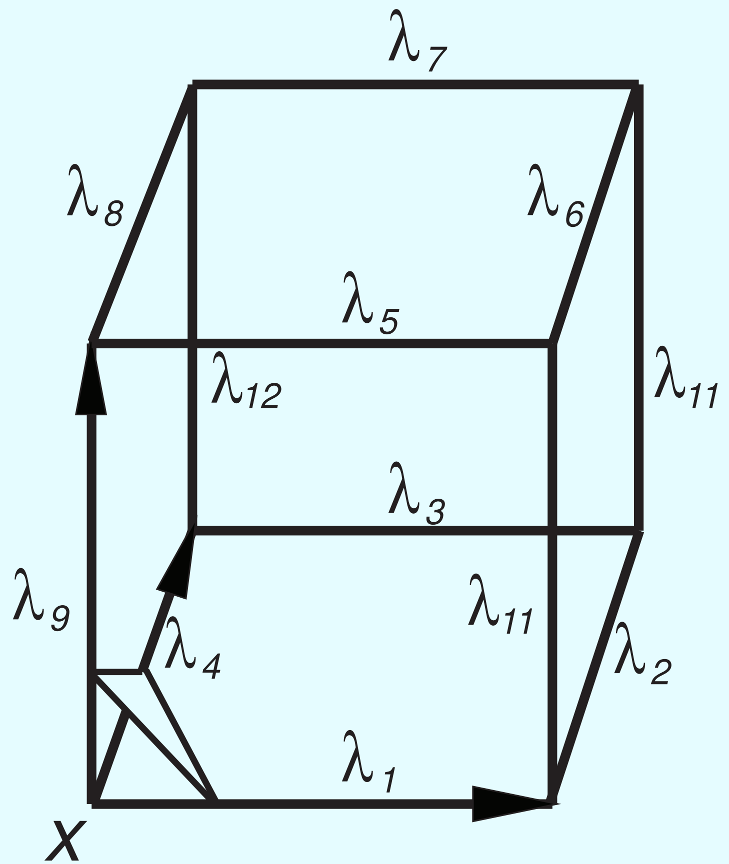
$$\prod_{f=1}^{N_f} \mathrm{Det}\,V_{tt'}(x) = \sum_{r,s} \chi_r(W(x))\,\chi_s(W^*(x))\,\chi_{r'}(h_+^f)\,\chi_{s'}(h_-^f) \qquad (8)$$

Polyakov loop model: $\beta_s = 0$, $\xi = 0$, arbitrary β_t

$$Z = \int \prod_x dW(x) \prod_{x,n} \left[\sum_\lambda D_\lambda^{N_t}(\beta_t) \chi_\lambda(W(x)) \chi_\lambda(W^\dagger(x+e_n)) \right]$$
$$\times \prod_{f=1}^{N_f} \prod_x \mathrm{Det}\, V_{tt'}(x),$$

$D_\lambda(\beta)$ - coefficients of the character expansion. Its dual is given by ($d = 3$, one flavour of staggered fermions)

$$Z = \sum_{\{q(x)\}=-\infty}^{\infty} \sum_{\{\rho(x)\}} \sum_{\{k(x),l(x)\}=0}^{N} \times \prod_x e^{-(k(x)+l(x))m - q(x)N\mu} \prod_{c_{\mathrm{even}}} B_c(\rho(x)) \,, \tag{9}$$



The order of coupling of the link representatons in the integrand of one-site group integrals: in three-dimensional theory representations are coupled inside every even cube of the lattice.

$$B_c(\rho(x)) = \sum_{\lambda_1=0}^{\infty} D_{\lambda_1}(\beta_t) \ldots \sum_{\lambda_{12}=0}^{\infty} D_{\lambda_{12}}(\beta_t) \sum_{\sigma_1,\gamma_1} \cdots \sum_{\sigma_8,\gamma_8}$$
$$\times C^{\sigma_1}_{\lambda_1\ \lambda_4} C^{\sigma_2}_{\lambda_2\ \bar{\lambda}_1} C^{\sigma_3}_{\bar{\lambda}_3\ \bar{\lambda}_2} C^{\sigma_4}_{\lambda_3\ \bar{\lambda}_4} C^{\sigma_5}_{\bar{\lambda}_5\ \bar{\lambda}_8} C^{\sigma_6}_{\lambda_5\ \bar{\lambda}_6} C^{\sigma_7}_{\lambda_6\ \lambda_7} C^{\sigma_8}_{\lambda_8\ \bar{\lambda}_7}$$
$$C^{\gamma_1}_{\sigma_1\ \lambda_9} C^{\gamma_5}_{\sigma_5\ \lambda_9} C^{\gamma_2}_{\sigma_2\ \lambda_{10}} C^{\gamma_6}_{\sigma_6\ \lambda_{10}} C^{\gamma_3}_{\sigma_3\ \lambda_{11}} C^{\gamma_7}_{\sigma_7\ \lambda_{11}} C^{\gamma_4}_{\sigma_4\ \lambda_{12}} C^{\gamma_8}_{\sigma_8\ \lambda_{12}}$$
$$\prod_{i=1}^{4} C^{\rho(x_i)+q(x_i)^N}_{\gamma_i\ 1^{k(x_i)}} \prod_{i=5}^{8} C^{\rho(x_i)}_{\gamma_i\ 1^{l(x_i)}} . \tag{10}$$

$C^{\gamma}_{\sigma\ \lambda}$ are the Littlewood-Richardson coefficients (positive integer numbers).

Polyakov loop model: $\beta_s = 0$, $\xi = 0$, $\beta_t < 1$, $m >> 1$

$$Z = \int \prod_x dW(x) e^{\beta_{eff} \sum_{x,\nu} \mathrm{ReTr} W(x) \mathrm{Tr} W^\dagger(x+e_\nu)} \tag{11}$$

$$\prod_x \prod_{f=1}^{N_f} \det[1 + h_+ W(x)] \det[1 + h_- W^\dagger(x)]$$

$$Z = \sum_{\{r(l)\}=-\infty}^{\infty} \sum_{\{s(l)\}=0}^{\infty} \prod_l \frac{\left(\frac{\beta}{2}\right)^{|r(l)|+2s(l)}}{(s(l)+|r(l)|)!s(l)!} \prod_x R_N(n_+(x), n_-(x)) \, ,$$

$$n_\pm(x) = \sum_{i=1}^{2d} \left(s(l_i) + \frac{1}{2}|r(l_i)| \right) \pm \frac{1}{2} \sum_{\nu=1}^{d} \left(r_\nu(x) - r_\nu(x - e_\nu) \right) . \tag{12}$$

Function $R_N(n,p)$ depends on N, N_f, $h_\pm$. For $N = 3, N_f = 1$

$$
\begin{aligned}
& R_3(n,p) = Q_3(n+1,p)\left(h_+ + h_-^2 + h_+h_-^3 + h_+^3h_-^2\right) \\
+ \;& Q_3(n,p)\left(1 + h_+^3 + h_-^3 + h_+^3h_-^3\right) \\
+ \;& Q_3(n,p+1)\left(h_- + h_+^2 + h_+^3h_- + h_+^2h_-^3\right) \\
+ \;& Q_3(n+1,p+1)\left(h_+h_- + h_+^2h_-^2\right) \\
+ \;& Q_3(n+2,p)h_+h_-^2 + Q_3(n,p+2)h_+^2h_- \,.
\end{aligned}
$$

$$
Q_N(n,p) \;=\; \sum_{\lambda\vdash \mathrm{min}(n,p)} d(\lambda)\; d(\lambda + |q|^N). \tag{13}
$$

$d(\lambda)$ is the dimension of the permutation group S_r in the representation λ, $q = (p-n)/N$ (when q is not an integer $Q_N(n,p) = 0$).

IV. Abelian models with static quarks

For $Z(N)$ and $U(1)$ models with the static staggered or Wilson fermions

$$Z_\Lambda = \sum_{\{s_l\}=0}^{N-1} e^{S_g(s_p)} \prod_{f=1}^{N_f} \mathrm{Det}\, V_{tt'}(x) \ , \tag{14}$$

$$Z_\Lambda = \int_0^{2\pi} \prod_l \frac{d\phi_l}{2\pi} e^{S_g(\phi_p)} \prod_{f=1}^{N_f} \mathrm{Det}\, V_{tt'}(x) \ . \tag{15}$$

the dual formulation with a positive weight can be constructed for any number of flavors, in any dimension and for arbitrary gauge action (Borisenko, et.al., '22)

$$e^{S_g(\phi_p)} = \sum_r C_r e^{ir\phi(p)} \ , \ C_r > 0 \ . \tag{16}$$

E.g., $2+1$-dimensional $U(1)$ model with N_f degenerate flavors

$$Z = \sum_{\{q(x)\}=-\infty}^{\infty} \prod_{l_t} C_{q(x)-q(x+e_0)}(\beta_s) \times \sum_{\rho_n(x)=-\infty}^{\infty} \prod_{l_s} C_{q(x)-q(x+e_n)+\rho_n(x)}(\beta_t) \prod_p K_{\rho(p)} \ , \tag{17}$$

$$K_\rho = \left(\frac{h_+}{h_-}\right)^{\frac{\rho}{2}} \frac{(gN_f)!}{(gN_f+\rho)!} P^{\rho}_{gN_f}\left(\frac{1+h_+h_-}{1-h_+h_-}\right) \ , \tag{18}$$

$$\rho(p) = \rho_1(x) + \rho_2(x+e_1) - \rho_1(x+e_2) - \rho_2(x).$$

$g = 1(2)$ for the staggered (Wilson) fermions, $P^{\rho}_n(x)$ is the associated Legendre function. Product $\prod_p$ runs over all space-like plaquettes of the dual lattice at a fixed time slice.

Complex masses in $Z(N)$ with two flavors

For $Z(N)$ model one has to make the following replacement

$$\sum_{r=-\infty}^{\infty} \to \sum_{r=0}^{N} \sum_{q=-\infty}^{\infty} \ , \ \ C_r(\beta) \to C_{r+qN}(\beta) \ . \tag{19}$$

This partition function can be evaluated as

$$Z = C_0^{LN_t}(\beta) \ \prod_{f=1}^{N_f} A_f^L \ \sum_{i=0} \ \lambda_i^L \ , \tag{20}$$

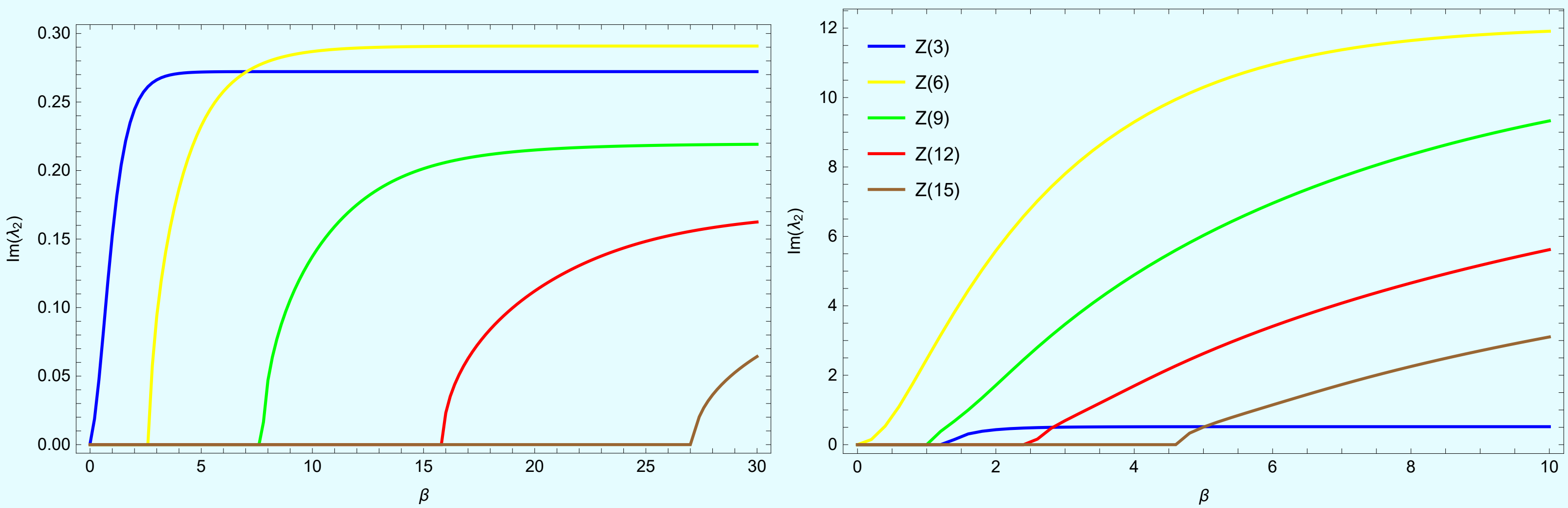
where λ_i are eigenvalues of the following transfer matrix

$$T_{r_1 r_2} = \sqrt{B_{r_1} B_{r_2}} \ \sum_{\substack{k_1=0 \\ k'_1=0}}^{1} \cdots \sum_{\substack{k_{N_f}=0 \\ k'_{N_f}=0}}^{1} \ \prod_{f=1}^{N_f} (h_+^f)^{k_f} \ (h_-^f)^{k'_f} \ . \tag{21}$$

$B_r = C_r^{N_t}(\beta)/C_0^{N_t}(\beta)$ and all configurations are subject to constraint $r_1 - r_2 + \sum_{f=1}^{N_f}\left(k_f - k'_f\right) = 0(\mathrm{mod}N)$. The Wilson action and two staggered fermion flavors. When chemical potentials are zero all eigenvalues are real. This leads to a familiar exponential decay of the connected part of the Polyakov loop correlation function. When non-zero chemical potentials are introduced, one finds such values of the coupling constant above which the eigenvalues become complex. The second and the third eigenvalues are conjugate to each other. This implies the following decay of the two-point correlation function of the Polyakov loops

$$\langle W(0)W^*(R)\rangle_c \approx e^{-m_r R}\cos m_i R\,. \quad (22)$$

In the limit $N \to \infty$: $Z(N) \to U(1)$ and $Im\lambda \to 0$.



Plots of the imaginary part of the 2nd eigenvalue of the transfer matrix of $Z(N)$ model with two flavors of staggered fermions as a function of the coupling constant. Left panel: $m_1 = 3, m_2 = 1$, $\mu_1 = -0.16, \mu_2 = 0.45$. Right panel: $m_1 = 1, m_2 = 0.1$, $\mu_1 = 2, \mu_2 = 1$.

V. Polyakov loop model: the 't Hooft-Veneziano limit

In the 't Hooft-Veneziano limit the problem is reduced to the calculation of the group integral

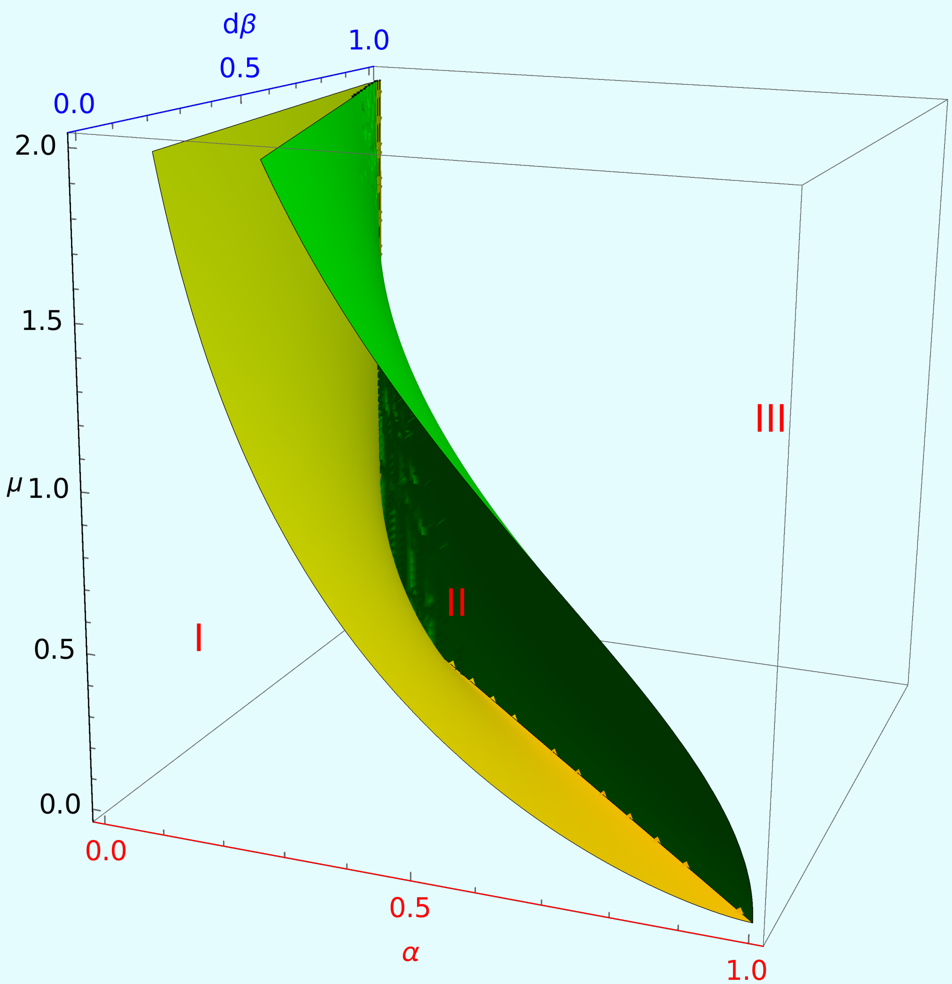
$$\int dW\ W^n (W^\dagger)^p \ \prod_{f=1}^{N_f} \mathrm{Det}\ V_{tt'}(W(x)) \tag{23}$$

in the limit $N \to \infty$, $N_f \to \infty$ such that $N_f/N = \kappa$ is fixed.

Phase diagram at $\mu \neq 0$

Critical surface (yellow) of the 3rd order phase transition:

$$\mu \ = \ \ln\left(1 + \sqrt{1 - z^2}\right) - \ln z - \sqrt{1 - z^2}\ , \ z = \frac{\kappa h}{1 - d\beta}\ . \tag{24}$$



Region I: no dependence on μ, $SU(N)$ free energy coincides with $U(N)$ one. Mass spectrum is real.

Region II: non-trivial dependence on μ. Non-zero particle density and complex masses.

Region III: Masses are real. $z \approx \cosh^{-1} \mu$.

Correlation functions and screening masses

The correlation function of an arbitrary form

$$\Gamma(\eta,\bar{\eta}) = \langle \prod_x W(x)^{\eta(x)} W^*(x)^{\bar{\eta}(x)} \rangle \tag{25}$$

is evaluated by integrating over Gaussian fluctuations around large N, N_f solution. Results below for $d=3$:

N-point function and baryon potential: pure gauge theory

$$\Gamma_N(\sigma) \sim \sum_x \prod_{i=1}^{N} G_{x,x(i)}(\sigma) \, , \; \sigma = \sqrt{\frac{2}{\beta}(1 - d\beta)} \, . \tag{26}$$

$x(i)$ - position of N static quarks. Green function

$$G_{x,x'} = \frac{\text{const}}{R^{\frac{d}{2}-1}} \, K_{\frac{d}{2}-1}(\sigma R) \, , \; R^2 = \sum_{n=1}^{d} (x_n - x'_n)^2 \, .$$

Calculating $\Gamma_N(\sigma)$ in the continuum reduces to the geometric median problem: find a point y which minimizes $\sum_{i=1}^{N} \sqrt{\sum_{n=1}^{d} (y_n - x_n(i))^2}$.
If $N = 3$ this gives famous Y law for the baryon potential

$$\Gamma_3(\sigma) \sim \exp[-\sigma Y] \, . \tag{27}$$

Complex masses and oscillating decay

Connected part of the Polyakov loop correlation in Regions II and III

$$\langle\, W(0)W^*(R)\,\rangle_c = MM^*(G_R(m_1) + G_R(m_2))\;. \tag{28}$$

M is magnetization, $G_R(m_i)$ are diagonal correlators in the correlation matrix

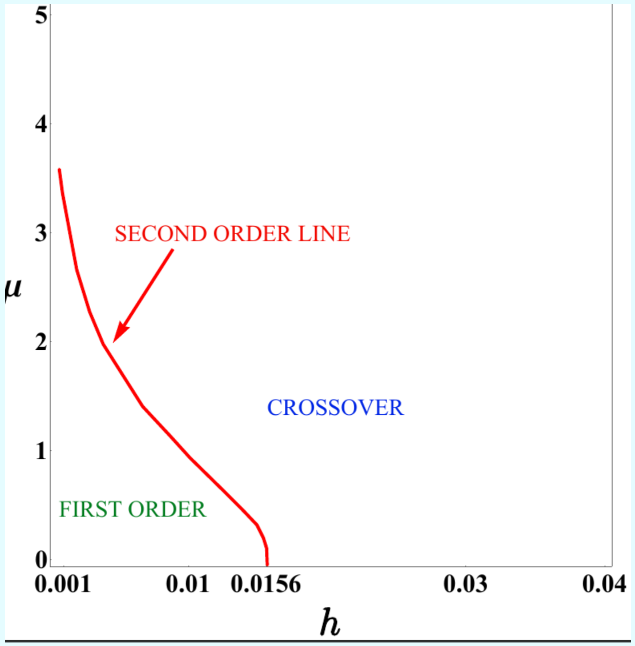
$$\Gamma(x,y) = \begin{pmatrix} \langle ReTrW(x)ReTrW(y)\rangle & \langle ReTrW(x)ImTrW(y)\rangle \\ \langle ImTrW(x)ReTrW(y)\rangle & \langle ImTrW(x)ImtrW(y)\rangle \end{pmatrix}$$

at $\mu \neq 0$. If $\mu = 0$, $m_{1,2}$ correspond to chromo-electric and chromo-magnetic masses. If $\mu \neq 0$, in the Region II the masses are complex: $m_1 = m_2^* = m_r + im_i$. This leads to an exponential oscillating decay of the correlations

$$\langle\, W(0)W^*(R)\,\rangle_c \;\sim\; e^{-m_r R}\, \cos m_i R\;. \tag{29}$$

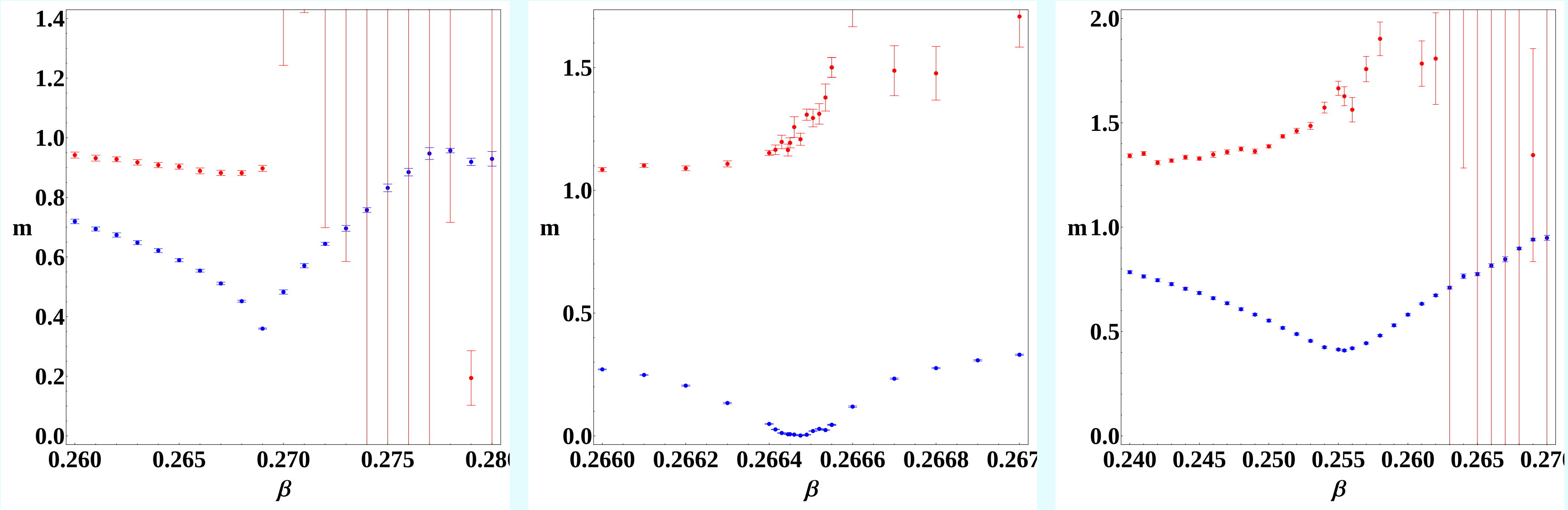
In the Region III the masses are real, $m_2 > m_1$. No phase transition separating Regions II and III has been found.

VI. Polyakov loop model: Monte-Carlo simulations

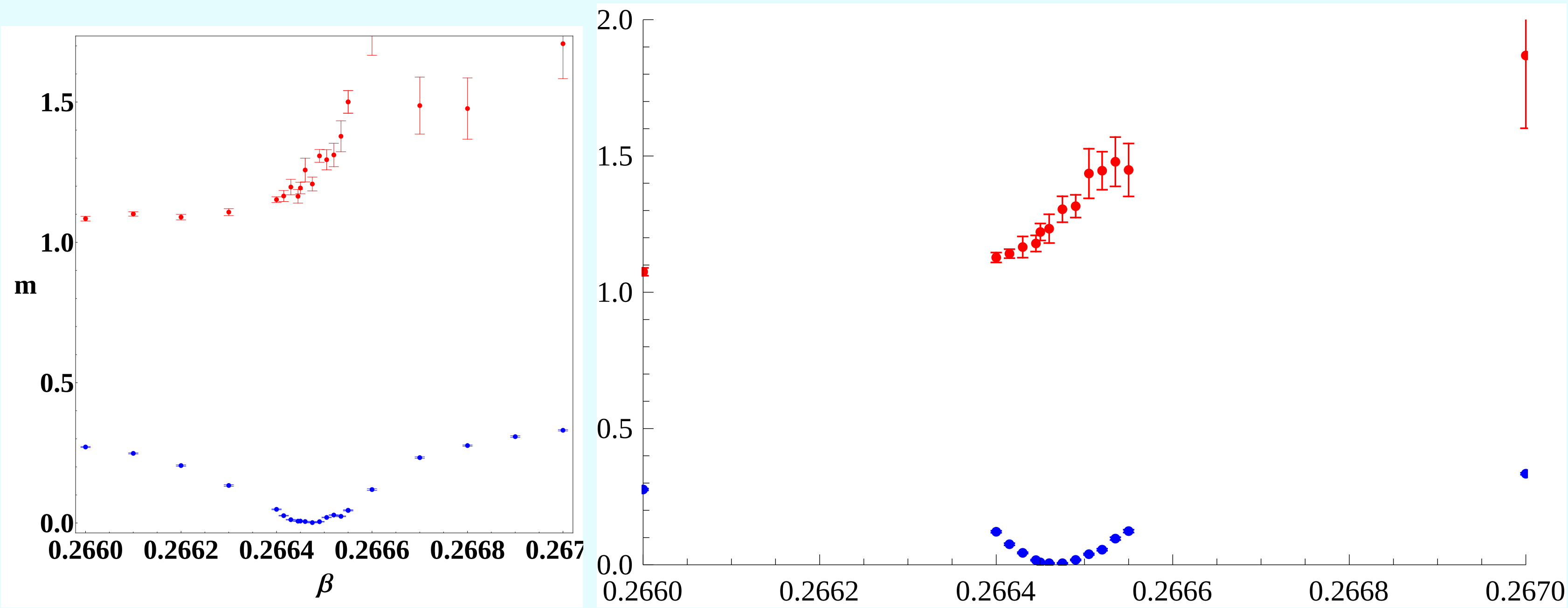


Estimated phase diagram. Data are well fitted by the function $\mu_C = -a \ln h + c - bh^2$, with $a = 0.988$, $c = -3.4$, $b = 1406$. The line of the second order phase transition might persist in the heavy-dense limit.

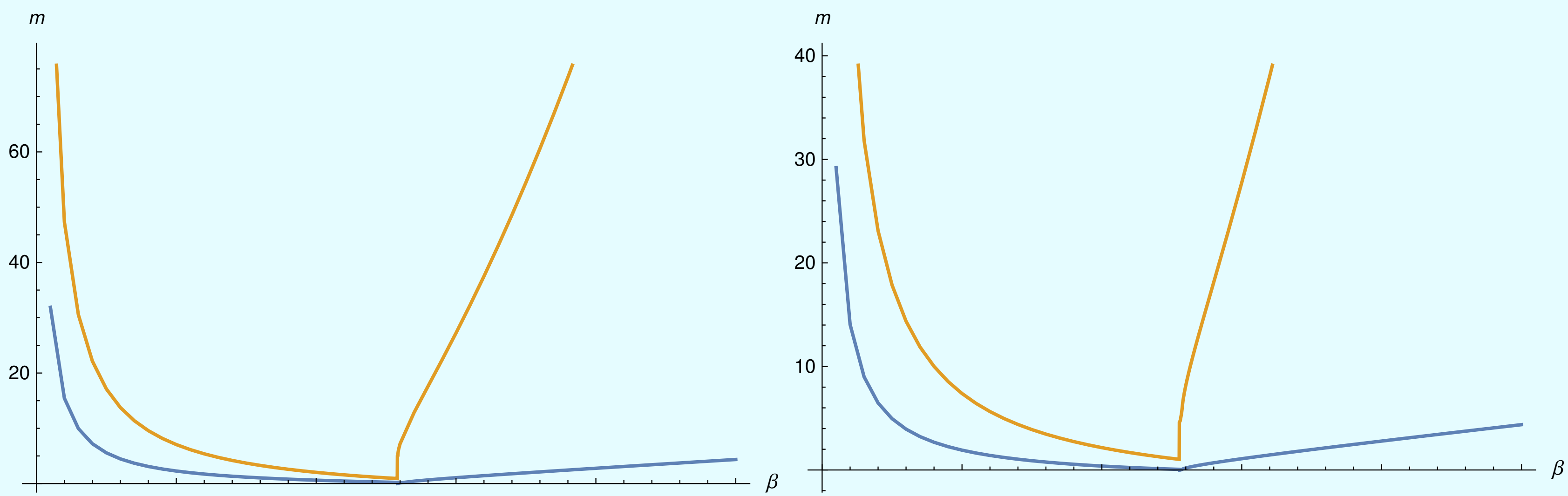
See also C. Gattringer et.al., NPB 862; M. Fromm et.al., JHEP 01, 042.



Behavior of the masses m_1 and m_2 on the lattice $L = 20$ as a function of β for different values of $h = e^{-m}$ and μ. Left: first order phase transition. Middle: second order phase transition. Right: crossover.



The same as before on the lattices $L = 24$ (left) and $L = 32$ (right) in the vicinity of the second order phase transition.



Behavior the masses m_1 (blue line) and m_2 (yellow line) as a function of β calculated in the mean-field approximation. Left panel: $m = 4.605$, $\mu = 0$; Right panel: $m = 4.605$, $\mu = 0.9635$. These values correspond to the second order phase transition.

VII. Summary

Complex masses was shown to exist in several lattice models if the fermion determinant is taken in the static approximation:

- $(1+1)$-dimensional QCD
- $Z(3)$ spin model in an external complex magnetic field
- Abelian $Z(N)$ LGTs with static quark determinant
- $(3+1)$-dimensional $SU(N)$ LGT in the combined $\beta_s = 0$ and the 't Hooft-Veneziano limits

The case of $SU(3)$ remains open
How to go beyond static approximation for quark determinant?

Existence of the complex spectrum at finite density:
another challenge for Monte-Carlo simulations of QCD